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Maximal estimates for nonsymmetric semigroups

by

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Abstract. Let $X_0, X_1, \ldots, X_k$ be left-invariant vector fields on a Lie group and let $L = \sum_{i=1}^k X_i^2 + X_0$. Then L is the infinitesimal generator of a semigroup $\{p_t\}_{t\ge 0}$ of probability measures on G. Let $P^*f(x) = \sup_{0<t<1} |f * p_t(x)|$. A necessary and sufficient condition for P^* to be of weak type $(1,1)$ is given.

1. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra identified with the left-invariant vector fields on G. Let $X_0, X_1, \ldots, X_k$ belong to $\mathfrak{g}$ and assume that they generate $\mathfrak{g}$ as a Lie algebra. Let

$$L = \sum_{i=1}^{k} X_i^2 + X_0 . \tag{1.1}$$

The operator L is the infinitesimal generator of a semigroup $\{p_t\}_{t\ge 0}$ of probability measures on G. We consider the *local maximal function*

$$P^*f(x) = \sup_{0<t<1} |f * p_t(x)| . \tag{1.2}$$

The aim of this note is to show that

$$P^* \text{ is of weak type } (1,1) \tag{1.3}$$

if and only if X_0 belongs to the linear span of $X_1, \ldots, X_k$ and of their brackets of length two.

The proof is obtained by showing that (1.3) is equivalent to a condition formulated in terms of a system of optimal control metrics defined by Hebisch in [He], which might be of independent interest.

We note that the *global maximal function*

$$P_*f(x) = \sup_{t>1} |f * p_t(x)|$$

is usually unbounded if $X_0 \ne 0$. Indeed, for the operator $\partial^2 + \partial$ on $\mathbb{R}$, we have

$$P_*\delta_0(x) = C|x|^{-1/2} .$$

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If $X_0 = 0$, then L is a symmetric, Markovian, nonpositive operator on $L^2(m)$ and, by a general theorem of E. M. Stein [S], P^* is bounded on $L^p(m)$, $1 < p \le \infty$.

Also it has been proved [Co] that, in this case, (1.3) holds. The basic tool used in [Co] were the Dunford–Schwartz ergodic theorem and the pointwise estimates of the semigroup kernels. This technique seems to be applicable in the proof of the sufficiency part of our theorem; however, we give here a more direct argument based on the classical covering properties.

Let d be the optimal control metric corresponding to $X_1, \ldots, X_k$ and

$$B(r) = \{x \in G : d(x) \le r\}.$$

Then, for m being the left invariant Haar measure (cf. e.g. [V] or [He]),

$$p_t(x) \le Cm(B(t))^{-1/2} \exp(-\varepsilon d(x)^2/t) \quad \text{for all } 0 < t \le 1,$$

and for some constants C and ε. From this (1.3) can be deduced as in the proof of Lemma (2.1) below.

Similarly, if $X_1, \ldots, X_k$ form a linear basis of $\mathfrak{g}$, and X_0 is arbitrary, then (1.3) holds.

However, in the case when $X_1, \ldots, X_k$ only generate $\mathfrak{g}$ as a Lie algebra, E. Damek and A. Hulanicki [DH] have noticed that (1.3) may not hold for an appropriate X_0.

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2. Let $X_0, X_1, \ldots, X_k$ belong to $\mathfrak{g}$ and assume that they generate $\mathfrak{g}$ as a Lie algebra. We define vector fields

$$Y_{i,j} = \mathrm{ad}^j_{X_0}(X_i), \quad i \in \{1, \ldots, k\},\ j \in \{0, 1, \ldots, \dim(G)\}.$$

Then the Lie algebra $\mathfrak{h}$ generated by the $Y_{i,j}$'s is either equal to $\mathfrak{g}$ or is of codimension 1 in $\mathfrak{g}$ depending on whether $X_0 \in \mathfrak{h}$ or $X_0 \notin \mathfrak{h}$.

