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DEPARTMENT OF MATHEMATICS & STATISTICS
COLLEGE OF SCIENCES
SHIRAZ UNIVERSITY
SHIRAZ 71454, ISLAMIC REPUBLIC OF IRAN

Received October 13, 1992 (3008)
Revised version July 5, 1993

STUDIA MATHEMATICA 108 (1) (1994)

Extension of multilinear mappings on Banach spaces

by

PABLO GALINDO, DOMINGO GARCÍA,
MANUEL MAESTRE (Valencia) and
JORGE MUJICA (Campinas)

Dedicated to the memory of Leopoldo Nachbin (1922–1993)

Abstract. By following an idea of Nicodemi we study certain sequences of extension operators for multilinear mappings on Banach spaces starting from any given extension operator for linear mappings. In this way we obtain several new properties of the extension operators previously studied by Aron, Berner, Cole, Davie and Gamelin. As an application of our methods we show the existence of plenty of unbounded scalar-valued homomorphisms on the locally convex algebra of all continuous polynomials on each infinite-dimensional Banach space. This improves a result of Dixon.

Introduction. The problem of extending holomorphic functions from a Banach space E to a larger Banach space F was first studied by Aron and Berner [3]. They showed that the holomorphic functions of bounded type on E extend in a natural way to E'', yielding an extension operator from $\mathcal{H}_b(E)$ into $\mathcal{H}_b(E'')$. To achieve their goal they constructed extension operators for the spaces of multilinear forms and then used Taylor series expansions to extend holomorphic functions.

It is in general possible to extend multilinear forms on E to E'' in many different ways. Davie and Gamelin [6], and Aron, Cole and Gamelin [4], have established important properties of the extension operators of Aron and Berner, and have given a different, much simpler, description of those operators. Very recently Lindström and Ryan [13] have constructed other extension operators for multilinear forms by using ultrapowers of Banach

1991 *Mathematics Subject Classification*: Primary 46A22, 46G20.
Key words and phrases: Banach space, multilinear mapping, holomorphic mapping, extension operator, algebra homomorphism.
The first three authors are supported in part by DGICYT pr. no. P.B.91-0326. The third author is also supported in part by DGICYT pr. no. P.B.91-0538.
The fourth author is supported in part by DGICYT (Spain) and FAPESP (Brazil).

spaces. Dineen and Timoney [7] have also used ultrapowers of Banach spaces to show that the bounded holomorphic functions on the open unit ball of E extend in a natural way to the open unit ball of E''.

In this paper we follow an idea of Nicodemi [15] to define a sequence of extension operators $R_m : \mathcal{L}({}^mE) \to \mathcal{L}({}^mF)$ starting from any given extension operator $R_1 : E' \to F'$. If we start, for instance, from the natural embedding $R_1 : E' \hookrightarrow E'''$, then we recover the extension operators of Aron and Berner. The main advantage of our approach is that the sequence (R_m) is defined inductively by means of an explicit algebraic formula. This allows us to establish many properties of the sequence (R_m) by straightforward induction.

This paper is organized as follows. In Section 0 we fix our notation and terminology. Sections 1 through 3 contain the preliminary material on the Nicodemi extension operators for multilinear mappings, polynomials and holomorphic mappings. In Section 4 we show how a given sequence of extension operators for multilinear forms yields in a natural way a sequence of extension operators for vector-valued multilinear mappings. In Section 5 we establish certain continuity properties of the sequence (R_m) that begins with the natural embedding $R_1 : E' \hookrightarrow E'''$. In Section 6 we give necessary and sufficient conditions for the extension operators for vector-valued multilinear mappings to be independent of the range space. The results in Section 7 on change of order of the variables are slight variations of results of Aron, Cole and Gamelin [4], and have been included to stress their relation to the results in the preceding section. In Section 8 we give examples of pairs of spaces E and G such that, for every $A \in \mathcal{L}({}^mE; G')$, the extension $\widetilde{R}_m A \in \mathcal{L}({}^mE''; G')$ is separately compact or separately integral. In Section 9 we give necessary and sufficient conditions for the coincidence of certain extensions to the fourth dual, thus answering questions raised by Aleksander Pełczyński and Richard Aron. Finally, in Section 10 we use an algebraic version of the Nicodemi extension operators to find many unbounded scalar-valued homomorphisms on the locally convex algebra $\mathcal{P}(E)$ of all continuous polynomials on each infinite-dimensional Banach space E. This improves a result of Dixon [8].

This paper was written when the fourth author was visiting the Department of Mathematical Analysis of the University of Valencia, whose hospitality is gratefully acknowledged.

0. Notation and terminology. The letters E, F, G always represent Banach spaces over the same field K, where K is $\mathbb{R}$ or $\mathbb{C}$. E^* denotes the algebraic dual of E, whereas E' denotes the topological dual of E. $\mathbb{N}$ denotes the set of all strictly positive integers.

We shall denote by $\mathcal{L}_{\mathrm{a}}({}^mE; G)$ the vector space of all m-linear mappings from E^m into G, whereas $\mathcal{L}({}^mE; G)$ denotes the subspace of all continuous members of $\mathcal{L}_{\mathrm{a}}({}^mE; G)$. $\mathcal{L}({}^mE; G)$ is a Banach space under its natural norm. The isomorphim

$$I_m : \mathcal{L}_{\mathrm{a}}({}^{m+n}E; G) \to \mathcal{L}_{\mathrm{a}}({}^mE; \mathcal{L}_{\mathrm{a}}({}^nE; G))$$

defined by $I_mA(x)(y) = A(x, y)$ for all $A \in \mathcal{L}_{\mathrm{a}}({}^{m+n}E; G)$, $x \in E^m$, $y \in E^n$, induces an isometry between $\mathcal{L}({}^{m+n}E; G)$ and $\mathcal{L}({}^mE; \mathcal{L}({}^nE; G))$. Likewise the isomorphism

$$\mathcal{L}_{\mathrm{a}}({}^mE; \mathcal{L}_{\mathrm{a}}({}^nF; G)) \ni A \to A^t \in \mathcal{L}_{\mathrm{a}}({}^nF; \mathcal{L}_{\mathrm{a}}({}^mE; G))$$

defined by $A^t(y)(x) = A(x)(y)$ for all $A \in \mathcal{L}_{\mathrm{a}}({}^mE; \mathcal{L}_{\mathrm{a}}({}^nF; G))$, $x \in E^m$, $y \in F^n$, induces an isometry between $\mathcal{L}({}^mE; \mathcal{L}({}^nF; G))$ and $\mathcal{L}({}^nF; \mathcal{L}({}^mE; G))$.

We shall denote by $\mathcal{P}_{\mathrm{a}}({}^mE; G)$ the vector space of all m-homogeneous polynomials from E into G, whereas $\mathcal{P}({}^mE; G)$ denotes the subspace of all continuous members of $\mathcal{P}_{\mathrm{a}}({}^mE; G)$. $\mathcal{P}({}^mE; G)$ is a Banach space under its natural norm. The natural surjective mapping

$$\mathcal{L}_{\mathrm{a}}({}^mE; G) \ni A \to \widehat{A} \in \mathcal{P}_{\mathrm{a}}({}^mE; G)$$

maps $\mathcal{L}({}^mE; G)$ onto $\mathcal{P}({}^mE; G)$. Let also $\mathcal{P}_{\mathrm{a}}(E; G) = \bigoplus_{m=0}^{\infty} \mathcal{P}_{\mathrm{a}}({}^mE; G)$ and $\mathcal{P}(E; G) = \bigoplus_{m=0}^{\infty} \mathcal{P}({}^mE; G)$.

In the case of complex Banach spaces, $\mathcal{H}(U; G)$ denotes the vector space of all holomorphic mappings from an open subset U of E into G, whereas $\mathcal{H}_{\mathrm{b}}(U; G)$ denotes the Fréchet space of all holomorphic mappings of bounded type from U into G (i.e. bounded on bounded subsets B of U satisfying $\mathrm{dist}(B, E \setminus U) > 0$). If $f \in \mathcal{H}(U; G)$ and $x \in U$, then $P^m f(x) \in \mathcal{P}({}^mE; G)$ denotes the mth term in the Taylor series expansion of f at x.

As is customary we will write $\mathcal{L}_{\mathrm{a}}({}^mE)$ instead of $\mathcal{L}_{\mathrm{a}}({}^mE; K)$, $\mathcal{P}_{\mathrm{a}}({}^mE)$ instead of $\mathcal{P}_{\mathrm{a}}({}^mE; K)$, etc.

We refer to [14] for the properties of multilinear mappings, polynomials and holomorphic mappings on Banach spaces.

1. Nicodemi sequences. Given a continuous linear operator

$$R_1 : \mathcal{L}(E; G) \to \mathcal{L}(F; G),$$

let

$$R_m : \mathcal{L}({}^mE; G) \to \mathcal{L}({}^mF; G)$$

be inductively defined by

$$R_{m+1}A = I_m^{-1}[R_m \circ (R_1 \circ I_mA)^t]^t \tag{1.1}$$

for all $A \in \mathcal{L}({}^{m+1}E; G)$ and $m \in \mathbb{N}$. Though it is not obvious at a first glance, the sequence of operators R_m thus defined is precisely the sequence of operators constructed by Nicodemi in [15, pp. 536–537], and will henceforth be referred to as the *Nicodemi sequence beginning with* R_1.

