Universal spaces in the theory of transfinite dimension, I

by

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Abstract. R. Pol has shown that for every countable ordinal α, there exists a universal space for separable metrizable spaces X with $\operatorname{ind} X = \alpha$.

We prove that for every countable limit ordinal λ, there is no universal space for separable metrizable spaces X with $\operatorname{Ind} X = \lambda$. This implies that there is no universal space for compact metrizable spaces X with $\operatorname{Ind} X = \lambda$. We also prove that there is no universal space for compact metrizable spaces X with $\operatorname{ind} X = \lambda$.

1. Introduction. Our terminology and notation follow the books [3] and [1], with the exception of the boundary, closure and interior of a subset A of a space X, which are denoted by $\operatorname{bd} A$, $\operatorname{cl} A$ and $\operatorname{int} A$, respectively, and the diameter of a subset A of a metric space (X, ϱ), which is denoted by $\operatorname{diam} A$ or $\operatorname{diam}_\varrho A$. If Y is a subspace of X, and A is contained in Y, then the boundary, closure and interior of A in Y are denoted by $\operatorname{bd}_Y A$, $\operatorname{cl}_Y A$ and $\operatorname{int}_Y A$, respectively.

The transfinite dimensions ind and Ind are transfinite extensions of the classical Menger–Urysohn small inductive dimension ind and Brouwer–Čech large inductive dimension Ind, respectively.

1.1. DEFINITION (see [4]). Let X denote a regular space, and let α be the integer -1, an ordinal, or the symbol ∞. The following conditions define the *small transfinite dimension* of X:

(1.1) $\operatorname{ind} X = -1$ if and only if $X = \emptyset$,

(1.2) $\operatorname{ind} X \leq \alpha \geq 0$ if for every $x \in X$ and every neighbourhood $V \subseteq X$ of x, there exists an open set $U \subseteq X$ such that $x \in U \subseteq V$ and $\operatorname{ind} \operatorname{bd} U < \alpha$ (i.e., $\operatorname{ind} \operatorname{bd} U \leq \beta$ for a $\beta < \alpha$),

(1.3) $\operatorname{ind} X = \alpha$ if $\operatorname{ind} X \leq \alpha$ and the inequality $\operatorname{ind} X < \alpha$ does not hold,

(1.4) $\operatorname{ind} X = \infty$ if $\operatorname{ind} X > \alpha$ (i.e., the inequality $\operatorname{ind} X \leq \alpha$ does not hold) for every ordinal α.

We say that *the small transfinite dimension* of a regular space X *at a point* x *does not exceed* an ordinal α, and we write $\text{ind}_x X \leq \alpha$, if for every neighbourhood $V \subseteq X$ of x, there exists an open set $U \subseteq X$ such that $x \in U \subseteq V$ and $\text{ind bd}\, U < \alpha$. If $\text{ind}_x X \leq \alpha$ and $\text{ind}_x X > \beta$ for every $\beta < \alpha$, then we say that *the small transfinite dimension* of X *at* x *is* α, and we write $\text{ind}_x X = \alpha$. We say that *the small transfinite dimension* of X *at* x *is* ∞ if $\text{ind}_x X > \alpha$ for every ordinal α.

It is easy to check that $\text{ind}\, X \leq \alpha$ if and only if $\text{ind}_x X \leq \alpha$ for every $x \in X$.

1.2. DEFINITION (see [16]). Let X denote a normal space, and let α be the integer -1, an ordinal, or the symbol ∞. The following conditions define the *large transfinite dimension* of X:

(1.5) $\text{Ind}\, X = -1$ if and only if $X = \emptyset$,

(1.6) $\text{Ind}\, X \leq \alpha \geq 0$ if for every closed set $A \subseteq X$ and every open set $V \subseteq X$ which contains A, there exists an open set $U \subseteq X$ such that $A \subseteq U \subseteq V$ and $\text{Ind bd}\, U < \alpha$,

(1.7) $\text{Ind}\, X = \alpha$ if $\text{Ind}\, X \leq \alpha$ and the inequality $\text{Ind}\, X < \alpha$ does not hold,

(1.8) $\text{Ind}\, X = \infty$ if $\text{Ind}\, X > \alpha$ for every ordinal α.

Observe that if we replace the ordinal α by a natural number n in Definitions 1.1 and 1.2, then we obtain the definitions of the small inductive dimension of Menger–Urysohn and of the large inductive dimension of Brouwer–Čech.

Let X be a topological space and A, B a pair of disjoint subsets of X; we say that a set $L \subseteq X$ is a *partition* (in X) *between* A *and* B if there exist open sets $U, V \subseteq X$ satisfying:

$$A \subseteq U, \quad B \subseteq V, \quad U \cap V = \emptyset \quad \text{and} \quad X - L = U \cup V.$$

One can check that a regular space X (a normal space X) satisfies $\text{ind}\, X \leq \alpha$ ($\text{Ind}\, X \leq \alpha$) if and only if for every $x \in X$ and each closed set $B \subseteq X$ such that $x \notin B$ (resp. for every pair A, B of disjoint closed subsets of X) there exists a partition L between x and A (a partition L between A and B) such that $\text{ind}\, L < \alpha$ ($\text{Ind}\, L < \alpha$).

For simplicity of notation we assume that $\alpha < \infty$, $\alpha + \infty = \infty + \alpha = \infty$, and $\alpha \cdot \infty = \infty \cdot \alpha = \infty$ for every ordinal α.

If X is a separable metrizable space and $-1 < \text{ind}\, X < \infty$ ($-1 < \text{Ind}\, X < \infty$), then $\text{ind}\, X$ ($\text{Ind}\, X$) is a countable ordinal, i.e., $\text{ind}\, X < \omega_1$ ($\text{Ind}\, X < \omega_1$) (see [4] and [16], or [2], Theorems 3.5 and 3.8). Obviously, $\text{ind}\, X \leq \text{Ind}\, X$ for every normal space X, but the inverse inequality does not hold; there exists a compact metrizable space X such that $\text{ind}\, X < \text{Ind}\, X$ (see [9]). For every ordinal α, there exist compact metrizable spaces X and Y such that

ind $X = \alpha$ and Ind $Y = \alpha$ (see Section 2). For a deeper discussion we refer the reader to R. Engelking's survey [2].

