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On the best constant in the Khinchin–Kahane inequality

by

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Abstract. We prove that if r_i is the Rademacher system of functions then

$$\Big(\int \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\|^2 dt\Big)^{1/2} \le \sqrt{2}\int \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\| dt$$

for any sequence of vectors x_i in any normed linear space F.

Introduction. The classical result of Khinchin [3] states that for each $p, q > 0$ there exists a constant $c_{p,q}$ such that for any real numbers $x_1, \ldots, x_n$,

$$(1)\qquad \Big(\int \Big|\sum_{i=1}^{n} x_i r_i(t)\Big|^p dt\Big)^{1/p} \le c_{p,q}\Big(\int \Big|\sum_{i=1}^{n} x_i r_i(t)\Big|^q dt\Big)^{1/q}.$$

The smallest constant $c_{p,q}$ will be denoted by $C_{p,q}^{\mathbb{R}}$. Obviously, $C_{p,q}^{\mathbb{R}} = 1$ for $p \le q$, but it took some effort to calculate the other best constants. The especially interesting case $p = 2, q = 1$ was first solved by S. J. Szarek [4], who proved $C_{2,1}^{\mathbb{R}} = \sqrt{2}$. A simpler proof was given by U. Haagerup [1] who also found $C_{p,2}^{\mathbb{R}}$ and $C_{2,p}^{\mathbb{R}}$ for each $p > 0$. A simple and elementary proof that $C_{2,1}^{\mathbb{R}} = \sqrt{2}$ was also presented by B. Tomaszewski [6].

J.-P. Kahane [2] generalized the result of Khinchin to sequences $x_1, \ldots, x_n$ in a normed linear space F, replacing in (1) the absolute value by the norm in F. Let $C_{p,q}$ denote the smallest constant in the vector-valued inequalities, over all normed linear spaces F. It is of interest to know if the constants are the same in the vector and real cases. As far as we know the best result for $p = 2$ and $q = 1$ known up to now was obtained by B. Tomaszewski [5], who proved that $C_{2,1} \le \sqrt{3}$. In this paper we show that $C_{2,1} = \sqrt{2}$; we think that our proof is simpler than the ones known for real numbers.

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Notation. For $\sigma = (\sigma_1, \dots, \sigma_n) \in \{0,1\}^n$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n), \eta = (\eta_1, \dots, \eta_n) \in \{-1,1\}^n$ and $x_1, \dots, x_n \in F$ let us define

- $|\sigma| = \sum_{i=1}^n \sigma_i$,
- $\alpha^\sigma = \prod_{i=1}^n \alpha_i^{\sigma_i}$ (where $x^0 = 1$ for any $x \in \mathbb{R}$),
- $-\varepsilon = (-\varepsilon_1, \dots, -\varepsilon_n)$,
- $X_\varepsilon = \|\sum_{i=1}^n \varepsilon_i x_i\|$,
- $d(\varepsilon, \eta) = \mathrm{card}\{i : \varepsilon_i \neq \eta_i\}$.

We will denote by $E(\cdot)$ the mean value of $(\cdot)$.

Results. We will prove the following theorem:

THEOREM 1. *Let $S = \sum_{i=1}^n \varepsilon_i x_i$, where ε_i are independent Bernoulli random variables and x_i are vectors of a normed linear space F. Then*

$$(E\|S\|^2)^{1/2} \le \sqrt{2} E\|S\| .$$

The constant $\sqrt{2}$ is the best possible.