1.1. EXAMPLE. Let $\pi \in \mathcal{L}(F;E)$, let $\pi_1 : \mathcal{L}(E;G) \to \mathcal{L}(F;G)$ be defined by $\pi_1 A(y) = A(\pi y)$ for all $A \in \mathcal{L}(E;G)$ and $y \in F$, and let (π_m) be the Nicodemi sequence beginning with π_1. Then we can readily prove by induction that

$$\pi_m A(y_1,\ldots,y_m) = A(\pi y_1,\ldots,\pi y_m)$$

for all $A \in \mathcal{L}(^mE;G)$ and $y_1,\ldots,y_m \in F$. Note that when $G = K$, then π_1 coincides with the dual mapping π' of π.

1.2. EXAMPLE. Let $R_m : \mathcal{L}(^mE) \to \mathcal{L}(^mE'')$ be the Nicodemi sequence beginning with the natural embedding $R_1 : E' \hookrightarrow E'''$. Though it is far from obvious, this sequence is precisely the sequence of operators constructed by Aron and Berner in [3, Proposition 2.1].

1.3. Remark. When $F = E''$ and $G = K$, then the Nicodemi sequences in the preceding two examples are always different, unless E is reflexive. Indeed, as Nicodemi has pointed out in [15, p. 539], the natural embedding $E' \hookrightarrow E'''$ cannot be a dual mapping, unless E is reflexive.

1.4. EXAMPLE. Let $R_1 : E' \hookrightarrow E'''$ and $S_1 : E''' \hookrightarrow E^{(5)}$ be the natural embeddings, let $T_1 := S_1 \circ R_1 : E' \to E^{(5)}$, and let $T_m : \mathcal{L}(^mE) \to \mathcal{L}(^mE^{(4)})$ be the Nicodemi sequence beginning with T_1. In a similar manner we may define a Nicodemi sequence of operators from $\mathcal{L}(^mE)$ into $\mathcal{L}(^mE^{(2k)})$ for each $k \in \mathbb{N}$.

1.5. Remark. Let (R_m) and (S_m) be the Nicodemi sequences beginning with the natural embeddings $R_1 : E' \hookrightarrow E'''$ and $S_1 : E''' \hookrightarrow E^{(5)}$, respectively. We will see in Section 9 that the sequence (T_m) from Example 1.4 is in general different from the sequence $(S_m \circ R_m)$. This will answer a question raised by Aleksander Pełczyński.

1.6. LEMMA. *Let $R_m : \mathcal{L}(^mE;G) \to \mathcal{L}(^mF;G)$ be a Nicodemi sequence. Then for all $A \in \mathcal{L}(^{m+n}E;G)$ and $m,n \in \mathbb{N}$ we have*

$$R_{m+n}A = I_m^{-1}[R_m \circ (R_n \circ I_m A)^t]^t . \tag{1.2}$$

Proof. By definition of (R_m), (1.2) is true for $n = 1$ and every $m \in \mathbb{N}$. Assuming (1.2) true for certain m and n, we will prove it for m and $n+1$. For $A \in \mathcal{L}(^{m+n+1}E;G)$, $u \in F^m$, $v \in F^n$ and $w \in F$ we have

$$R_{m+n+1}A(u,v,w) = R_{m+n}[(R_1 \circ I_{m+n}A)^t(w)](u,v) .$$

Set $B_w = (R_1 \circ I_{m+n}A)^t(w) \in \mathcal{L}(^{m+n}E;G)$. Then by using the induction hypothesis we get

$$R_{m+n+1}A(u,v,w) = R_{m+n}B_w(u,v) = R_m[(R_n \circ I_m B_w)^t(v)](u) . \tag{1.3}$$

On the other hand,

$$[R_m \circ (R_{n+1} \circ I_m A)^t]^t(u)(v,w) = R_m[(R_{n+1} \circ I_m A)^t(v,w)](u) . \tag{1.4}$$

By (1.3) and (1.4), to complete the proof it suffices to show that

$$(R_n \circ I_m B_w)^t(v) = (R_{n+1} \circ I_m A)^t(v,w) . \tag{1.5}$$

Both sides of (1.5) belong to $\mathcal{L}(^mE;G)$. Now for $x \in E^m$ we have

$$(R_n \circ I_m B_w)^t(v)(x) = R_n[I_m B_w(x)](v) . \tag{1.6}$$

On the other hand,

$$\begin{aligned} (R_{n+1} \circ I_m A)^t(v,w)(x) &= R_{n+1}[I_m A(x)](v,w) \\ &= R_n[(R_1 \circ I_n\{I_m A(x)\})^t(w)](v) . \end{aligned} \tag{1.7}$$

By (1.6) and (1.7), to prove (1.5) it suffices to show that

$$I_m B_w(x) = (R_1 \circ I_n\{I_m A(x)\})^t(w) . \tag{1.8}$$

Both sides of (1.8) belong to $\mathcal{L}(^nE;G)$. Now for $y \in E^n$ we have

$$I_m B_w(x)(y) = (R_1 \circ I_{m+n}A)^t(w)(x,y) = R_1[I_{m+n}A(x,y)](w) . \tag{1.9}$$

On the other hand,

$$(R_1 \circ I_n\{I_m A(x)\})^t(w)(y) = R_1[I_n\{I_m A(x)\}(y)](w) . \tag{1.10}$$

Since

$$I_{m+n}A(x,y)(z) = A(x,y,z) = I_n\{I_m A(x)\}(y)(z)$$

for every $z \in E$, (1.8) follows from (1.9) and (1.10). This proves (1.5) and the lemma. ∎

1.7. PROPOSITION. *Let $R_m : \mathcal{L}(^mE) \to \mathcal{L}(^mF)$ be a Nicodemi sequence. Then for all $A \in \mathcal{L}(^mE)$ and $B \in \mathcal{L}(^nE)$ we have*

$$R_{m+n}(A \otimes B) = R_m A \otimes R_n B . \tag{1.11}$$

Proof. Let $u \in F^m$ and $v \in F^n$. By using Lemma 1.6 we get

$$R_{m+n}(A \otimes B)(u,v) = R_m[(R_n \circ I_m\{A \otimes B\})^t(v)](u) . \tag{1.12}$$

Note that $(R_n \circ I_m\{A \otimes B\})^t(v) \in \mathcal{L}(^mE)$. Now for $x \in E^m$ we have

$$\begin{aligned} (R_n \circ I_m\{A \otimes B\})^t(v)(x) &= R_n[I_m\{A \otimes B\}(x)](v) \\ &= R_n[A(x)B](v) = A(x)R_nB(v) . \end{aligned}$$

Thus by substituting into (1.12) we get

$$R_{m+n}(A \otimes B)(u,v) = R_m[R_nB(v)A](u) = R_nB(v)R_mA(u) ,$$

as we wanted. ∎

2. Nicodemi sequences of extension operators. The most interesting Nicodemi sequences are the *Nicodemi sequences of extension operators*, that is, those that satisfy the hypothesis in the following proposition.

2.1. PROPOSITION. *Let $R_m : \mathcal{L}(^mE;G) \to \mathcal{L}(^mF;G)$ be a Nicodemi sequence. If there exists $J \in \mathcal{L}(E;F)$ such that $R_1A(Jx) = A(x)$ for all $A \in \mathcal{L}(E;G)$ and $x \in E$, then*

$$R_mA(Jx_1,\ldots,Jx_m) = A(x_1,\ldots,x_m) \tag{2.1}$$

for all $A \in \mathcal{L}(^mE;G)$ and $x_1,\ldots,x_m \in E$.

Proof. For convenience define $J^m \in \mathcal{L}(E^m;F^m)$ by $J^mx = (Jx_1,\ldots,Jx_m)$ for every $x = (x_1,\ldots,x_m) \in E^m$. By hypothesis, (2.1) is true for $m = 1$. Assuming it true for some m, we will prove it for $m+1$. Let $A \in \mathcal{L}(^{m+1}E;G)$, $x \in E^m$ and $y \in E$. By using the induction hypothesis and the case $m = 1$ we get

$$\begin{aligned} R_{m+1}A(J^mx,Jy) &= R_m[(R_1 \circ I_mA)^t(Jy)](J^mx) \\ &= (R_1 \circ I_mA)^t(Jy)(x) = R_1[I_mA(x)](Jy) \\ &= I_mA(x)(y) = A(x,y), \end{aligned}$$

as we wanted. ∎

Under the conditions of Proposition 2.1 we will say that (R_m) is a *Nicodemi sequence of extension operators for J*.

With the notation of Example 1.1, Proposition 2.1 says that $J_m \circ R_m(A) = A$ for every $A \in \mathcal{L}(^mE;G)$. In particular, $\mathcal{L}(^mE;G)$ is topologically isomorphic to a complemented subspace of $\mathcal{L}(^mF;G)$.

Observe that the sequence (R_m) from Example 1.2 is a Nicodemi sequence of extension operators for the natural embedding $J : E \hookrightarrow E''$. Indeed, if $R_1 : E' \hookrightarrow E'''$ is the natural embedding, then $\langle R_1x', Jx\rangle = \langle x', x\rangle$ for all $x' \in E'$ and $x \in E$.

We will need the following refinement of Proposition 2.1.