We shall only be concerned with the question of universal spaces. Let us recall that a space X is *universal* in a class $\mathcal{C}$ of spaces if X belongs to $\mathcal{C}$ and every space of $\mathcal{C}$ is embeddable in X. The following questions are natural in this context (α is a countable ordinal).

1. Does there exist a universal space for compact metrizable spaces X with ind $X \leq \alpha$?

2. Does there exist a universal space for compact metrizable spaces X with Ind $X \leq \alpha$?

3. Does there exist a universal space for separable metrizable spaces X with ind $X \leq \alpha$?

4. Does there exist a universal space for separable metrizable spaces X with Ind $X \leq \alpha$?

For finite α, the answers to the four questions are affirmative and belong to the classical results of dimension theory (see for instance [1], §1.11). For infinite α, questions 3 and 4 were asked e.g. by R. Engelking ([2], Problem 5.11), and the same set of questions were raised by L. Luxemburg ([11], Problem 8.4); question 1 was recently recalled in R. Pol's article [15] (Question 14.1 (425)).

Observe first that questions 2 and 4 are equivalent. This follows from the compactification theorem of L. Luxemburg (see [8], and [10] for the proof).

1.A. THEOREM ([8]). *Every separable metrizable space X has a metrizable compactification Z such that* Ind $Z = $ Ind X.

Now, let us state the results relating to our questions that can be found in the literature. Question 3 was answered in the affirmative by R. Pol:

1.B. THEOREM ([14]). *For every countable ordinal α, there exists a universal space in the class of separable metrizable spaces X with* ind $X \leq \alpha$.

A partial positive result related to questions 1 and 2 can be found in R. Pol's paper [13]: for each countable ordinal α, there exist a compact metrizable space X_α with ind $X_\alpha < \infty$ and a compact metrizable space Y_α with Ind $Y_\alpha < \infty$ such that X_α contains topologically all compact metrizable spaces X with ind $X \leq \alpha$ and Y_α contains topologically all compact metrizable spaces Y with Ind $Y \leq \alpha$.

On the other hand, some negative results concerning questions 1–4 were announced by L. Luxemburg (see [7]); to wit, for any infinite α, there is no universal space in the class of all completely metrizable separable spaces X such that ind $X \leq \alpha$ and X is the union of a countable family of finite-dimensional closed sets, nor in the class of all compact metrizable spaces X

such that $\operatorname{ind} X \leq \alpha$ ($\operatorname{Ind} X \leq \alpha$) and X is the union of a countable family of finite-dimensional closed sets.

The present paper is devoted to the particular case of our questions, namely to the case when $\alpha = \lambda$ is a limit ordinal. We shall prove that in this case, the answers to questions 1 and 4, and thus also to question 2, are negative.

Questions 1, 2 and 4 remain open for non-limit ordinals $\alpha \geq \omega$. We know neither the least possible transfinite dimension $\operatorname{ind} X_\lambda$ of a compact metrizable space X_λ containing topologically all compact metrizable spaces X with $\operatorname{ind} X = \lambda$, nor the least possible transfinite dimension $\operatorname{Ind} Y_\lambda$ of a compact metrizable space Y_λ containing topologically all compact metrizable spaces Y with $\operatorname{Ind} Y = \lambda$ (see [13], §4, 1). The answers to these questions, even in the case $\lambda = \omega_0$, seem to have interesting applications in the theory of transfinite dimension.

Acknowledgements. The paper contains some of the results of my Ph.D. thesis supervised by Professor R. Engelking, whom I would like to thank for his comments and improvements. I am also grateful to Professor R. Pol who suggested the use of the spaces C_λ^σ and D_λ^σ (see Section 4) instead of the more complicated spaces originally used to the same end.

2. Smirnov's spaces. Yu. M. Smirnov [16] defined a sequence $\{S_\alpha : \alpha < \omega_1\}$ of compact metrizable spaces. The definition is by transfinite induction on α: S_0 is a one-point space; if $\alpha = \beta + 1$ for a $\beta < \alpha$, then $S_\alpha = S_\beta \times I$; if $\alpha = \lambda$ is a limit ordinal, then S_α is the one-point compactification of the sum of topological spaces $\bigoplus\{S_\beta : \beta < \alpha\}$ (denote by p_λ the unique point of the remainder). The space S_{ω_0+1} is exhibited in Fig. 2.1.

Yu. M. Smirnov evaluated the large transfinite dimension of S_α.

2.A. THEOREM ([16]). *For every* $\alpha < \omega_1$, *we have* $\operatorname{Ind} S_\alpha = \alpha$.

Smirnov's spaces are also a source of examples of compact metrizable spaces with given small transfinite dimension. In order to get these examples, let us quote the following theorems.

2.B. THEOREM ([6]). *For every hereditarily normal compact space* X, $\operatorname{Ind} X \leq \omega_0 \cdot \operatorname{ind} X$.

2.C. THEOREM ([17]). *For every metrizable space* X, $\operatorname{Ind}(X \times I) \leq \operatorname{Ind} X + 1$.

Combining Theorems 2.A–2.C and the obvious equality

(2.1) $\operatorname{ind} S_\lambda = \sup\{\operatorname{ind} S_\alpha : \alpha < \lambda\}$,

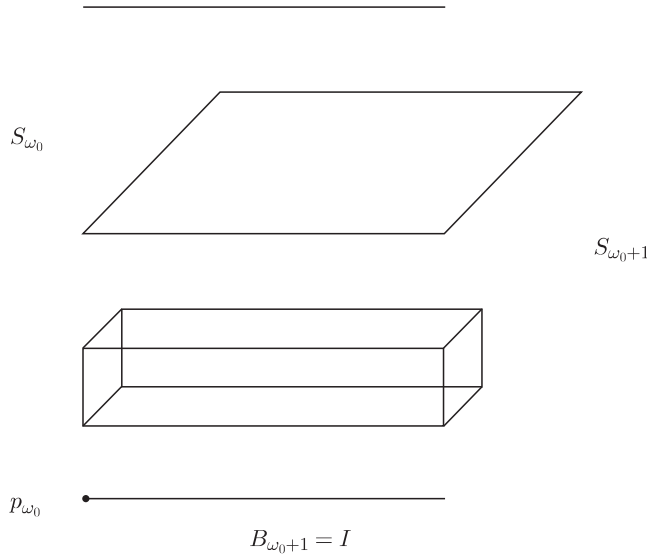for every limit ordinal $\lambda < \omega_1$, we obtain the following corollary.