Let $H = \exp \mathfrak{h}$. An absolutely continuous curve $\gamma : (0,1) \to H$ is called *admissible* if

$$\dot\gamma(s) = \sum_{i=1}^{k} \sum_{j=0}^{\dim(G)} a_{i,j}(s) Y_{i,j}(\gamma(s))$$

where $a_{i,j} \in L^2(0,1)$.

Following W. Hebisch [He] we define the length of γ and the distance on H:

$$|\gamma|_t^2 = K \sum_{i=1}^{k} \sum_{j=0}^{\dim(G)} \int_0^1 |a_{i,j}(s)|^2\, ds\, t^{-1-2j},$$

$$d_t(x) = \inf\{|\gamma|_t : \gamma \text{ admissible}, \gamma(0) = e,\ \gamma(1) = x\}$$

where the constant K will be specified later.

We extend the metric d_t to the whole of G putting $d_t(x) = \infty$ for $x \in G \setminus H$. The basic properties of the metrics d_t are collected in the two lemmas below.

(2.1) Lemma. *Let $d = 2\dim(G) + 1$. For $s \ge 1$, $t \in [0,1]$, we have*

$$s^{-d} d_t(x)^2 \le d_{st}(x)^2 \le s^{-1} d_t(x)^2.$$

Proof. It suffices to prove (2.1) with d_t replaced by $|\gamma|_t$. Let $d = 2\dim(G) + 1$ and $s \ge 1$. From the definition of $|\gamma|_t$ we obtain

$$|\gamma|_{st}^2 = \sum_{i=1}^{k} \sum_{j=0}^{\dim(G)} \int_0^1 a_{i,j}(u)^2\, du\, (st)^{-1-2j}$$
$$\ge \sum_{i=1}^{k} \sum_{j=0}^{\dim(G)} \int_0^1 a_{i,j}(u)^2\, du\, s^{-d} t^{-1-2j} = |\gamma|_t^2 s^{-d}.$$

The second inequality can be proved in the same way. ■

Let $B_t(R) = \{x : d_t(x) < R\}$.

(2.2) Corollary. $B_t(k) \subseteq B_{k^2 t}(1)$ *for* $k = 1, 2, 3, \ldots$

Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$. Define a family of linear transformations of $\mathfrak{g}$, called *dilations*, by

$$\delta_t(e_i) = t^{\alpha_i} e_i.$$

We say that a vector $X \in \mathfrak{g}$ is *homogeneous* of degree α if

$$\delta_t(X) = t^\alpha X.$$

A norm $\|\cdot\|$ on $\mathfrak{g}$ is called *homogeneous* if

$$\|\delta_t(X)\| = t\|X\|.$$

Denote by m the left invariant Haar measure on G.

We identify a neighbourhood of e in $\mathfrak{g}$ with a neighbourhood of 0 in G via the exponential map. The following facts have been proved in [He].

(2.3) Lemma. *There exists a homogeneous norm $\|\cdot\|$ on H and a family of dilations $\{\delta_t\}_{t>0}$ such that*

(i) $B_t(R) \supseteq \{x : \|x\| < Ct^{1/2}\}$ *for* $1 > t > 0$,

(ii) $\{x : \|x\| < C_1 t^{1/2}\} \supseteq B_t(R)$ *for* $1 > t > 0$,

where C, C_1 depend on R,

(iii) *X_0 is the sum of homogeneous vectors of degree* 1 *or* 2 (*in the sense defined above*) *if and only if*

$$X_0 = \sum_{i=1}^{k} a_i X_i + \sum_{i,j=1}^{k} b_{i,j}[X_i, X_j],$$

(iv) *there exist positive constants q, C, c such that*

$$ct^{q/2} < m(B_t(1)) < Ct^{q/2} \quad \text{for } 0 < t < 1 .$$

The equivalence of (i) and (ii) has been proved in [He, Lemmas (4.1) and (4.10)].

The property (iii) follows easily from the construction of the norm $\| \|$ and the construction of gradations defined in the proof of [He, Lemma (4.1)].

Our basic tool is the following semigroup kernel estimate proved in [He].