2.2. LEMMA. *Let $R_m : \mathcal{L}(^mE;G) \to \mathcal{L}(^mF;G)$ be a Nicodemi sequence of extension operators for some $J \in \mathcal{L}(E;F)$. Then*

$$R_mA(J^{m-k}x,y) = R_k[I_{m-k}A(x)](y) \tag{2.2}$$

for all $A \in \mathcal{L}(^mE;G)$, $x \in E^{m-k}$, $y \in F^k$ and $0 \le k \le m$, with the obvious interpretations when $k = 0$ or $k = m$.

Proof. By Proposition 2.1, (2.2) is true for $k = 0$ and every $m \in \mathbb{N}$, and it is obviously true whenever $k = m \in \mathbb{N}$. In particular, (2.2) is true for $m = 1$ and $0 \le k \le 1$. Assuming (2.2) true for some $m \in \mathbb{N}$ and $0 \le j < m$, we will prove it for $m+1$ and $0 < k < m+1$. Let $A \in \mathcal{L}(^{m+1}E;G)$, $x \in E^{m+1-k}$, $y = (u,v) \in F^{k-1} \times F$. By the induction hypothesis we have

$$\begin{aligned} R_{m+1}A(J^{m+1-k}x,u,v) &= R_m[(R_1 \circ I_mA)^t(v)](J^{m+1-k}x,u) \\ &= R_{k-1}[I_{m+1-k}\{(R_1 \circ I_mA)^t(v)\}(x)](u). \end{aligned} \tag{2.3}$$

On the other hand,

$$R_k[I_{m+1-k}A(x)](u,v) = R_{k-1}[(R_1 \circ I_{k-1}\{I_{m+1-k}A(x)\})^t(v)](u). \tag{2.4}$$

By (2.3) and (2.4), to complete the proof it suffices to show that

$$I_{m+1-k}\{(R_1 \circ I_mA)^t(v)\}(x) = (R_1 \circ I_{k-1}\{I_{m+1-k}A(x)\})^t(v). \tag{2.5}$$

Both sides of (2.5) belong to $\mathcal{L}(^{k-1}E;G)$. Now for $s \in E^{k-1}$ we have

$$\begin{aligned} (R_1 \circ I_{k-1}\{I_{m+1-k}A(x)\})^t(v)(s) &= R_1[I_{k-1}\{I_{m+1-k}A(x)\}(s)](v) \\ &= R_1[I_mA(x,s)](v) = (R_1 \circ I_mA)^t(v)(x,s). \end{aligned}$$

This proves (2.5) and the lemma. ∎

2.3. LEMMA. *Assume there are $J \in \mathcal{L}(E;F)$ and $R_1 \in \mathcal{L}(\mathcal{L}(E;G);\mathcal{L}(F;G))$ such that $R_1A(Jx) = A(x)$ for all $A \in \mathcal{L}(E;G)$ and $x \in E$. Then $\|x\| \le \|R_1\|\,\|Jx\|$ for every $x \in E$. In particular, J has a continuous inverse.*

Proof. Fix $c \in G$, $\|c\| = 1$, so that $\|x' \otimes c\| = \|x'\|$ and $R_1(x' \otimes c)(Jx) = \langle x', x\rangle c$ for all $x' \in E'$ and $x \in E$. Then

$$\|x\| = \sup_{\|x'\| \le 1} |\langle x', x\rangle| = \sup_{\|x'\| \le 1} \|R_1(x' \otimes c)(Jx)\| \le \|R_1\|\,\|Jx\|. \ ∎$$

3. Extension of holomorphic mappings. Given a Nicodemi sequence $R_m : \mathcal{L}(^mE;G) \to \mathcal{L}(^mF;G)$ we define

$$\widehat{R}_m : \mathcal{P}(^mE;G) \to \mathcal{P}(^mF;G)$$

by $\widehat{R}_m\widehat{A} = \widehat{R_mA}$ for every symmetric $A \in \mathcal{L}(^mE;G)$. Note that, when $G = K$, it follows from Proposition 1.7 that

$$\widehat{R}_{m+n}(PQ) = \widehat{R}_mP \cdot \widehat{R}_nQ \tag{3.1}$$

for all $P \in \mathcal{P}(^mE)$ and $Q \in \mathcal{P}(^nE)$. This result is due to Nicodemi [15, Lemma 4].

We can readily see that $\|R_m\| \le \|R_1\|^m$ and $\|\widehat{R}_m\| \le e^m\|R_1\|^m$ for every $m \in \mathbb{N}$. Hence in the complex case we may also define

$$\widehat{R} : \mathcal{H}_b(E;G) \to \mathcal{H}_b(F;G)$$

by $\widehat{R}f = \sum_{m=0}^{\infty} \widehat{R}_m(P^mf(0))$ for every $f \in \mathcal{H}_b(E;G)$. The operator $\widehat{R}$ is linear and continuous. In the case $G = \mathbb{C}$ the multiplicativity (3.1) on polynomials implies that $\widehat{R} : \mathcal{H}_b(E) \to \mathcal{H}_b(F)$ is an algebra homomorphism.

3.1. Example. Let $\pi \in \mathcal{L}(F;E)$ and let $\pi_m : \mathcal{L}(^mE;G) \to \mathcal{L}(^mF;G)$ be the Nicodemi sequence of Example 1.1. Then we can readily see that $\widehat{\pi}_m P = P \circ \pi$ for every $P \in \mathcal{P}(^mE;G)$ and $\widehat{\pi} f = f \circ \pi$ for every $f \in \mathcal{H}_b(E;G)$. Moreover, given an open set U in E, let $V = \pi^{-1}(U)$ and let

$$\widehat{\pi} : \mathcal{H}(U;G) \to \mathcal{H}(V;G)$$

be defined by $\widehat{\pi} f = f \circ \pi$ for every $f \in \mathcal{H}(U;G)$. Then we can readily see that $P^m(\widehat{\pi} f)(y) = \widehat{\pi}_m(P^m f(\pi y))$ for every $f \in \mathcal{H}(U;G)$, $y \in V$ and $m \in \mathbb{N}$.

3.2. Example. Let $R_m : \mathcal{L}(^mE;G) \to \mathcal{L}(^mF;G)$ be a Nicodemi sequence of extension operators for some $J \in \mathcal{L}(E;F)$. Then we can readily see that $\widehat{R}_m P(Jx) = P(x)$ for every $P \in \mathcal{P}(^mE;G)$ and $x \in E$, and $\widehat{R} f(Jx) = f(x)$ for every $f \in \mathcal{H}_b(E;G)$ and $x \in E$. Moreover, we have the following theorem.

3.3. Theorem. *Let $R_m : \mathcal{L}(^mE;G) \to \mathcal{L}(^mF;G)$ be a Nicodemi sequence of extension operators for some $J \in \mathcal{L}(E;F)$. Then for each open set U in E and each mapping $f \in \mathcal{H}(U;G)$, there are an open set V in F containing $J(U)$ and a mapping $\widetilde{f} \in \mathcal{H}(V;G)$ such that $\widetilde{f}(Jx) = f(x)$ for every $x \in U$. We can take $V = \bigcup_{x\in U} B(Jx; r_b f(x)/(e\|R_1\|))$, where $r_b f(x)$ denotes the radius of boundedness of f at x.*

This theorem generalizes a result of Aron and Berner [3, Corollary 2.1], and their proof also works in our case. We should remark that their proof is slightly incorrect, since they claim that the extension $R_m A$ of each symmetric $A \in \mathcal{L}(^mE;G)$ is again symmetric, and this is not true in general (see Remark 7.2(b)). However, the extension $R_m A$ of each symmetric $A \in \mathcal{L}(^mE;G)$ is "sufficiently symmetric" to make the proof work. This is the content of Lemma 3.4. We should remark that Lemmas 2.2 and 2.3 play also a key role in the proof of Theorem 3.3.

Given $A \in \mathcal{L}(^mE;G)$ and a permutation σ of $\{1,\dots,m\}$, let $A^\sigma \in \mathcal{L}(^mE;G)$ be defined by

$$A^\sigma(x_1,\dots,x_m) = A(x_{\sigma(1)},\dots,x_{\sigma(m)})$$

for all $x_1,\dots,x_m \in E$. With this notation we have the following lemma.

3.4. Lemma. *Let $R_m : \mathcal{L}(^mE;G) \to \mathcal{L}(^mF;G)$ be a Nicodemi sequence of extension operators for some $J \in \mathcal{L}(E;F)$. Then*

$$(R_m A)^\sigma(u) = R_m A^\sigma(u)$$

for every $A \in \mathcal{L}(^mE;G)$, every transposition of the form $\sigma = (j\ j+1)$, and every $u \in F^m$ such that $u_j \in J(E)$.

Proof. We first consider the case $m = 2$. Let $A \in \mathcal{L}(^2E;G)$, let $u = (Jx, y) \in J(E) \times F$, and let $\sigma = (1\ 2)$. Then

$$(R_2A)^\sigma(Jx,y) = R_2A(y,Jx) = R_1[(R_1 \circ I_1A)^t(Jx)](y).$$

Now for $s \in E$ we have

$$(R_1 \circ I_1A)^t(Jx)(s) = R_1[I_1A(s)](Jx) = I_1A(s)(x) = I_1A^\sigma(x)(s).$$

Thus

$$\begin{aligned}(R_2A)^\sigma(Jx,y) &= R_1[I_1A^\sigma(x)](y) = (R_1 \circ I_1A^\sigma)^t(y)(x)\\ &= R_1[(R_1 \circ I_1A^\sigma)^t(y)](Jx) = R_2A^\sigma(Jx,y).\end{aligned}$$

Assuming the conclusion true for some $m \ge 2$, we will prove it for $m+1$. Let $A \in \mathcal{L}(^{m+1}E;G)$, let $u \in F^{m+1}$ with $u_j \in J(E)$, let $v = (u_{\sigma(1)},\dots,u_{\sigma(m+1)})$ and let $\sigma = (j\ j+1)$. Then either $j+1 \le m$ or $j \ge 2$.