S_{ω_0}

S_{ω_0+1}

p_{ω_0}

$B_{\omega_0+1} = I$

Fig. 2.1

COROLLARY. *For every countable ordinal α, there exists a countable ordinal $\beta \geq \alpha$ such that $\operatorname{ind} S_\beta = \alpha$.*

We denote by $\beta(\alpha)$ the smallest ordinal β such that $\operatorname{ind} S_\beta = \alpha$. Note that, as shown in [9], $\beta(\alpha)$ is greater than α for some ordinals α, and $\operatorname{ind} S_\alpha$ is unknown for some α (see [2], Problems 2.3 and 2.4). From (2.1) it follows immediately that

(2.2)   if α is a non-limit ordinal, then so is $\beta(\alpha)$.

The rest of this section is devoted to a property of Smirnov's spaces (see Theorem 2.1) which will be of importance in the proofs of Theorems 5.1 and 5.2. In order to formulate and prove Theorem 2.1, we quote two results and introduce a notion.

The following lemma is a consequence of Lemma 7 of [16] and Lemma 1.2.9 of [1]; recall that if $\alpha = \lambda + n$, where λ is a limit ordinal and n is a natural number, then $S_\alpha = S_\lambda \times I^n$.

2.D. LEMMA ([16]). *Let $\alpha = \lambda + n$, where λ is a limit ordinal and n is a natural number. For every partition L in S_α between $S_\lambda \times A$ and $S_\lambda \times B$, where A, B is a pair of opposite $(n-1)$-dimensional faces of I^n, we have $\operatorname{Ind} L \geq \lambda + (n-1)$.*

Every non-limit ordinal $\alpha \geq \omega_0$ can be uniquely represented as the sum $\lambda + n$ of a limit ordinal λ and a natural number n. From the construction of S_α it follows that $S_\alpha = S_\lambda \times I^n$. Denote by B_α the set $\{p_\lambda\} \times I^n$; we call B_α the *base* of S_α. Sometimes, we will identify B_α and the n-dimensional cube

I^n; in particular, we will write $S_\alpha = S_\lambda \times B_\alpha$. Let $B_\alpha = S_\alpha$ for $\alpha < \omega_0$, i.e., if $\alpha = n$, then $B_\alpha = S_\alpha = I^n$. Thus for every non-limit ordinal $\alpha < \omega_1$, the base of S_α is a finite-dimensional cube.

2.E. THEOREM ([17]). *If a hereditarily normal space X can be represented as the union $A_1 \cup A_2$ of closed subspaces, and there is a homeomorphism $h : A_1 \to A_2$ such that $h(x) = x$ for every $x \in A_1 \cap A_2$, then* ind $X =$ ind $A_1 =$ ind A_2.

The above theorem appears in the section of [17] which considers only hereditarily normal spaces; one can verify, however, that the proof still goes when we only assume that X is regular. In the present paper, we only need Theorem 2.E in the form given above.

In the proof of the following theorem and in the sequel, we will use the monotonicity of ind and of Ind with respect to closed subspaces (see [2], Proposition 3.3).

2.1. THEOREM. *Let $\alpha = \lambda + n$, where λ is a countable limit ordinal, and n is a natural number. For each partition K in S_α between any distinct points $a, b \in B_\alpha$ we have* ind $K \geq$ ind $S_\alpha - 1$ *and* Ind $K \geq$ Ind $S_\alpha - 1 = \lambda + (n-1)$.

P r o o f. We prove the first inequality, and then indicate the modifications needed to show the second.

First, we show that

(2.3) $\text{ind}_x S_\alpha \leq \text{ind} K + 1$ for each $x \in B_\alpha$,

i.e., for every $x \in B_\alpha$ and each closed set $F \subseteq S_\alpha$ such that $x \notin F$ there exists a partition L between x and F such that ind $L \leq$ ind K. The reasoning is divided into a few steps.

First, we show that

(2.4)   if x belongs to the geometrical boundary of B_α, and F is the union of $(n-1)$-dimensional faces of B_α not containing a, then there exists a partition with the required property.

Let U and V be disjoint open subsets of S_α such that $a \in U$, $b \in V$ and $K = S_\alpha - (U \cup V)$. Consider a subspace $Q \subseteq B_\alpha$ homeomorphic to the n-dimensional cube, and such that a belongs to the geometrical boundary of Q and the union E of $(n-1)$-dimensional faces of Q not containing a is contained in V; then $K \cap (S_\lambda \times Q)$ is a partition in $S_\lambda \times Q$ between a and E, and ind $K \cap (S_\lambda \times Q) \leq$ ind K. The existence of the required partition in S_α between x and F follows from the existence of a homeomorphism of $S_\lambda \times Q$ onto $S_\alpha = S_\lambda \times B_\alpha$ sending a to x and E onto F.

In the second step, we show that the assumption that x belongs to the geometrical boundary of B_α can be omitted in (2.4), i.e.,

(2.5)   if F is the union of the $(n-1)$-dimensional faces of B_α not containing x, then there exists a partition L between x and F with $\operatorname{ind} L \leq \operatorname{ind} K$.

Assume that $x = (p_\lambda, (x_1,, \ldots, x_n))$ belongs to the geometrical interior of $B_\alpha = \{p_\lambda\} \times I^n$.

Let

$$A_1 = \{(z, (y_1, y_2, \ldots, y_n)) \in S_\lambda \times I^n : y_1 \leq x_1\},$$
$$A_2 = \{(z, (y_1, y_2, \ldots, y_n)) \in S_\lambda \times I^n : y_1 \geq x_1\};$$

further, let $h : S_\alpha \to S_\alpha$ be given by

$$h((z, (y_1, y_2, \ldots, y_n))) = (z, (f(y_1), y_2, \ldots, y_n)),$$

where

$$f(y_1) = \begin{cases} (x_1 - 1)y_1/x_1 + 1 & \text{if } y_1 \leq x_1, \\ x_1(y_1 - 1)/(x_1 - 1) & \text{if } y_1 \geq x_1, \end{cases}$$

for $(z, (y_1, y_2, \ldots, y_n)) \in S_\lambda \times I^n = S_\alpha$.