Proof. Differentiating in t both sides of the equality

$$t^2 \prod_{i=1}^n (1 + t^{-1}\alpha_i) = \sum_{\sigma \in \{0,1\}^n} t^{2-|\sigma|}\alpha^\sigma$$

and setting $t = 1$ we get

$$2\prod_{i=1}^n (1+\alpha_i) - \sum_{j=1}^n \alpha_j \prod_{i=1, i\neq j}^n (1+\alpha_i) = \sum_{\sigma\in\{0,1\}^n} (2 - |\sigma|)\alpha^\sigma .$$

Hence setting $\alpha_i = \varepsilon_i\eta_i$ and summing over ε and η we obtain

$$\begin{aligned}(2)\quad \sum_{\varepsilon,\eta\in\{-1,1\}^n} &\Big(2\prod_{i=1}^n (1+\varepsilon_i\eta_i) - \sum_{j=1}^n \varepsilon_j\eta_j \prod_{i=1, i\neq j}^n (1+\varepsilon_i\eta_i)\Big) X_\varepsilon X_\eta \\ &= \sum_{\varepsilon,\eta\in\{-1,1\}^n} \sum_{\sigma\in\{0,1\}^n} (2-|\sigma|)\varepsilon^\sigma\eta^\sigma X_\varepsilon X_\eta \\ &= \sum_{\sigma\in\{0,1\}^n} (2-|\sigma|)\Big(\sum_{\varepsilon\in\{-1,1\}^n} \varepsilon^\sigma X_\varepsilon\Big)^2 \le 2\Big(\sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon\Big)^2 .\end{aligned}$$

The last inequality holds because $X_\varepsilon = X_{-\varepsilon}$ for each ε, so that obviously

$$\sum_{\varepsilon\in\{-1,1\}^n} \varepsilon^\sigma X_\varepsilon = 0$$

for each σ with $|\sigma| = 1$.

Since $\prod_i (1+\varepsilon_i\eta_i) \neq 0$ iff $\varepsilon = \eta$, and $\prod_{i=1, i\neq j}^n (1+\varepsilon_i\eta_i) \neq 0$ iff $\varepsilon_i = \eta_i$ for all $i \neq j$, the left-hand side of (2) is equal to

$$2^{n+1} \sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon^2 - n2^{n-1} \sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon^2 + 2^{n-1} \sum_{\varepsilon,\eta\in\{-1,1\}^n, d(\varepsilon,\eta)=1} X_\varepsilon X_\eta$$

we arrive at

$$\begin{aligned}(3)\quad 2^n \sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon^2 + 2^{n-1} \sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon\Big(\sum_{\eta\in\{-1,1\}^n, d(\varepsilon,\eta)=1} X_\eta - (n-2)X_\varepsilon\Big) \\ \le 2\Big(\sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon\Big)^2 .\end{aligned}$$

By the triangle inequality for each fixed ε we get

$$(n-2)X_\varepsilon \le \sum_{\eta\in\{-1,1\}^n, d(\varepsilon,\eta)=1} X_\eta .$$

So inequality (3) yields

$$2^n \sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon^2 \le 2\Big(\sum_{\varepsilon\in\{-1,1\}^n} X_\varepsilon\Big)^2 .$$

Dividing by 2^{2n} we get

$$E\|S\|^2 \le 2(E\|S\|)^2 .$$

To see that the constant $\sqrt{2}$ is the best possible it suffices to take $n = 2$, $x_1 = x_2 \neq 0$.

Remark 1. If we replace in the above proof X_ε by X_ε^p and use the inequality

$$X_\varepsilon \le \frac{1}{n-2} \sum_{\eta\in\{-1,1\}^n, d(\varepsilon,\eta)=1} X_\eta \le \frac{n}{n-2}\Big(\frac{1}{n} \sum_{\eta\in\{-1,1\}^n, d(\varepsilon,\eta)=1} X_\eta^p\Big)^{1/p}$$

we will obtain, for $p \in [1,2)$,

$$(E\|S\|^{2p})^{1/(2p)} \le (1-p/2)^{-1/(2p)} (E\|S\|^p)^{1/p}$$

but we do not think that the above constants are optimal for $p > 1$.

Remark 2. Since for each bounded real random variable X the function $f(r) = r \ln E|X|^{1/r}$ is convex Theorem 1 yields that for all $q \in (0,1]$ and $p \in (0,2]$ with $q \le p$ the following inequality holds:

$$(E\|S\|^p)^{1/p} \le 2^{1/q-1/p} (E\|S\|^q)^{1/q}$$

and the constants $2^{1/q-1/p}$ are optimal.

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