If $j+1 \le m$, then we can write $u = (u', u_{m+1})$, $v = (v', u_{m+1}) \in F^m \times F$. Since σ leaves $m+1$ fixed, we get

$$\begin{aligned}(R_{m+1}A)^\sigma(u',u_{m+1}) &= R_{m+1}A(v',u_{m+1}) = R_m[(R_1 \circ I_mA)^t(u_{m+1})](v')\\ &= R_m[(R_1 \circ I_mA)^t(u_{m+1})]^\sigma(u')\\ &= R_m[(R_1 \circ I_mA^\sigma)^t(u_{m+1})](u') = R_{m+1}A^\sigma(u',u_{m+1}).\end{aligned}$$

Finally, if $j \ge 2$ then we can write $u = (u_1, u')$, $v = (u_1, v') \in F \times F^m$. Then

$$(R_{m+1}A)^\sigma(u_1,u') = R_{m+1}A(u_1,v') = R_1[(R_m \circ I_1A)^t(v')](u_1).$$

Now $(R_m \circ I_1A)^t(v') \in \mathcal{L}(E;G)$. Since σ leaves 1 fixed, for $s \in E$ we get

$$\begin{aligned}(R_m \circ I_1A)^t(v')(s) &= R_m[I_1A(s)](v') = R_m[I_1A(s)]^\sigma(u')\\ &= R_m[I_1A^\sigma(s)](u') = (R_m \circ I_1A^\sigma)^t(u')(s).\end{aligned}$$

Thus

$$(R_{m+1}A)^\sigma(u_1,u') = R_1[(R_m \circ I_1A^\sigma)^t(u')](u_1) = R_{m+1}A^\sigma(u_1,u')$$

and the proof is complete. ■

4. Scalar-valued and vector-valued Nicodemi sequences. Thus far we have considered Nicodemi sequences for multilinear mappings with values in a fixed Banach space. In this section we will see how each Nicodemi sequence for multilinear forms yields in a natural way a Nicodemi sequence for multilinear mappings with values in a dual Banach space.

From now on, δ_z will denote the evaluation at z defined in the usual way.

4.1. Proposition. *Given $R_1 \in \mathcal{L}(E';F')$, let $\widetilde{R}_1 \in \mathcal{L}(\mathcal{L}(E;G');\mathcal{L}(F;G'))$ be defined by*

$$\widetilde{R}_1A(y)(z) = R_1(\delta_z \circ A)(y)$$

for all $A \in \mathcal{L}(E;G')$, $y \in F$ and $z \in G$. If (R_m) and $(\widetilde{R}_m)$ are the corresponding Nicodemi sequences, then

$$\widetilde{R}_mA(y)(z) = R_m(\delta_z \circ A)(y) \tag{4.1}$$

for all $A \in \mathcal{L}(^mE;G')$, $y \in F^m$ and $z \in G$.

Proof. By hypothesis, (4.1) is true for $m = 1$. Assuming it true for a certain m, we will prove it for $m+1$. Now let $A \in \mathcal{L}(^{m+1}E; G')$, $u \in F^m$, $v \in F$ and $z \in G$. By using the induction hypothesis we get

$$\widetilde{R}_{m+1}A(u,v)(z) = \widetilde{R}_m[(\widetilde{R}_1 \circ I_m A)^t(v)](u)(z) \tag{4.2}$$
$$= R_m\{\delta_z \circ [(\widetilde{R}_1 \circ I_m A)^t(v)]\}(u).$$

On the other hand,

$$R_{m+1}(\delta_z \circ A)(u,v) = R_m\{[R_1 \circ I_m(\delta_z \circ A)]^t(v)\}(u). \tag{4.3}$$

By (4.2) and (4.3), to complete the proof it suffices to show that

$$\delta_z \circ [(\widetilde{R}_1 \circ I_m A)^t(v)] = [R_1 \circ I_m(\delta_z \circ A)]^t(v). \tag{4.4}$$

Both sides of (4.4) belong to $\mathcal{L}(^mE)$. Now for $x \in E^m$ we have

$$\delta_z \circ [(\widetilde{R}_1 \circ I_m A)^t(v)](x) = (\widetilde{R}_1 \circ I_m A)^t(v)(x)(z) = \widetilde{R}_1[I_m A(x)](v)(z)$$
$$= R_1\{\delta_z \circ [I_m A(x)]\}(v)$$
$$= R_1\{I_m(\delta_z \circ A)(x)\}(v) = [R_1 \circ I_m(\delta_z \circ A)]^t(v)(x).$$

This shows (4.4) and the proposition. ∎

4.2. Corollary. *Let $R_m : \mathcal{L}(^mE) \to \mathcal{L}(^mF)$ be a Nicodemi sequence for multilinear forms, and let $\widetilde{R}_m : \mathcal{L}(^mE; G') \to \mathcal{L}(^mF; G')$ be the corresponding sequence for vector-valued multilinear mappings. Then:*

(a) *For every $A \in \mathcal{L}(^mE; G')$ we have*

$$\widetilde{R}_m A = (R_m \circ A^t)^t. \tag{4.5}$$

(b) *For every $A \in \mathcal{L}(^{m+1}E)$ we have*

$$R_{m+1}A = I_m^{-1}[\widetilde{R}_m(R_1 \circ I_m A)] = I_1^{-1}[R_1 \circ \widetilde{R}_m(I_1 A)^t]^t. \tag{4.6}$$

Proof. (4.5) follows directly from (4.1). The first equality in (4.6) follows from (1.1) and (4.5), whereas the second one follows from (1.2) and (4.5). ∎

Formulas (4.6) are very useful to establish properties of the sequences (R_m) and $(\widetilde{R}_m)$ by an inductive procedure. It is often easy to derive a property of $\widetilde{R}_m$ from the corresponding property of R_m. These formulas are also often useful to derive a property of R_{m+1} from the corresponding property of $\widetilde{R}_m$.

We omit the straightforward proof of the following proposition.

4.3. Proposition. *Let $R_m : \mathcal{L}(^mE) \to \mathcal{L}(^mF)$ be a Nicodemi sequence of extension operators for some $J \in \mathcal{L}(E;F)$. Then the corresponding sequence $\widetilde{R}_m : \mathcal{L}(^mE; G') \to \mathcal{L}(^mF; G')$ is also a Nicodemi sequence of extension operators for the same J.*

4.4. Remark. We point out that the mappings in $\mathcal{L}(^mE; G)$ may be regarded as mappings with values in G'' and thus be subject to the methods in this section. In general it is not possible to extend the mappings in $\mathcal{L}(E; G)$ to E'' in such a way that the extensions still take their values in G. The most natural counterexample is the identity mapping $I : c_0 \to c_0$. Indeed, if $J : c_0 \hookrightarrow l^\infty$ is the natural embedding, then, by a result of Phillips [18], there is no $P \in \mathcal{L}(l^\infty; c_0)$ such that $P \circ J = I$. This justifies our choice of setting in this section.

5. Extensions to the bidual. Throughout this section let $J : E \hookrightarrow E''$ and $R_1 : E' \hookrightarrow E'''$ be the natural embeddings, let $R_m : \mathcal{L}(^mE) \to \mathcal{L}(^mE'')$ be the Nicodemi sequence beginning with R_1, and let $\widetilde{R}_m : \mathcal{L}(^mE; G') \to \mathcal{L}(^mE''; G')$ be the corresponding Nicodemi sequence for vector-valued mappings.

Proposition 5.1. *If $A \in \mathcal{L}(^mE)$, then the linear functional*

$$E'' \ni x_j'' \to R_m A(Jx_1, \ldots, Jx_{j-1}, x_j'', x_{j+1}'', \ldots, x_m'') \in K$$

is $\sigma(E'', E')$-continuous for fixed $x_1, \ldots, x_{j-1} \in E$ and $x_{j+1}'', \ldots, x_m'' \in E''$.

Proof. The conclusion is obviously true for $m = 1$. Assuming the conclusion true for some $m \in \mathbb{N}$, we will prove it for $m+1$. Now for $A \in \mathcal{L}(^{m+1}E)$, $x'' \in (E'')^m$ and $y'' \in E''$ we have

$$R_{m+1}A(x'', y'') = R_m[(R_1 \circ I_m A)^t(y'')](x'').$$

Then it follows from the induction hypothesis that the linear functional

$$E'' \ni x_j'' \to R_{m+1}A(Jx_1, \ldots, Jx_{j-1}, x_j'', x_{j+1}'', \ldots, x_m'', y'') \in K$$

is $\sigma(E'', E')$-continuous for fixed $x_1, \ldots, x_{j-1} \in E$ and $x_{j+1}'', \ldots, x_m'', y'' \in E''$, when $1 \le j \le m$. Finally, if $x \in E^m$, then by Lemma 2.2 or by direct computation we have

$$R_{m+1}A(J^m x, y'') = R_1[I_m A(x)](y''),$$

whence the linear functional

$$E'' \ni y'' \to R_{m+1}A(J^m x, y'') \in K$$

is also $\sigma(E'', E')$-continuous. ∎

5.2. Corollary. *If $A \in \mathcal{L}(^mE; G')$, then the linear functional*

$$E'' \ni x_j'' \to \widetilde{R}_m A(Jx_1, \ldots, Jx_{j-1}, x_j'', x_{j+1}'', \ldots, x_m'') \in G'$$

is $\sigma(E'', E')$-$\sigma(G', G)$-continuous for fixed $x_1, \ldots, x_{j-1} \in E$ and $x_{j+1}'', \ldots, x_m'' \in E''$.