Observe that h is a homeomorphism of S_α mapping A_1 onto A_2 and such that $h(x) = x$ for every $x \in A_1 \cap A_2$. By (2.4), there exists a partition L_1 in A_1 between x and $F \cap A_1$ such that $\operatorname{ind} L_1 \leq \operatorname{ind} K$; let $L_2 = h(L_1)$. It is easily seen that $L = L_1 \cup L_2$ is a partition in S_α between x and F; by Theorem 2.E, $\operatorname{ind} L = \operatorname{ind} L_1$.

In the third step, we show that

(2.6)   for every closed set $F \subseteq B_\alpha$, there exists a partition L between x and F such that $\operatorname{ind} L \leq \operatorname{ind} K$.

Let E be the union of $(n-1)$-dimensional faces of B_α not containing x. By (2.5), there exists a partition M in S_α between x and E with $\operatorname{ind} M \leq \operatorname{ind} K$; let U and V be disjoint open subsets of S_α such that $x \in U$, $E \subseteq V$ and $M = S_\alpha - (U \cup V)$.

Consider a homeomorphism $h : B_\alpha \to B_\alpha$ with $h(x) = x$ and $h(F) \subseteq V \cap B_\alpha$; it can be defined by taking a neighbourhood of x disjoint from F and expanding it, without moving x, until it covers the set $B_\alpha - V$. Let $g : S_\alpha \to S_\alpha$ be given by $g(y, z) = (y, h(z))$ for $(y, z) \in S_\lambda \times B_\alpha = S_\alpha$. It is easy to observe that $L = g^{-1}(M)$ is a partition in S_α between x and F, and $\operatorname{ind} L = \operatorname{ind} M \leq \operatorname{ind} K$.

Having disposed of these preliminary steps, we can now prove (2.3).

Let $x \in B_\alpha$ be an arbitrary point and $F \subseteq S_\alpha$ an arbitrary closed set not containing x. Since $S_\alpha = B_\alpha$ for $\alpha < \omega_0$, we can assume that $\alpha \geq \omega_0$. Let $E = F \cap B_\alpha$ (see Fig. 2.2, where $\alpha = \omega_0 + 1$). By (2.6), there exists a partition M between x and E such that $\operatorname{ind} M \leq \operatorname{ind} K$; let $U, V \subseteq S_\alpha$ be disjoint open sets such that $x \in U$, $E \subseteq V$ and $M = S_\alpha - (U \cup V)$.
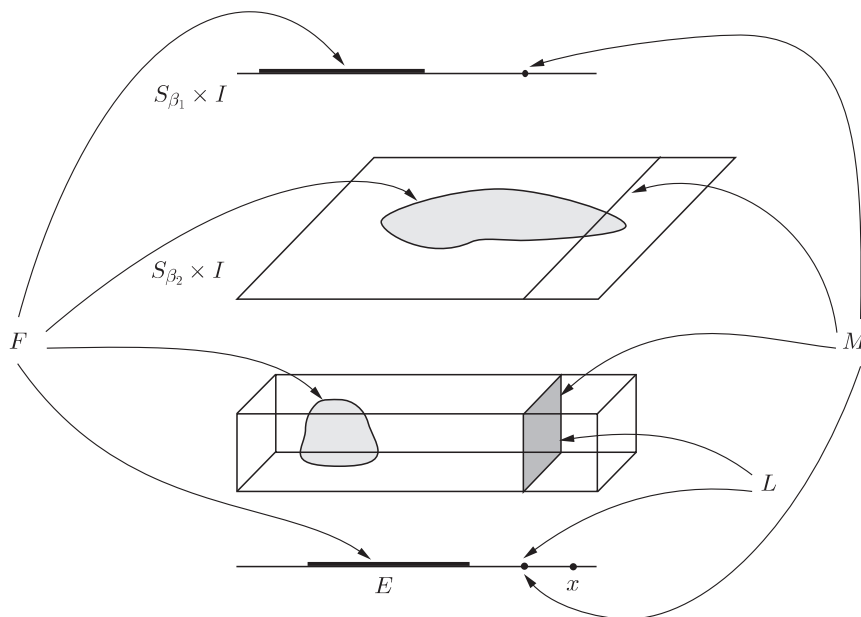
Fig. 2.2

By construction of S_α, we have

$$S_\alpha = B_\alpha \cup \bigcup \{S_\beta \times I^n : \beta < \lambda\}\,;$$

also, each compact subset of S_α disjoint from B_α meets only finitely many sets $S_\beta \times I^n$. Thus

$$[F \cap (U \cup M)] \cap [S_\beta \times I^n] \neq \emptyset$$

only for finitely many β; assume that the above intersection is empty for $\beta \neq \beta_1, \ldots, \beta_k$ (see Fig. 2.2, where $k = 2$).

It is easily seen that $L = M - \bigcup_{i=1}^{k}(S_{\beta_i} \times I^n)$ is a partition between x and F (see Fig. 2.2), and $\operatorname{ind} L \leq \operatorname{ind} M \leq \operatorname{ind} K$.

This proves (2.3). We now prove $\operatorname{ind} S_\alpha \leq \operatorname{ind} K + 1$ by induction on α. For $\alpha < \omega_0$ the inequality is equivalent to (2.3); assume that $\alpha \geq \omega_0$ and the inequality holds for $\beta < \alpha$. By (2.3), we only have to show that $\operatorname{ind}_x S_\alpha \leq \operatorname{ind} K + 1$ for every $x \in S_\alpha - B_\alpha$.

Observe that K contains a partition in $S_\beta \times I^n = S_{\beta+n}$ between distinct points of the base $B_{\beta+n}$ for all but a finite number of ordinals $\beta < \lambda$. Thus, by the inductive assumption, $\operatorname{ind} S_{\beta+n} \leq \operatorname{ind} K + 1$ for those β. Since $\operatorname{ind} S_\nu \leq \operatorname{ind} S_\mu$ whenever $\nu \leq \mu$, we have $\operatorname{ind} S_{\beta+n} \leq \operatorname{ind} K + 1$ for all $\beta < \lambda$; therefore every $x \in S_\alpha - B_\alpha$ has a neighbourhood U in S_α with $\operatorname{ind} U \leq \operatorname{ind} K + 1$, which completes the proof of $\operatorname{ind} S_\alpha \leq \operatorname{ind} K + 1$.

We now sketch the proof of $\operatorname{Ind} K \geq \lambda + (n-1)$.