(2.4) THEOREM. *Let p_t be the kernel of the semigroup generated by L, and let*

$$q_t(x) = p_t * \delta_{\exp(-tX_0)}(x) .$$

Then there exist C and K (see definition of d_t) such that for q defined in (2.3)(iv) *and all $t < 1$ we have*

$$q_t(x) \le Ct^{-q/2} \exp(-d_t(x)^2) .$$

3. First let $G = H$. We will now control our maximal operator by a local maximal operator of Hardy–Littlewood type

$$M^* f(x) = \sup_{0<t<1} m(B_t(1))^{-1} \int_{B_t(1)} |f(xy^{-1})| \, dy \tag{3.1}$$

where dy is the right-invariant Haar measure on G.

(3.2) LEMMA. *Assume that*

$$\sup_{0<t<1} d_t(\exp(tX_0)) < \infty . \tag{3.3}$$

Then there is an operator S bounded on $L^1(m)$ and a constant C such that

$$P^* f(x) \le CM^* f(x) + Sf(x)$$

for $x \in G$.

Proof. Denote by I_A the indicator function of the set A. From Theorem (2.4) and Corollary (2.2) we obtain

$$\begin{aligned} e_t(x) &= Ct^{-q/2} \exp(-d_t(x)^2) \le Ct^{-q/2} \sum_{k=0}^{\infty} \exp(-k^2) I_{B_t(k+1)}(x) \\ &\le Ct^{-q/2} \sum_{k^2 \le t^{-1}} \exp(-k^2) I_{B_{k^2t}(1)}(x) \\ &\quad + Ct^{-q/2} \exp(-(2t)^{-1}) \sum_{k^2 > t^{-1}} \exp(-k^2/2) I_{B_1(k)}(x) \\ &\le CC_1 \sum_{k^2 t \le 1} \exp(-k^2) k^q m(B_{k^2t}(1))^{-1} I_{B_{k^2t}(1)}(x) \\ &\quad + \sup_{0<t<1} t^{-q/2} \exp(-(2t)^{-1}) \sum_{k=0}^{\infty} \exp(-k^2/2) I_{B_1(k)}(x) . \end{aligned}$$

Hence, for $f \ge 0$,

$$\begin{aligned} f * e_t(x) &\le CC_1 \sum_{k^2 t \le 1} k^q \exp(-k^2) m(B_{k^2t}(1))^{-1} f * I_{B_{k^2t}(1)}(x) \\ &\quad + C_2 \sum_{k=0}^{\infty} \exp(-k^2/2) f * I_{B_1(k)}(x) \end{aligned}$$

and so

$$\begin{aligned} \sup_{0<t<1} |f * e_t(x)| &\le CC_1 \sum_{k=1}^{\infty} k^q \exp(-k^2) M^* f(x) \\ &\quad + C_2 \sum_{k=1}^{\infty} \exp(-k^2/2) |f| * I_{B_1(k)}(x) . \end{aligned}$$

Let

$$Sf = |f| * \sum_{k=1}^{\infty} \exp(-k^2/2) I_{B_1(k)} .$$

Since $m(B_1(k)) = m(B(1))^k \le e^{Ck}$, S is bounded on $L^1(m)$.

From (3.3) and (2.2) we obtain

$$\begin{aligned} e_t(x \exp(tX_0)) &\le t^{-q/2} \exp\{-d_t(x)^2/2 + 8d_t(\exp(tX_0))^2\} \\ &\le C_3 t^{-q/2} \exp(-d_t(x)^2/2) = C_4 e_{ct}(x) , \end{aligned}$$

which completes the proof of the lemma. ■

(3.4) LEMMA. *There exist R, t_0 such that for $t < t_0$ we have*

$$\int_{B_t(R)} q_t(y) \, dy \ge 1/2 .$$

Proof. As in the proof of Lemma (3.2), for $x \notin B_t(R)$ and $t < t_0$ we have

$$e_t(x) \le C \sum_{R^2<k^2<t^{-1}} \exp(-k^2)k^q m(B_{k^2t}(1))^{-1} I_{B_{k^2t}(1)}(x)$$
$$+ C \sup_{0<s<t} (s^{-q/2}\exp(-(2s)^{-1})) \sum_{k=0}^{\infty} \exp(-k^2/2) I_{B_1(k)}(x).$$