Proposition 5.1 tells us that the Nicodemi sequence beginning with the natural embedding $R_1 : E' \hookrightarrow E'''$ is precisely the sequence of extension operators considered by Davie and Gamelin in [6] and by Aron, Cole and Gamelin in [4].

Let $L : G \hookrightarrow G''$ denote the natural embedding. In the case of a bilinear mapping $A \in \mathcal{L}(^2E; G)$, our extension $\widetilde{R}_2(L \circ A) \in \mathcal{L}(^2E''; G'')$ coincides with the extension considered by Arens [2], who seems to be the first to have studied the extension of bilinear mappings to the bidual. Corollary 5.2, for this case, is also due to Arens [2, Theorem 3.2].

6. Independence of the range space. Throughout this section let $J : E \hookrightarrow E''$ and $R_1 : E' \hookrightarrow E'''$ be the natural embeddings, let (R_m) be the Nicodemi sequence beginning with R_1, and let $(\widetilde{R}_m)$ be the corresponding Nicodemi sequence for vector-valued mappings.

Since every $A \in \mathcal{L}(^mE; G')$ may also be regarded as a mapping with values in G''', we may ask whether the two extensions coincide. In other words, we ask whether $\widetilde{R}_m(T_1 \circ A) = T_1 \circ \widetilde{R}_m A$, where $T_1 : G' \hookrightarrow G'''$ is the natural embedding. The answer to this question is given by Theorem 6.3. But first we need some preparatory results.

6.1. Lemma. *Let $A \in \mathcal{L}(E; G')$. Then:*

(a) $A' = (T_1 \circ A)^t$.

(b) $A^{t\prime} = (R_1 \circ A^t)^t = \widetilde{R}_1 A$.

(c) $A'' = [R_1 \circ (T_1 \circ A)^t]^t = \widetilde{R}_1(T_1 \circ A)$.

Proof. (a) is immediate, whereas (b) and (c) follow from (a) and Corollary 4.2. ∎

With the aid of Corollary 5.2 we can generalize this result as follows.

6.2. Proposition. *Let $A \in \mathcal{L}(^mE; G')$, and let $A_j \in \mathcal{L}(E; G')$, $B_j \in \mathcal{L}(E''; G')$, and $C_j \in \mathcal{L}(E''; G''')$ be defined by*

$$\begin{aligned} &A_j(x_j) = \widetilde{R}_m A(Jx_1, \ldots, Jx_{j-1}, Jx_j, x''_{j+1}, \ldots, x''_m), \\ (6.1)\qquad &B_j(x''_j) = \widetilde{R}_m A(Jx_1, \ldots, Jx_{j-1}, x''_j, x''_{j+1}, \ldots, x''_m), \\ &C_j(x''_j) = \widetilde{R}_m(T_1 \circ A)(Jx_1, \ldots, Jx_{j-1}, x''_j, x''_{j+1}, \ldots, x''_m), \end{aligned}$$

for each choice of the points $x_1, \ldots, x_{j-1} \in E$ and $x''_{j+1}, \ldots, x''_m \in E''$. Then:

(a) $B_j = A_j^{t\prime} = \widetilde{R}_1 A_j$ *for every* $j = 1, \ldots, m$.

(b) $C_m = A''_m = \widetilde{R}_1(T_1 \circ A_m)$.

Proof. Since $B_j \circ J = A_j = A_j^{t\prime} \circ J$, and both B_j and $A_j^{t\prime}$ are $\sigma(E'', E')$-$\sigma(G', G)$-continuous, (a) follows. And since $C_m \circ J = T_1 \circ A_m = A''_m \circ J$, and both C_m and A''_m are $\sigma(E'', E')$-$\sigma(G''', G'')$- continuous, (b) follows. ∎

6.3. Theorem. *Let $A \in \mathcal{L}(^mE; G')$, and let $A_j \in \mathcal{L}(E; G')$, and $B_j \in \mathcal{L}(E''; G')$ be defined by*

$$A_j(x_j) = \widetilde{R}_m A(Jx_1, \ldots, Jx_{j-1}, Jx_j, x''_{j+1}, \ldots, x''_m),$$

$$B_j(x''_j) = \widetilde{R}_m A(Jx_1, \ldots, Jx_{j-1}, x''_j, x''_{j+1}, \ldots, x''_m),$$

for each choice of the points $x_1, \ldots, x_{j-1} \in E$ and $x''_{j+1}, \ldots, x''_m \in E''$. Then the following conditions are equivalent:

(a) $\widetilde{R}_m(T_1 \circ A)((E'')^m) \subset T_1(G')$.

(b) $\widetilde{R}_m(T_1 \circ A) = T_1 \circ \widetilde{R}_m A$.

(c) *Each B_j is $\sigma(E'', E')$-$\sigma(G', G'')$-continuous.*

(d) *Each B_j is weakly compact.*

(e) *Each A_j is weakly compact.*

Proof. As in Proposition 6.2, let $C_j \in \mathcal{L}(E''; G''')$ be defined by

$$C_j(x''_j) = \widetilde{R}_m(T_1 \circ A)(Jx_1, \ldots, Jx_{j-1}, x''_j, x''_{j+1}, \ldots, x''_m)$$

for each choice of the points $x_1, \ldots, x_{j-1} \in E$ and $x''_{j+1}, \ldots, x''_m \in E''$.

(a)⇒(b). By (a) for each $x'' \in (E'')^m$ there exists $z' \in G'$ such that $\widetilde{R}_m(T_1 \circ A)(x'') = T_1 z'$. We claim that $z' = \widetilde{R}_m A(x'')$. Indeed, if $L : G \hookrightarrow G''$ is the natural embedding, then for every $z \in G$ we have

$$\begin{aligned} \langle z', z \rangle &= \langle T_1 z', Lz \rangle = \langle \widetilde{R}_m(T_1 \circ A)(x''), Lz \rangle \\ &= R_m(\delta_{Lz} \circ T_1 \circ A)(x'') = R_m(\delta_z \circ A)(x'') = \langle \widetilde{R}_m A(x''), z \rangle . \end{aligned}$$

(b)⇒(c). By (b) we have $C_j = T_1 \circ B_j$ for each choice of the points $x_1, \ldots, x_{j-1} \in E$ and $x''_{j+1}, \ldots, x''_m \in E''$. Since C_j is $\sigma(E'', E')$-$\sigma(G''', G'')$-continuous, it follows that B_j is $\sigma(E'', E')$-$\sigma(G', G'')$-continuous.

The implication (c)⇒(d) follows from the Alaoglu theorem, and (d)⇒(e) is obvious.

(e)⇒(a). Since $C_m = A''_m$, by Proposition 6.2(b), a theorem of Gantmacher (see [9, p. 482, Theorem 2]) implies that $C_m(E'') \subset T_1(G')$ for each choice of the points $x_1, \ldots, x_{m-1}$ in E. Then the proof of (a)⇒(b) shows that $C_m = T_1 \circ B_m$ and hence it follows easily that $C_{m-1} \circ J = T_1 \circ A_{m-1} = A''_{m-1} \circ J$. As in the proof of Proposition 6.2(b) we conclude that $C_{m-1} = A''_{m-1}$, and hence as before $C_{m-1} = T_1 \circ B_{m-1}$. Proceeding inductively we conclude that $C_1 = T_1 \circ B_1$, in particular (a) holds. ∎

6.4. Corollary. *For each $A \in \mathcal{L}(E; G')$ the following conditions are equivalent:*

(a) $\widetilde{R}_1(T_1 \circ A)(E'') \subset T_1(G')$.

(b) $\widetilde{R}_1(T_1 \circ A) = T_1 \circ \widetilde{R}_1 A$.

(c) *$\widetilde{R}_1 A$ is $\sigma(E'', E')$-$\sigma(G', G'')$-continuous.*

(d) *$\widetilde{R}_1 A$ is weakly compact.*

(e) *A is weakly compact.*

6.5. Remark. The equivalent conditions of Theorem 6.3 are clearly satisfied by every $A \in \mathcal{L}(^m E; G')$ if every $B \in \mathcal{L}(E; G')$ is weakly compact.

6.6. Remark. It follows from Lemma 6.1 that $I_1(R_2 A) = (I_1 A)''$ for every $A \in \mathcal{L}(^2 E)$. Then a tedious proof by induction shows that the sequence (R_m) coincides with the sequence of extension operators constructed by Aron and Berner in [3, Proposition 2.1]. We refrain from giving the details.

7. Change of order of the variables. Throughout this section let $J : E \hookrightarrow E''$ and $R_1 : E' \hookrightarrow E'''$ be the natural embeddings, let (R_m) be the Nicodemi sequence beginning with R_1, and let $(\widetilde{R}_m)$ be the corresponding Nicodemi sequence for vector-valued mappings.