Of course, the inequality holds for $\alpha < \omega_0$; thus assume that $\alpha \geq \omega_0$. Let A, B be a pair of opposite $(n-1)$-dimensional faces of $B_\alpha = I^n$. A reasoning similar to that in the proof of (2.4) shows that

(2.7)   there exists a partition L in S_α between A and B such that $\operatorname{Ind} L \leq \operatorname{Ind} K$.

Consider disjoint open sets $U, V \subseteq S_\alpha$ such that $A \subseteq U$, $B \subseteq V$, and $L = S_\alpha - (U \cup V)$. It follows immediately from the definition of S_α that there exist $\beta_1, \ldots, \beta_k < \lambda$ with the property that

$$\left(\bigcup \{ S_\beta \times I^n : \beta < \lambda \text{ and } \beta \neq \beta_1, \ldots, \beta_k \} \right) \times A \subseteq U \,,$$

$$\left(\bigcup \{ S_\beta \times I^n : \beta < \lambda \text{ and } \beta \neq \beta_1, \ldots, \beta_k \} \right) \times B \subseteq V \,.$$

Let f be a 1-1 mapping of the set of all ordinals $\beta < \lambda$ to the set of all ordinals $\beta < \lambda$ such that $\beta \neq \beta_1, \ldots, \beta_k$ with the property that $\beta \leq f(\beta)$ for every $\beta < \lambda$. Since S_ν is embeddable in S_μ whenever $\nu \leq \mu$, it follows that S_β is embeddable in $S_{f(\beta)}$. Hence there exists an embedding of S_λ in its subspace $\bigcup \{ S_\beta : \beta < \lambda \text{ and } \beta \neq \beta_1, \ldots, \beta_k \}$, and so there also exists an embedding $h : S_\alpha \to S_\alpha$ such that

$$h(S_\lambda \times A) \subseteq \left(\bigcup \{ S_\beta \times I^n : \beta < \lambda \text{ and } \beta \neq \beta_1, \ldots, \beta_k \} \right) \times A \subseteq U \,,$$

$$h(S_\lambda \times B) \subseteq \left(\bigcup \{ S_\beta \times I^n : \beta < \lambda \text{ and } \beta \neq \beta_1, \ldots, \beta_k \} \right) \times B \subseteq V \,.$$

The set $h^{-1}(L)$ is a partition in S_α between $S_\lambda \times A$ and $S_\lambda \times B$, and so, by Lemma 2.D,

$$\lambda + (n-1) \leq \operatorname{Ind} h^{-1}(L) \leq \operatorname{Ind} L \leq \operatorname{Ind} K \,.$$

3. Two lemmas. In this section we state two fairly straightforward, but important lemmas. They indicate a property of compact metrizable spaces X with $\operatorname{ind} X \leq \lambda$ and a property of separable metrizable spaces X with $\operatorname{Ind} X \leq \lambda$, respectively, which preclude the existence of universal spaces in these classes.

3.1. LEMMA. *Let (X, ϱ) be a compact metric space with $\operatorname{ind} X \leq \lambda$, where λ is a limit ordinal, and let ε be a positive real number. Then there exists an ordinal $\alpha < \lambda$ such that*

(3.1)   *for every subspace $A \subseteq X$ and for any $x, y \in A$ such that $\varrho(x, y) \geq \varepsilon$, there exists a partition in A between x and y with small transfinite dimension not greater than α.*

P r o o f. Let $V_1, \ldots, V_k$ be a finite open covering of X such that $\operatorname{diam} V_i < \varepsilon$ and $\operatorname{ind} \operatorname{bd} V_i < \lambda$ for $i = 1, \ldots, k$; the existence of such a covering follows from the definition of ind and the compactness of X.

Let $\alpha = \max\{\operatorname{ind} \operatorname{bd} V_i : i = 1, \ldots, k\}$.

Consider a subspace $A \subseteq X$ and points $x, y \in A$ such that $\varrho(x, y) \geq \varepsilon$. Then $x \in V_i$ for some i, and since $\operatorname{diam} V_i < \varepsilon$ and $\varrho(x, y) \geq \varepsilon$, we have $y \notin \operatorname{cl} V_i$. Thus $A \cap \operatorname{bd} V_i$ is a partition in A between x and y; obviously,

$$\operatorname{ind}(A \cap \operatorname{bd} V_i) \leq \operatorname{ind} \operatorname{bd} V_i \leq \alpha.$$

3.2. LEMMA. *Let (X, ϱ) be a totally bounded metric space with $\operatorname{Ind} X \leq \lambda$, where λ is a limit ordinal, and let ε be a positive real number. Then there exists an ordinal $\alpha < \lambda$ such that*

(3.2)   *for every subspace $A \subseteq X$ and for any $x, y \in A$ such that $\varrho(x, y) \geq \varepsilon$, there exists a partition in A between x and y with large transfinite dimension not greater than α.*

P r o o f. Let $W_1, \ldots, W_k$ be a finite open covering of X such that $\operatorname{diam} W_i < \varepsilon$ for $i = 1, \ldots, k$, and $F_1, \ldots, F_k$ its closed shrinking. Since $\operatorname{Ind} X \leq \lambda$, it follows that for $i = 1, \ldots, k$, there exists an open set V_i such that $F_i \subseteq V_i \subseteq \operatorname{cl} V_i \subseteq W_i$ and $\operatorname{Ind} \operatorname{bd} V_i < \lambda$; obviously, $V_1, \ldots, V_k$ is also a finite open covering of X and $\operatorname{diam} \operatorname{cl} V_i < \varepsilon$ for $i = 1, \ldots, k$.

A similar argument to that in the proof of Lemma 3.1 shows that $\alpha = \max\{\operatorname{Ind} \operatorname{bd} V_i : i = 1, \ldots, k\}$ has the required property.

4. The compacta C_λ^σ and D_λ^σ. Let $\lambda < \omega_1$ be a limit ordinal; denote by Λ the set of all non-limit ordinals α such that $0 < \alpha < \lambda$. In this section we associate with each sequence $\sigma : \mathbb{N} \to \Lambda$ compact metrizable spaces C_λ^σ and D_λ^σ such that $\operatorname{ind} C_\lambda^\sigma \leq \lambda$ and $\operatorname{Ind} D_\lambda^\sigma \leq \lambda$.

First, we define a certain subspace Z of the square I^2.