Hence

$$(3.5)\qquad \int_{B_t(R)^c} q_t(x)\,dx \le \int_{B_t(R)^c} e_t(x)\,dx$$
$$\le C\sum_{k=R}^{\infty} k^q \exp(-k^2) + C \sup_{0<s<t} (s^{-q/2}\exp((2s)^{-1})).$$

On the other hand,

$$\int q_t(x)\,dx \to 1 \quad \text{as } t \to 0^+,$$

which, in view of (3.5), implies the lemma. ■

(3.6) Corollary. *Let $f_T = I_{B_T(2R)}$. Then $f_T * q_t(x) \ge 1/2$ for $x \in B_T(R)$ and $t < T$.*

Proof. We have

$$f_T * q_t(x) = \int_{B_T(2R)x} q_t(y)\,dy \ge \int_{B_t(R)} q_t(y)\,dy \ge 1/2,$$

whence, since for $x \in B_t(R)$ and $t < T$ we have $B_T(2R)x \supseteq B_T(R) \supseteq B_t(R)$, the corollary follows. ■

As a consequence we obtain

(3.7) Corollary. *For $x \in B_t(R)\exp(-tX_0)$ and $t \le T$,*

$$f_T * q_t(x\exp(tX_0)) > 1/2.$$

(3.8) Lemma. *Assume that $\sup_{0<t<1} d_t(\exp(tX_0)) = \infty$. Then for fixed N there exists $T_N \le 1$ such that*

$$d_{T_N}(\exp(lN^{-1}T_NX_0)) > 4R \quad \textit{for } l \in \{1,\dots,N\}.$$

Proof. Let $t_n \searrow 0$ be such that $d_{t_n}(\exp(t_nX_0)) \to \infty$ as $n \to \infty$, $c_n = ((N-1)!)^{-1}t_n$ and $1 \le l \le N$. Then

$$d_{t_n}(\exp(t_nX_0)) = d_{(N-1)!c_n}(\exp((N-1)!c_nX_0))$$
$$= d_{(N-1)!c_n}((\exp(lN^{-1}c_nX_0))^{N!/l})$$
$$\le N!l^{-1} d_{(N-1)!c_n}(\exp(lN^{-1}c_nX_0))$$

$$\le \frac{N!}{((N-1)!)^{1/2}l} d_{c_n}(\exp(lN^{-1}c_nX_0)).$$

Hence

$$d_{c_n}(\exp(lN^{-1}c_nX_0)) \to \infty \quad \text{as } n \to \infty. \ ■$$

(3.9) Lemma. *Let $\tau_j = j(2N)^{-1}T_{2N}$, where $N \le j \le 2N$. Under the hypothesis of Lemma (3.8), the balls*

$$B_j = B_{\tau_j}(R)\exp(-\tau_jX_0), \qquad N \le j \le 2N,$$

are disjoint.

Proof. Suppose

$$y\exp(-\tau_iX_0) = x\exp(-\tau_jX_0), \qquad y \in B_{\tau_i}(R),\ x \in B_{\tau_j}(R),\ i > j.$$

Then $x^{-1}y = \exp(l(2N)^{-1}T_{2N}X_0)$, $l = i-j \in \{1,\dots,N\}$ and, by Lemma (3.8),

$$d_{T_{2N}}(x^{-1}y) = d_{T_{2N}}(\exp(l(2N)^{-1}T_{2N}X_0)) > 4R.$$

On the other hand,

$$d_{T_{2N}}(x^{-1}y) \le d_{T_{2N}}(x^{-1}) + d_{T_{2N}}(y) \le d_{\tau_j}(x) + d\tau_i(y) \le 2R$$

and the lemma follows. ■

Now we are ready to prove our main result.