The results in this section are slight variations of results of Aron, Cole and Gamelin (see [4, Theorem 8.3]). We include them here to stress their relation to the results in the preceding section.

7.1. Lemma. *For $A \in \mathcal{L}(^2 E)$ the following conditions are equivalent:*

(a) $(R_2 A)^\sigma = R_2 A^\sigma$ *if σ is the transposition* $(1\ 2)$.
(b) $R_2 A$ *is separately $\sigma(E'', E')$-continuous.*
(c) $I_1 A \in \mathcal{L}(E; E')$ *is weakly compact.*

Proof. (a)$\Leftrightarrow$(c). By Corollary 4.2,

$$I_1(R_2 A)^\sigma = [\widetilde{R}_1(R_1 \circ I_1 A)]^t, \quad I_1(R_2 A^\sigma) = [R_1 \circ \widetilde{R}_1(I_1 A)]^t .$$

Thus $(R_2 A)^\sigma = R_2 A^\sigma$ if and only if $\widetilde{R}_1(R_1 \circ I_1 A) = R_1 \circ \widetilde{R}_1(I_1 A)$. By Corollary 6.4 this occurs if and only if $I_1 A$ is weakly compact.

(b)$\Leftrightarrow$(c). By Proposition 5.1, $R_2 A(x'', y'')$ is always a $\sigma(E'', E')$-continuous function of x''. On the other hand, by Corollary 4.2,

$$R_2 A(x'', y'') = \langle \widetilde{R}_1(R_1 \circ I_1 A)(x''), y'' \rangle .$$

Thus $R_2 A(x'', y'')$ is a $\sigma(E'', E')$-continuous function of y'' if and only if $\widetilde{R}_1(R_1 \circ I_1 A)(E'') \subset R_1(E')$. By Corollary 6.4 this occurs if and only if $I_1 A$ is weakly compact. ∎

7.2. Remarks. (a) Lemma 7.1 is well-known. The implications (c)$\Rightarrow$(a) and (c)$\Rightarrow$(b) were already noticed by Grothendieck [11, p. 26].

(b) Let $L : G \hookrightarrow G''$ be the natural embedding. According to Arens [2], a mapping $A \in \mathcal{L}(^2 E; G)$ is said to be *regular* if $[\widetilde{R}_2(L \circ A)]^\sigma = \widetilde{R}_2(L \circ A)^\sigma$ where σ is the transposition $(1\ 2)$. Arens [2, Theorem 4.4] proved that every element of $\mathcal{L}(^2 c_0; G)$ is regular and gave an example of a symmetric form $A \in \mathcal{L}(^2 l^1)$ which is not regular (see [2, pp. 847–848]). Likewise Aron, Cole and Gamelin [4, p. 83] gave an example of a symmetric form $B \in \mathcal{L}(^2 l^1)$ such that $I_1 B \in \mathcal{L}(l^1; l^\infty)$ is not weakly compact. Thus neither $R_2 A$ nor $R_2 B$ is symmetric.

7.3. Theorem. *Assume that every $B \in \mathcal{L}(E; E')$ is weakly compact. Then:*

(a) $(\widetilde{R}_m A)^\sigma = \widetilde{R}_m A^\sigma$ *for every $A \in \mathcal{L}(^m E; G')$ and every permutation σ of $\{1, \ldots, m\}$.*

(b) $\widetilde{R}_m A$ *is separately $\sigma(E'', E')$-$\sigma(G', G)$-continuous for every $A \in \mathcal{L}(^m E; G')$.*

Proof. (a) To begin with observe that if $(R_m A)^\sigma = R_m A^\sigma$ for every $A \in \mathcal{L}(^m E)$, then $(\widetilde{R}_m A)^\sigma = \widetilde{R}_m A^\sigma$ for every $A \in \mathcal{L}(^m E; G')$. Thus (a) is true for $m = 2$ by Lemma 7.1, and we shall prove (a) for $m+1$ whenever it is true for some $m \geq 2$. Let $A \in \mathcal{L}(^{m+1} E)$ and let $x'' \in (E'')^m$ and $y'' \in E''$. By Corollary 4.2,

$$R_{m+1} A(x'', y'') = \widetilde{R}_m(R_1 \circ I_m A)(x'')(y'') .$$

If σ is a permutation of $\{1, \ldots, m+1\}$ that leaves $m+1$ fixed, then it follows at once from the induction hypothesis that $(\widetilde{R}_{m+1} A)^\sigma = \widetilde{R}_{m+1} A^\sigma$. If σ does not leave $m+1$ fixed, then by the preceding case we may restrict our attention to the transposition $\sigma = (1\ m+1)$. Then by Corollary 4.2 and Theorem 6.3 we have

$$\begin{aligned} (7.1) \quad R_{m+1} A(y'', x'') &= [R_1 \circ \widetilde{R}_m (I_1 A)^t](x'')(y'') \\ &= \widetilde{R}_m [R_1 \circ (I_1 A)^t](x'')(y'') = R_{m+1} B(x'', y'') \end{aligned}$$

where $B(x, y) = A(y, x)$ for all $x \in E^m$ and $y \in E$. Hence $B = A^\tau$, where $\tau(1) = m+1$ and $\tau(j) = j-1$ for $2 \leq j \leq m+1$. Thus (7.1) tells us that $(R_{m+1} A)^\tau = R_{m+1} A^\tau$. Let $\varrho = \sigma \circ \tau^{-1}$. Then $\sigma = \varrho \circ \tau$ and ϱ leaves $m+1$ fixed. Since (a) is true for ϱ and for τ, it is also true for σ. This proves (a).

(b) To prove that $\widetilde{R}_m A$ is separately $\sigma(E'', E')$-$\sigma(G', G)$-continuous for every $A \in \mathcal{L}(^m E; G')$, it certainly suffices to prove that $R_m A$ is separately $\sigma(E'', E')$-continuous for every $A \in \mathcal{L}(^m E)$. Now by Proposition 5.1, $R_m A(x_1'', \ldots, x_m'')$ is a $\sigma(E'', E')$-continuous function of x_1'' for every $A \in \mathcal{L}(^m E)$. If σ is the transposition $(1\ j)$, then by using (a) we see that

$$R_m A(x_1'', \ldots, x_j'', \ldots, x_m'') = R_m A^\sigma (x_j'', \ldots, x_1'', \ldots, x_m'')$$

is a $\sigma(E'', E')$-continuous function of x_j''. This proves (b). ∎

We end this section with some examples of nonreflexive Banach spaces E with the property that every $B \in \mathcal{L}(E; E')$ is weakly compact.

7.4. Example. Aron, Cole and Gamelin [4, p. 83] have observed that if X is a compact, Hausdorff space, then every continuous linear operator from $C(X)$ into $C(X)'$ is weakly compact.

7.5. Example. More generally, if E is a C^*-algebra, then every $B \in \mathcal{L}(E; E')$ is weakly compact. Indeed, Akemann [1, Corollary II.9] has shown that every continuous linear operator from a C^*-algebra to the predual of a W^*-algebra is weakly compact. Since the bidual of a C^*-algebra is a W^*-algebra (see [20, p. 43, Theorem 1.17.2]), the desired conclusion follows.

7.6. EXAMPLE. In [16] Pełczyński introduced the properties (V) and (V*). He proved that $C(X)$ has property (V) for each compact, Hausdorff space X, and proved that if E has property (V), then E' has property (V*). In [10] Godefroy and Iochum proved that if E' has property (V*) (and hence if E has property (V)), then every $B \in \mathcal{L}(E; E')$ is weakly compact. Godefroy and Iochum also proved that the dual of every C^*-algebra, as well as the dual of the disc algebra $A(\Delta)$, have property (V*). Very recently Pfitzner [17] proved that every C^*-algebra has property (V).

8. Separate compactness of multilinear mappings. Throughout this section let $J : E \hookrightarrow E''$ and $R_1 : E' \hookrightarrow E'''$ be the natural embeddings, let (R_m) be the Nicodemi sequence beginning with R_1, and let $(\widetilde{R}_m)$ be the corresponding Nicodemi sequence for vector-valued mappings.

Theorems 6.3 and 7.3 yield the following corollary.

8.1. COROLLARY. *Assume that every $B \in \mathcal{L}(E; E')$ as well as every $C \in \mathcal{L}(E; G')$ are weakly compact. Then for every $A \in \mathcal{L}(^m E; G')$ the extension $\widetilde{R}_m A \in \mathcal{L}(^m E''; G')$ is separately $\sigma(E'', E')$-$\sigma(G', G'')$-continuous, and hence separately weakly compact.*

Proof. By Theorem 6.3, $\widetilde{R}_m A(x_1'', \ldots, x_m'')$ is a $\sigma(E'', E')$-$\sigma(G', G'')$-continuous function of x_1''. If σ is the transposition $(1\ j)$, then by using Theorem 7.3 we see that

$$\widetilde{R}_m A(x_1'', \ldots, x_j'', \ldots, x_m'') = \widetilde{R}_m A^\sigma(x_j'', \ldots, x_1'', \ldots, x_m'')$$

is a $\sigma(E'', E')$-$\sigma(G', G'')$-continuous function of x_j''. ∎

8.2. PROPOSITION. *Let $A \in \mathcal{L}(^m E; G')$, and let $A_j \in \mathcal{L}(E; G')$ and $B_j \in \mathcal{L}(E''; G')$ be defined by*

$$A_j(x_j) = \widetilde{R}_m A(Jx_1, \ldots, Jx_{j-1}, Jx_j, x_{j+1}'', \ldots, x_m''),$$

$$B_j(x_j'') = \widetilde{R}_m A(Jx_1, \ldots, Jx_{j-1}, x_j'', x_{j+1}'', \ldots, x_m''),$$

for each choice of the points $x_1, \ldots, x_{j-1} \in E$ and $x_{j+1}'', \ldots, x_m'' \in E''$. Then A_j is compact (resp. integral) if and only if the corresponding B_j is compact (resp. integral).