Denote by S_n the set of all pairs (i, n), where $i = 1, 2, \ldots, 2^n$ and $n = 1, 2, \ldots$, and put $S = \{0\} \cup \bigcup_{n=1}^\infty S_n$. Note that the sets S_n are pairwise disjoint and $0 \notin S_n$ for all n.

Let Z_0 stand for the lower edge of I^2, i.e., the set of all $(x, y) \in I^2$ such that $y = 0$; for $s = (i, n) \in S_n$, let a_s denote the point $((2i-1)/2^{n+1}, 0) \in I^2$, i.e., the centre of the interval $\{(x, 0) \in I^2 : (i-1)/2^n \leq x \leq i/2^n\}$ contained in the lower edge of I^2, and let Z_s stand for the semicircle with centre a_s and radius $1/2^{n+1}$ contained in our square, i.e., $Z_s = \{z \in I^2 : \varrho(z, a_s) = 1/2^{n+1}\}$, where ϱ denotes the standard metric on I^2.

Set $Z = \bigcup\{Z_s : s \in S\}$. It is a simple matter to check that Z is a closed subspace of I^2. The space Z is exhibited in Fig. 4.1.

Let us introduce the following notation: $z_0 = (0, 0)$, $z_1 = (1, 0)$ and $z_i^n = (i/2^n, 0)$ for $n \in \mathbb{N}$ and $i = 0, 1, \ldots, 2^n$. Since $z_i^n = z_j^m$ whenever $i/2^n = j/2^m$, and $z_0 = z_0^n$ and $z_1 = z_{2^n}^n$ for every $n \in \mathbb{N}$, some symbols stand for the same point, but that will cause no confusion. Note that z_{i-1}^n, z_i^n are the endpoints of the arc Z_s for $s = (i, n) \in S_n$.
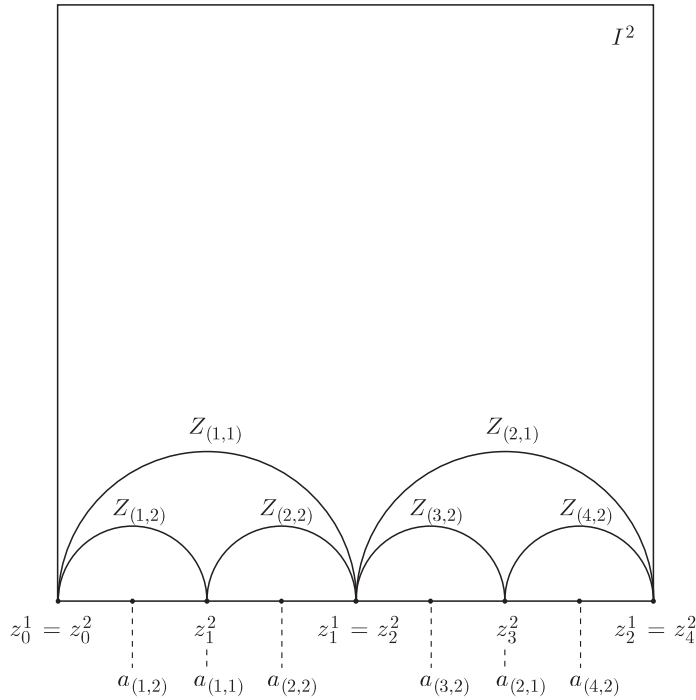
$$Z_{(1,1)} \qquad Z_{(2,1)}$$
$$Z_{(1,2)} \quad Z_{(2,2)} \qquad Z_{(3,2)} \quad Z_{(4,2)}$$
$$z_0^1 = z_0^2 \qquad z_1^2 \qquad z_1^1 = z_2^2 \qquad z_3^2 \qquad z_2^1 = z_4^2$$
$$a_{(1,2)} \quad a_{(1,1)} \quad a_{(2,2)} \qquad a_{(3,2)} \quad a_{(2,1)} \quad a_{(4,2)}$$

Fig. 4.1

We now pass to the construction of the spaces C_λ^σ and D_λ^σ.

Let S and S_n, for $n \in \mathbb{N}$, be the sets described in the definition of Z. Recall that $\beta(\alpha)$ denotes the smallest ordinal β such that $\operatorname{ind} S_\beta = \alpha$; since $\sigma(n) \in \Lambda$, it is a non-limit ordinal, and so is $\beta(\sigma(n))$ (see Section 2). Denote by $\gamma(n)$ the predecessor of $\beta(\sigma(n))$; by the definition of Smirnov's spaces, $S_{\beta(\sigma(n))} = S_{\gamma(n)} \times I$. Set $X_s = S_{\gamma(n)}$ and $b_s = p_{\gamma(n)}$ for $s \in S_n$, where $p_{\gamma(n)}$ is the distinguished point of $S_{\gamma(n)}$ (see Section 2). It will cause no confusion to identify Z with the subspace of $Z \times \mathbb{P}\{X_s : s \in S - \{0\}\}$ consisting of all points $(x_0, \{x_s : s \in S - \{0\}\})$ such that $x_s = b_s$ for all $s \in S - \{0\}$. For $t \in S - \{0\}$, let C_t be the subspace of the Cartesian product consisting of all $(x_0, \{x_s : s \in S - \{0\}\})$ such that $x_0 \in Z_t$ and $x_s = b_s$ for $s \neq t$.

Note that since Z_t is an arc, C_t is homeomorphic to $S_{\gamma(n)} \times I = S_{\beta(\sigma(n))}$; moreover, there exists a homeomorphism of C_s onto $S_{\beta(\sigma(n))}$ mapping Z_s onto an edge of the base $B_{\beta(\sigma(n))}$.

Let $C_\lambda^\sigma = Z \cup \bigcup \{C_t : t \in S - \{0\}\}$. It is easily seen that C_λ^σ is a closed subspace of $Z \times \mathbb{P}\{X_s : s \in S - \{0\}\}$, and therefore it is compact and metrizable.

Loosely speaking, in order to obtain C_λ^σ we stick, for every $n \in \mathbb{N}$, a copy of $S_{\beta(\sigma(n))}$ to each arc Z_s, where $s \in S_n$, along an edge of the base $B_{\beta(\sigma(n))}$ in such a way that the space C_λ^σ so obtained is compact, i.e., the diameters of the stuck copies of Smirnov's spaces converge to 0 as n tends to ∞.