(3.10) Theorem. *The following conditions are equivalent:*

(i) *P^* is of weak type (p,p) for a p, $1 \le p < \infty$.*
(ii) $\sup_{0<t<1} d_t(\exp(tX_0)) < \infty$.
(iii) $X_0 = \sum_{i=1}^k a_iX_i + \sum_{i,j=1}^k b_{i,j}[X_i,X_j]$.

Proof. The equivalence of (ii) and (iii) follows easily from Lemma (2.3) and from the definition of d_t.

To prove (i)⇒(ii) we fix a sufficiently large N. From Corollary (3.7) and Lemma (3.8) we infer that

$$m(\{x \in G : P^*f_{T_{2N}}(x) \ge 1/2\}) \ge m\Big(\bigcup_{j=N}^{2N} B_j\Big) = \sum_{j=N}^{2N} m(B_{\tau_j})$$
$$\ge Nm(B_{2^{-1}T_{2N}}(R)) \ge Nm(B_{2^{-1}T_{2N}}(1)).$$

But $\|f_{T_{2N}}\|^p_{L^p(H)} = m(B_{T_{2N}}(2R)) \le Cm(B_{2^{-1}T_{2N}}(1))$, whence

$$(3.11)\qquad m(\{x \in G : P^*f_{T_{2N}}(x) \ge 1/2\}) \ge N\|f_{T_{2N}}\|^p_{L^p(H)},$$

which completes the proof that P^* on $L^p(H)$ is not of weak type (p,p) in the case when $\mathfrak{g} = \mathfrak{h}$.

In the case when $\mathfrak{g} \ne \mathfrak{h}$ we represent elements x from a small neighbourhood of e in G in the unique form $x = h\exp(-tX_0)$ where $h \in H$. Let f be a Borel function on G such that $f \in L^p(G)$ for some $p < \infty$ and f equals $+\infty$ on H. Then, from Theorem (2.4), we have

$$T_t f(x) = \int f(x \exp(tX_0)y^{-1})\, d\mu_t(y) = \infty$$

where μ_t is a measure supported on H with the density q_t. This completes the proof of (i)$\Rightarrow$(ii).

The proof of (ii)$\Rightarrow$(i) is classical and well known [CW]. We only briefly sketch two main steps.

Let $0 \le f \in C_c^\infty(G)$ and $K = \{x : M^* f(x) > \lambda\}$. By the definition of K, for every $x \in K$, there exists $0 < t_x < 1$ such that

$$\int\limits_{B_{t_x}(1)} f(xy^{-1})\, dy > \lambda m(B_{t_x}(1))\,. \tag{3.12}$$

Denote by Δ the modular function $d_l y/dy$ (where $d_l y$ is the left invariant Haar measure on G). From (3.12) one easily derives

$$\int\limits_{xB_{t_x}(1)^{-1}} f(y)\Delta(x^{-1}y)\, dy > \lambda m(xB_{t_x}(1))\,. \tag{3.13}$$

Hence

$$C \int\limits_{xB_{t_x}(1)} f(y)\, dy > \lambda m(xB_{t_x}(1)) \tag{3.14}$$

because the balls $B_t(1)$ are symmetric and $|\Delta(x^{-1}y)| < C$ for $y \in xB_{t_x}(1)$.

The required weak type (1,1) estimate follows, by a standard argument, from (3.14) and the following covering property:

Let $K = \bigcup_{i=1}^n x_i B_{t_i}(1)$ where $t_i < 1$. Then there exists a constant C and numbers $i_1, \ldots, i_s$ such that

(i) the sets $x_{i_k} B_{t_{i_k}}(1)$ are pairwise disjoint,

(ii) $m(\bigcup_{k=1}^s x_{i_k} B_{t_{i_k}}(1)) \ge C m(K)$.

We omit a standard proof of this property.

Remark. Since the crucial estimate also holds for operators of the form

$$L = \sum (-1)^{n_j} X_j^{2n_j} + X_0 \tag{3.15}$$

minor modifications of our argument prove the equivalence of (i) and (ii) of Theorem (3.10) for operators (3.15). Also an analogue of (iii) can be formulated.

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