Proof. By Proposition 6.2, $B_j = A_j^{t\prime}$. Note that $A_j^t = A_j' \circ L$, where $L : G \hookrightarrow G''$ is the natural embedding. Thus in the case of compact operators it suffices to apply Schauder's Theorem, and in the case of integral operators, a result of Grothendieck (see [12, p. 313]). ∎

8.3. COROLLARY. *Assume that every $B \in \mathcal{L}(E; E')$ is weakly compact and every $C \in \mathcal{L}(E; G')$ is compact (resp. integral). Then for every $A \in \mathcal{L}(^m E; G')$, the extension $\widetilde{R}_m A \in \mathcal{L}(^m E''; G')$ is separately compact (resp. separately integral).*

Proof. The proof of Corollary 8.1 applies, but using Proposition 8.2 instead of Theorem 6.3. ∎

8.4. EXAMPLE. Every weakly compact linear operator from c_0 into a Banach space is compact (see [12, p. 208]). Hence the spaces $E = G = c_0$, as well as $E = c_0$, G a reflexive Banach space, satisfy the hypotheses of Corollary 8.3 in the compact case.

8.5. EXAMPLE. Pisier [19, Theorem 3.2] has constructed a separable, infinite-dimensional Banach space P such that $P \otimes_\pi P = P \otimes_\varepsilon P$. By duality every $B \in \mathcal{L}(P; P')$ is integral and in particular weakly compact. Thus the spaces $E = G = P$ satisfy the hypotheses of Corollary 8.3 in the integral case.

9. Extensions to the fourth dual. Let $R_m : \mathcal{L}(^m E) \to \mathcal{L}(^m E'')$ be the Nicodemi sequence beginning with the natural embedding $R_1 : E' \hookrightarrow E'''$, let $S_m : \mathcal{L}(^m E'') \to \mathcal{L}(^m E^{(4)})$ be the Nicodemi sequence beginning with the natural embedding $S_1 : E''' \hookrightarrow E^{(5)}$, and let $T_m : \mathcal{L}(^m E) \to \mathcal{L}(^m E^{(4)})$ be the Nicodemi sequence beginning with the composite mapping $T_1 := S_1 \circ R_1 : E' \to E^{(5)}$. Aleksander Pełczyński asked us whether $T_m = S_m \circ R_m$ for every $m \in \mathbb{N}$.

Since the mapping $\pi := R_1' : E^{(4)} \to E''$ is a projection, there are two natural Nicodemi sequences on E''. One is the sequence (S_m) and the other is the Nicodemi sequence $\pi_m : \mathcal{L}(^m E'') \to \mathcal{L}(^m E^{(4)})$ beginning with the mapping $\pi_1 := \pi' = R_1'' : E''' \to E^{(5)}$. We already know from Remark 1.3 that $S_1 \neq \pi_1$ unless E is reflexive, but we may still ask whether $S_m \circ R_m = \pi_m \circ R_m$ for every $m \in \mathbb{N}$. This question, in the case of symmetric multilinear forms, was raised by Richard Aron in an attempt to find different types of nontrivial homomorphisms on $\mathcal{H}_b(E)$.

Theorem 9.3 and Proposition 9.4 will answer Pełczyński's question and Aron's question at the same time.

We begin with the following auxiliary lemma.

9.1. LEMMA. *Let $R_m : \mathcal{L}(^m E) \to \mathcal{L}(^m F)$ be a Nicodemi sequence for multilinear forms, and let $(\widetilde{R}_m)$ be the corresponding Nicodemi sequence for vector-valued multilinear mappings. If $\pi \in \mathcal{L}(G''; G)$, then $\widetilde{R}_m(\pi' \circ A) = \pi' \circ \widetilde{R}_m A$ for every $A \in \mathcal{L}(^m E; G')$.*

Proof. For all $y \in F^m$ and $z'' \in G''$ we have

$$\widetilde{R}_m(\pi' \circ A)(y)(z'') = R_m(\delta_{z''} \circ \pi' \circ A)(y).$$

On the other hand,

$$(\pi' \circ \widetilde{R}_m A)(y)(z'') = \widetilde{R}_m A(y)(\pi z'') = R_m(\delta_{\pi z''} \circ A)(y).$$

Since we can readily see that $\delta_{z''} \circ \pi' \circ A = \delta_{\pi z''} \circ A$, the desired conclusion follows. ∎

In the statements of Theorems 9.2 and 9.3, and Proposition 9.4, (R_m), (S_m), (T_m) and (π_m) are the Nicodemi sequences defined at the beginning of this section.

9.2. THEOREM. *$\widetilde{T}_m(A) = \widetilde{\pi}_m \circ \widetilde{R}_m(A)$ for every $A \in \mathcal{L}(^mE; G')$.*

Proof. We use induction on m. To begin with we show that $T_1(x') = \pi_1 \circ R_1(x')$ for all $x' \in E'$. Indeed, for all $y^{(4)} \in E^{(4)}$ we have

$$\langle \pi_1 \circ R_1 x', y^{(4)}\rangle = \langle R_1'' \circ R_1 x', y^{(4)}\rangle = \langle R_1 x', R_1' y^{(4)}\rangle = \langle R_1' y^{(4)}, x'\rangle$$
$$= \langle y^{(4)}, R_1 x'\rangle = \langle S_1 \circ R_1 x', y^{(4)}\rangle = \langle T_1' x', y^{(4)}\rangle.$$

We can readily see that if $T_m(A) = \pi_m \circ R_m(A)$ for all $A \in \mathcal{L}(^mE)$, then $\widetilde{T}_m(A) = \widetilde{\pi}_m \circ \widetilde{R}_m(A)$ for all $A \in \mathcal{L}(^mE; G')$. Thus the key step in the proof is deriving the desired result for $(m+1)$-linear forms from the corresponding result for vector-valued m-linear mappings.

Now let $A \in \mathcal{L}(^{m+1}E)$, so that $I_m A \in \mathcal{L}(^mE; E')$. By Corollary 4.2, the induction hypothesis and the case $m = 1$ we have

$$I_m(T_{m+1}A) = \widetilde{T}_m(T_1 \circ I_m A) = \widetilde{\pi}_m \circ \widetilde{R}_m(\pi_1 \circ R_1 \circ I_m A).$$

On the other hand, by Corollary 4.2 again,

$$I_m[\pi_{m+1}(R_{m+1}A)] = \widetilde{\pi}_m[\pi_1 \circ I_m(R_{m+1}A)] = \widetilde{\pi}_m[\pi_1 \circ \widetilde{R}_m(R_1 \circ I_m A)].$$

By Lemma 9.1, $\widetilde{R}_m(\pi_1 \circ R_1 \circ I_m A) = \pi_1 \circ \widetilde{R}_m(R_1 \circ I_m A)$ and the proof is complete. ∎

9.3. THEOREM. *Assume that every $B \in \mathcal{L}(E; E')$ is weakly compact. Then*

$$\widetilde{S}_m \circ \widetilde{R}_m(A) = \widetilde{T}_m(A) = \widetilde{\pi}_m \circ \widetilde{R}_m(A)$$

for every $A \in \mathcal{L}(^mE; G')$.

Proof. By Theorem 9.2 it suffices to prove the first equality. We proceed by induction on m. By definition $T_1(x') = S_1 \circ R_1(x')$ for all $x' \in E'$. We can readily see that if $T_m(A) = S_m \circ R_m(A)$ for all $A \in \mathcal{L}(^mE)$, then $\widetilde{T}_m(A) = \widetilde{S}_m \circ \widetilde{R}_m(A)$ for all $A \in \mathcal{L}(^mE; G')$. Thus the key step in the proof is deriving the desired result for $(m+1)$-linear forms from the corresponding result for vector-valued m-linear mappings.

Now let $A \in \mathcal{L}(^{m+1}E)$. By Corollary 4.2, the induction hypothesis and the case $m = 1$ we have

$$(9.1) \qquad I_m(T_{m+1}A) = \widetilde{T}_m(T_1 \circ I_m A) = \widetilde{S}_m \circ \widetilde{R}_m(S_1 \circ R_1 \circ I_m A).$$

On the other hand, by Corollary 4.2 again,

$$(9.2) \quad I_m[S_{m+1}(R_{m+1}A)] = \widetilde{S}_m[S_1 \circ I_m(R_{m+1}A)] = \widetilde{S}_m[S_1 \circ \widetilde{R}_m(R_1 \circ I_m A)].$$

We claim that

$$(9.3) \qquad \widetilde{R}_m(R_1 \circ I_m A) = R_1 \circ \widetilde{R}_m(I_m A),$$

$$(9.4) \qquad \widetilde{R}_m(S_1 \circ R_1 \circ I_m A) = S_1 \circ \widetilde{R}_m(R_1 \circ I_m A).$$

Since every $B \in \mathcal{L}(E; E')$ is weakly compact, (9.3) follows directly from Theorem 6.3. But then (9.4) also follows from Theorem 6.3 since, by (9.3), the mapping $R_1 \circ I_m A$ satisfies condition (e) in that theorem. It follows from (9.1), (9.2) and (9.4) that $T_{m+1}A = S_{m+1}(R_{m+1}A)$, as we wanted. ∎

Our next result tells us that the hypothesis in Theorem 9.3 is indeed necessary.