The space D_λ^σ is defined similarly, except that $S_{\beta(\sigma(n))}$ is replaced by $S_{\sigma(n)}$; recall that $\operatorname{Ind} S_{\sigma(n)} = \sigma(n)$.

The space C_λ^σ, where $\lambda = \omega_0$ and σ given by $\sigma(n) = n + 1$ for every $n \in \mathbb{N}$, is exhibited in Fig. 4.2. Note that if $\lambda = \omega_0$, then $C_\lambda^\sigma = D_\lambda^\sigma$ for every σ.
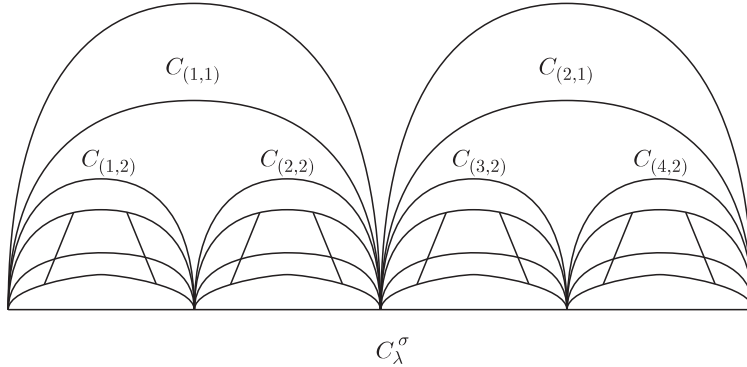

$$C_\lambda^\sigma$$

Fig. 4.2

We now evaluate $\operatorname{ind} C_\lambda^\sigma$ and $\operatorname{Ind} D_\lambda^\sigma$. To this end, we need the following theorem which is a consequence of a theorem obtained independently by M. Landau and A. R. Pears (see [5] and [12], and also [2], Theorem 5.17); it also follows from a theorem of B. T. Levshenko (see [6], and [2], Theorem 5.15).

4.A. THEOREM ([5], [12] and [6]). *If a hereditarily normal space X can be represented as the union $A_1 \cup A_2$ of its closed subspaces with $\operatorname{Ind} A_1 < \lambda$ and $\operatorname{Ind} A_2 < \lambda$, where λ is a limit ordinal, then $\operatorname{Ind} X < \lambda$.*

4.1. LEMMA. *For every limit ordinal $\lambda < \omega_1$ and every sequence $\sigma : \mathbb{N} \to \Lambda$, we have $\operatorname{ind} C_\lambda^\sigma \leq \lambda$ and $\operatorname{Ind} D_\lambda^\sigma \leq \lambda$.*

P r o o f. We show that $\operatorname{ind}_z C_\lambda^\sigma \leq \lambda$ for every $z \in C_\lambda^\sigma$.

Note that $C_\lambda^\sigma = Z_0 \cup \bigcup \{C_s - Z_0 : s \in S - \{0\}\}$. Since $C_s - Z_0$ is an open subset of C_λ^σ and $\operatorname{ind}(C_s - Z_0) \leq \operatorname{ind} C_s < \lambda$, it follows that $\operatorname{ind}_z C_\lambda^\sigma < \lambda$ for every $z \in C_\lambda^\sigma - Z_0$.

Thus consider a $z \in Z_0$. Assume first that $z \neq z_i^n$ for any $n \in \mathbb{N}$ and $i = 1, 2, \ldots, 2^n$. Let $A \subseteq C_\lambda^\sigma$ be a closed set such that $z \notin A$. Take $n \in \mathbb{N}$ and $i \leq 2^n$ with the property that x belongs to the segment with endpoints z_{i-1}^n and z_i^n contained in Z_0, and A meets neither this segment nor any C_s, where $s \in \bigcup \{S_m : m \geq n\}$, such that the endpoints of Z_s belong to this segment. One can readily check that $L = \{z_{i-1}^n, z_i^n\}$ is a partition between z and A (see Fig. 4.3).
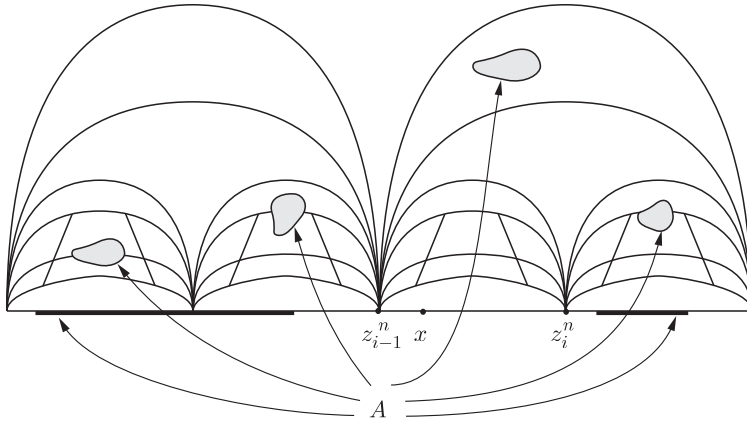
Fig. 4.3

Suppose now that $z = z_i^n$ for some $n \in \mathbb{N}$ and $i \leq 2^n$; obviously, one can assume that n is the smallest number for which there exists an $i \leq 2^n$ such that $z = z_i^n$. Let $A \subseteq C_\lambda^\sigma$ be a closed set such that $z \notin A$. Then there exist $m > n$ and $j \leq 2^m$ with $z_i^n = z_j^m$, and A meets neither the segment with endpoints z_{j-1}^m and z_{j+1}^m contained in Z_0 nor any C_s, where $s \in \bigcup\{S_k : k > m\}$, such that the endpoints of Z_s belong to this segment. For every $k \in \{n, n+1, \ldots, m\}$, $z = z_i^n$ is an endpoint of Z_s for two $s \in S_k$; denote by z_s the other endpoint of Z_s. Let L_k^1 be a partition in C_s between z_i^n and $(A \cap C_s) \cup \{z_s\}$ such that $\operatorname{ind} L_k^1 < \lambda$ for one of these two indices s, and L_k^2 a partition with the same properties for the other one. It followsimmediately that $L = \{z_{j-1}^m, z_{j+1}^m\} \cup \bigcup_{k=n}^m (L_k^1 \cup L_k^2)$
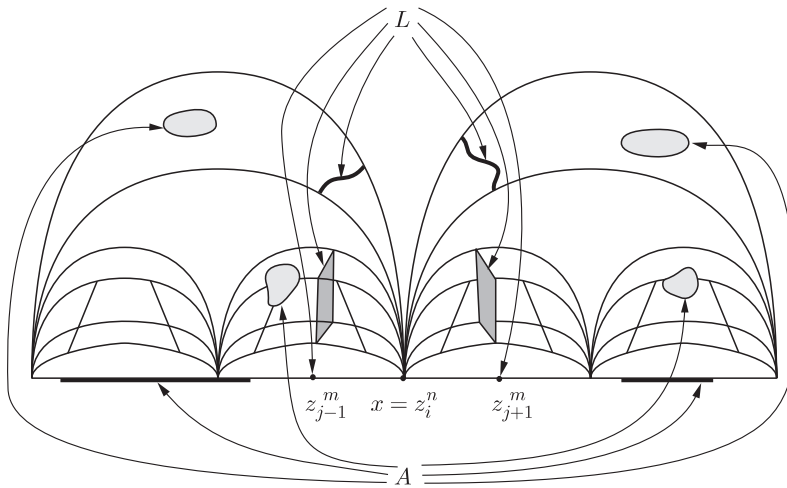

Fig. 4.4

is a partition in C_λ^σ between $z = z_i^n$ and A (see Fig. 4.4). Since the sets $\{z_{j-1}^m, z_{j+1}^m\}, L_n^1, L_n^2, L_{n+1}^1, L_{n+1}^2, \ldots, L_m^1, L_m^2$ are pairwise disjoint, compact and have small transfinite dimension less than λ, we have $\operatorname{ind} L < \lambda$.

Thus $\operatorname{ind} C_\lambda^\sigma \le \lambda$. In order to prove $\operatorname{Ind} D_\lambda^\sigma \le \lambda$, observe that a similar reasoning shows that for every $z \in D_\lambda^\sigma$ and every closed set $B \subseteq D_\lambda^\sigma$ such that $z \notin B$, there exists a partition L between z and B with $\operatorname{Ind} L < \lambda$.

Let A be a closed subset of D_λ^σ and V an open set containing A. For every $z \in A$, consider a partition L_z between z and $D_\lambda^\sigma - V$ with $\operatorname{Ind} L_z < \lambda$; obviously, one can assume that $L_z = \operatorname{bd} W_z$ for some neighbourhood $W_z \subseteq V$ of z. By compactness of A, there exists a finite family $\mathcal{W} \subseteq \{W_z : z \in A\}$ such that $A \subseteq \bigcup \mathcal{W}$; let $U = \bigcup \mathcal{W}$. Then $A \subseteq U \subseteq V$ and $\operatorname{bd} U \subseteq \bigcup\{\operatorname{bd} W : W \in \mathcal{W}\}$; in particular, $\operatorname{Ind} \operatorname{bd} U \le \operatorname{Ind}(\bigcup\{\operatorname{bd} W : W \in \mathcal{W}\})$. Now Theorem 4.A shows that $\operatorname{Ind} \operatorname{bd} U < \lambda$.

5. There is no universal space for spaces with limit transfinite dimension. In this section we prove the theorems announced in Section 1.

5.1. THEOREM. *Let $\lambda < \omega_1$ be an arbitrary limit ordinal. There is no universal space for compact metrizable spaces X with $\operatorname{ind} X \le \lambda$.*

Proof. It suffices to show that

(5.1)   for every compact metrizable space X with $\operatorname{ind} X \le \lambda$, there exists a sequence $\sigma : \mathbb{N} \to \Lambda$ such that C_λ^σ is not homeomorphic to any subspace of X.

Let ϱ be an arbitrary metric on X. It follows from Lemma 3.1 that for every $n \in \mathbb{N}$, there exists an ordinal $\alpha_n < \lambda$ satisfying (3.1) for $\varepsilon = 2^{-2n}$. Set $\sigma(n) = \alpha_n + 2$ for $n \in \mathbb{N}$. Since λ is a limit ordinal, $\sigma(n) < \lambda$ for each $n \in \mathbb{N}$, and therefore $\sigma(n) \in \Lambda$ for every $n \in \mathbb{N}$. Suppose that there exists an embedding $h : C_\lambda^\sigma \to X$.

Since there exists a homeomorphism of C_s, $s \in S_n$, onto $S_{\beta(\sigma(n))}$ mapping Z_s onto an edge of the cube $B_{\beta(\sigma(n))} \subseteq S_{\beta(\sigma(n))}$, there is no partition in C_s between any two distinct points of Z_s with small transfinite dimension less than the predecessor of $\sigma(n)$ (see the first inequality of Theorem 2.1). Thus there is no partition in $h(C_s)$ between any two distinct points of $h(Z_s)$ with small transfinite dimension not greater than α_n. From (3.1), we have $\operatorname{diam} h(Z_s) < 2^{-2n}$.

Hence, by the triangle inequality,

$$\varrho(h(z_0), h(z_1)) \le \sum_{i=1}^{2^n} \varrho(h(z_{i-1}^n), h(z_i^n))$$

$$\le \sum_{s \in S_n} \operatorname{diam} h(Z_s) < 2^n \cdot 2^{-2n} = 2^{-n};$$

recall that z_0 and z_1 are the endpoints of the interval $Z_0 \subseteq Z$, and $\{z_i^n : i = 1, 2, \ldots, 2^n\}$ stands for the sequence of points of Z with the property that z_{i-1}^n, z_i^n are the endpoints of the arc Z_s for $s = (i, n) \in S_n$.

Since n is an arbitrary natural number, we conclude that $h(z_0) = h(z_1)$, which contradicts the assumption that h is a homeomorphism.

5.2. THEOREM. *Let $\lambda < \omega_1$ be an arbitrary limit ordinal. There is no universal space for separable metrizable spaces X with* $\operatorname{Ind} X \leq \lambda$.

P r o o f. It suffices to show that

(5.2)   for every separable metrizable space X with $\operatorname{Ind} X \leq \lambda$, there exists a sequence $\sigma : \mathbb{N} \to \Lambda$ such that D_λ^σ is not homeomorphic to any subspace of X.

To this end, take a totally bounded metric ϱ on X and apply a reasoning similar to that in the proof of Theorem 5.1 using Lemma 3.2 instead of Lemma 3.1 and the second inequality of Theorem 2.1 instead of the first one.

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