9.4. PROPOSITION. *For $A \in \mathcal{L}(^2E)$ the following conditions are equivalent:*

(a) $S_2 \circ R_2(A) = T_2(A)$.
(b) $S_2 \circ R_2(A) = \pi_2 \circ R_2(A)$.
(c) $I_1 A \in \mathcal{L}(E; E')$ *is weakly compact.*

Proof. By Theorem 9.2 it suffices to prove that (a)⇔(c). On the one hand, by Corollary 4.2 we have

$$I_1[S_2(R_2 A)] = \widetilde{S}_1[S_1 \circ I_1(R_2 A)] = \widetilde{S}_1[S_1 \circ \widetilde{R}_1(R_1 \circ I_1 A)].$$

On the other hand, by using Corollary 4.2 again, and the fact that the equality $\widetilde{T}_1 = \widetilde{S}_1 \circ \widetilde{R}_1$ is always true, we have

$$I_1(T_2 A) = \widetilde{T}_1(T_1 \circ I_1 A) = \widetilde{S}_1 \circ \widetilde{R}_1(S_1 \circ R_1 \circ I_1 A).$$

Since $\widetilde{S}_1$ is an extension operator, it is injective. Hence $S_2(R_2 A) = T_2 A$ if and only if $S_1 \circ \widetilde{R}_1(R_1 \circ I_1 A) = \widetilde{R}_1(S_1 \circ R_1 \circ I_1 A)$. By Corollary 6.4 this holds if and only if the operator $R_1 \circ I_1 A$ is weakly compact. And clearly this holds if and only if the operator $I_1 A$ is weakly compact. ∎

9.5. Remark. Theorem 9.2 tells us, in particular, that $(\pi_m \circ R_m)$ is always a Nicodemi sequence. On the other hand, Theorem 9.3 and Proposition 9.4 tell us, in particular, that $(S_m \circ R_m)$ is a Nicodemi sequence if and only if every $B \in \mathcal{L}(E; E')$ is weakly compact.

9.6. Remark. By following an entirely different approach, Aron *et al.* [5] have proved that $S_m \circ R_m(A) = \pi_m \circ R_m(A)$ for every symmetric $A \in \mathcal{L}(^mE)$ if and only if every symmetric $B \in \mathcal{L}(E; E')$ is weakly compact. Here, according to Aron, Cole and Gamelin [4, p. 81], an operator $B \in \mathcal{L}(E; E')$ is called *symmetric* if the corresponding bilinear form $I_1^{-1}B \in \mathcal{L}(^2E)$ is symmetric.

10. Unbounded homomorphisms on algebras of polynomials. Dixon [8, Theorem 4.2] has given an example of a complete, commutative, locally convex algebra, with jointly continuous multiplication, on which not every scalar-valued homomorphism is bounded. His example is a suitable algebra of polynomials of infinitely many variables. In this section we use an algebraic version of Nicodemi sequences to show that whenever E is an infinite-dimensional Banach space, then there are plenty of unbounded scalar-valued homomorphisms on the algebra $\mathcal{P}(E) = \bigoplus_{m=0}^{\infty} \mathcal{P}(^mE)$, with the locally convex direct sum topology. This is an immediate consequence of Theorem 10.1. It follows from another result of Dixon [8, Lemma 4.1] that $\mathcal{P}(E)$, with the locally convex direct sum topology, is always a complete, commutative, locally convex algebra, with jointly continuous multiplication.

To begin with we present an algebraic version of Nicodemi sequences. Given a linear mapping $R_1 : \mathcal{L}_a(E; G) \to \mathcal{L}_a(F; G)$, let $R_m : \mathcal{L}_a(^mE; G) \to \mathcal{L}_a(^mF; G)$ be inductively defined by

$$R_{m+1}A = I_m^{-1}[R_m \circ (R_1 \circ I_m A)']^t$$

for all $A \in \mathcal{L}_a(^{m+1}E; G)$ and $m \in \mathbb{N}$. As in Section 1 we will refer to (R_m) as the *Nicodemi sequence beginning with* R_1. Many of the results established in this paper for the case of continuous multilinear mappings, apply equally well, with the obvious changes, for the case of arbitrary multilinear mappings.

Now let

$$R_m : \mathcal{L}_a(^mE) \to \mathcal{L}_a(^mE^{**})$$

be the Nicodemi sequence beginning with the natural embedding $R_1 : E^* \hookrightarrow E^{***}$. As we did in Section 3 for the case of continuous mappings, let

$$\widehat{R}_m : \mathcal{P}_a(^mE) \to \mathcal{P}_a(^mE^{**})$$

be defined by $\widehat{R}_m\widehat{A} = \widehat{R_mA}$ for every symmetric $A \in \mathcal{L}_a(^mE)$. Since the conclusion of Proposition 1.7 is true in our new situation, it follows that

$$\widehat{R}_{m+n}(PQ) = \widehat{R}_mP \cdot \widehat{R}_nQ \tag{10.1}$$

for all $P \in \mathcal{P}_a(^mE)$ and $Q \in \mathcal{P}_a(^nE)$. Finally, let

$$\widehat{R} : \mathcal{P}_a(E) \to \mathcal{P}_a(E^{**})$$

be defined by $\widehat{R}(\sum_{k=0}^m P_k) = \sum_{k=0}^m \widehat{R}_kP_k$ if $P_k \in \mathcal{P}_a(^kE)$ for $k = 0, 1, \ldots, m$. It follows from (10.1) that $\widehat{R}$ is an algebra homomorphism. Now we can prove the following theorem.

10.1. THEOREM. *For each linear functional $\omega_1 : E' \to K$, there is an algebra homomorphism $\omega : \mathcal{P}(E) \to K$ such that $\omega(x') = \omega_1(x')$ for all $x' \in E'$.*

Proof. Let $x^{**} : E^* \to K$ be any linear functional that extends ω_1 from E' to E^*. Then the composite mapping

$$\omega : \mathcal{P}(E) \hookrightarrow \mathcal{P}_a(E) \xrightarrow{\widehat{R}} \mathcal{P}_a(E^{**}) \xrightarrow{\delta_{x^{**}}} K$$

is the required homomorphism. ∎

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DEPARTAMENTO DE ANÁLISIS MATEMÁTICO
UNIVERSIDAD DE VALENCIA
DOCTOR MOLINER 50
46100 BURJASOT (VALENCIA)
SPAIN

INSTITUTO DE MATEMÁTICA
UNIVERSIDADE ESTADUAL DE CAMPINAS
CAIXA POSTAL 6065
13081 CAMPINAS SP
BRAZIL

Received April 9, 1992
Revised version May 12, 1993 (3027)

STUDIA MATHEMATICA 108 (1) (1994)

Spectrum of multidimensional dynamical systems with positive entropy

by

B. KAMIŃSKI (Toruń) and P. LIARDET (Marseille)

Abstract. Applying methods of harmonic analysis we give a simple proof of the multidimensional version of the Rokhlin–Sinaĭ theorem which states that a Kolmogorov $\mathbb{Z}^d$-action on a Lebesgue space has a countable Lebesgue spectrum. At the same time we extend this theorem to $\mathbb{Z}^\infty$-actions. Next, using its relative version, we extend to $\mathbb{Z}^\infty$-actions some other general results connecting spectrum and entropy.

1. Introduction. One of the important results in classical ergodic theory is the Rokhlin–Sinaĭ theorem which states that every Kolmogorov automorphism ($\mathbb{Z}$-action) of a Lebesgue space has a countable Lebesgue spectrum (cf. [RS]). This theorem has been extended to measure-preserving $\mathbb{Z}^d$-actions in [Ka]. The main tool used in the proofs of these theorems are perfect σ-algebras. The proof of their existence is complicated and it seems that it is very difficult to extend it to measure-preserving actions of general groups. It is worth mentioning that it is still an open question, asked by Thouvenot, whether Kolmogorov actions of any countable abelian group have a countable Haar spectrum.

In this paper we give a simple proof of the above mentioned multidimensional version of the Rokhlin–Sinaĭ theorem by a construction of two groups of unitary operators satisfying a commutation relation of the Weyl type. This method allows us also to extend this theorem to the case $d = \infty$.

Our method is similar to that used by Helson in the investigation of invariant subspaces (cf. [H]) and by Mandrekar and Nadkarni (cf. [MN]) to simplify the proof of the generalized F. and M. Riesz theorem concerning the quasi-invariance of analytic measures on compact groups.

The idea of our proof may be used without major changes to prove the following relative version of the result mentioned above. Every ergodic and relatively Kolmogorov $\mathbb{Z}^d$-action T ($1 \le d \le \infty$) on a Lebesgue space

1991 *Mathematics Subject Classification*: Primary 28D15.
Research of the first author supported by KBN grant 211109101.
Research of the second author supported by DRET under contract 901636/A000.