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Some spectral inequalities involving generalized scalar operators

by

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Abstract. In 1971, Allan Sinclair proved that for a hermitian element h of a Banach algebra and λ complex we have $\|\lambda + h\| = r(\lambda + h)$, where r denotes the spectral radius. Using Levin's subordination theory for entire functions of exponential type, we extend this result locally to a much larger class of generalized spectral operators. This fundamental result improves many earlier results due to Gelfand, Hille, Colojoară–Foiaş, Vidav, Dowson, Dowson–Gillespie–Spain, Crabb–Spain, I. & V. Istrăţescu, Barnes, Pytlik, Boyadzhiev and others.

1. Introduction. The study of invertible operators with polynomial growth, that is, $T \in \mathcal{B}(X)$ satisfying, for some $k \geq 0$,

$$\|T^n\| = O(|n|^k) \quad \text{as } |n| \text{ goes to infinity,} \tag{1}$$

began with the work of E. R. Lorch in 1939, in the case of reflexive spaces and with the work of B. Sz.-Nagy in 1947, in the case of Hilbert space and $k = 0$ (see [32], §144 and Appendix). Later on F. Wolf [40] made a systematic study of these operators for k arbitrary, the case $k = 1$ being investigated by G. K. Leaf [29].

Extending the work of N. Dunford on spectral operators, C. Foiaş introduced in 1960 the wider class of generalized spectral operators. These are operators in $\mathcal{B}(X)$ having a spectral distribution instead of a spectral measure. This class can be built up with real generalized spectral operators, that is, operators $H \in \mathcal{B}(X)$ such that e^{iH} satisfies (1), which is equivalent to saying that

$$\|e^{itH}\| = O(|t|^k) \quad \text{as } t \text{ real goes to infinity.} \tag{2}$$

This class was intensively investigated by B. G. Tillman [36], D. R. Smart [35], S. Kantorovitz [28] and mainly Colojoară–Foiaş [15], also recently by B. A. Barnes [5].

The pioneering work of O. Tœplitz and F. Hausdorff around 1918 on the numerical range of matrices, extended by M. H. Stone in 1932 to operators

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on Hilbert space, lead I. Vidav [37], F. L. Bauer and G. Lumer around 1960 to extend this notion to Banach spaces and to define the notions of an abstract hermitian operator and abstract normal operator on a Banach space (for a complete history of this and references see [9]). These hermitian operators are characterized by the condition

$$\|e^{itH}\| = 1 \quad \text{when } t \in \mathbb{R}, \tag{3}$$

so they are real generalized spectral operators. For these hermitian operators I. Vidav proved that $\|H\| = \nu(H)$, where ν denotes the radius of the numerical range and this is equivalent to saying that $\|e^H\| = r(e^H)$ (see [9], p. 54). This implies in particular that $\|H\|/e \le r(H)$. All this theory was greatly improved by the fundamental result of Allan Sinclair [34] saying that $\|\lambda + H\| = r(\lambda + H)$ for H hermitian and λ complex. The original proof used a version of the Phragmén–Lindelöf theorem established by Duffin and Schaeffer. For $\lambda = 0$, more elementary proofs have been given by A. Browder [14], V. È. Katsnel'son [27] and F. F. Bonsall [10]. For more details see [9, 10].

It is very surprising to see these two lonely fields live in a parallel manner with practically no interconnections. The aim of this paper is to show that the two fields are in fact intimately related.

F. F. Bonsall and J. Duncan [10] say on page 73 that B. Bollobás, in a lecture, established a proof of Allan Sinclair's result using the subordination theory of Levin. This is really the spark which gave us the idea to extend Sinclair's theorem both locally and to the class of generalized spectral operators. We had in mind the possible generalization of many results obtained previously in the first field.

After having proved Theorem 2.5 we discovered many applications given in §3. We believe that there are still many others to be found.

When we submitted this work for publication, J. Zemánek mentioned to us that similar results were obtained by K. N. Boyadzhiev in 1987 [12, 13] also using Levin's subordination theory. Our work was done independently of Boyadzhiev's papers and was motivated by the ideas of B. Bollobás. The results contained in [12, 13] are much less general than ours. For instance in [12], our Corollary 2.6 is proved but only for $\lambda = 0$ and the main result of [13] is our Theorem 2.5 for $\lambda = 0$. The case $\lambda = 0$ is much easier to prove and has fewer consequences.

2. The fundamental inequalities. For the standard definitions and tools needed in operator theory see [3].

Let $T \in \mathcal{B}(X)$ and $x \in X$. We define Ω_x to be the set of $\alpha \in \mathbb{C}$ for which there exists a neighbourhood V_α of α and u analytic on V_α with values in X such that $(\lambda - T)u(\lambda) = x$ on V_α. This set is open and contains the

complement of the spectrum of T. By definition the *local spectrum of T at x*, denoted by $\mathrm{Sp}_x(T)$, is the complement of Ω_x.

In general this closed set may be empty (take the left shift operator on l^2 with $e_1 = (1, 0, \ldots)$). But for $x \neq 0$, the local spectrum of T is nonempty if T has the *uniqueness property for the local resolvent*, that is, $(\lambda - T)v(\lambda) = 0$ implies $v = 0$ for any analytic function v defined on any domain D of $\mathbb{C}$ with values in X. It is easy to see that an operator T having spectrum without interior points has this property. In particular, the operators of the class $\mathcal{S}$ defined below have this property because their spectra are real by Lemma 2.2. For operators with this property there is a unique local resolvent which is the analytic extension of $(\lambda - T)^{-1}x$ to Ω_x. In this case the *local spectral radius* $r_x(T)$ is equal to $\limsup_{k\to\infty} \|T^k x\|^{1/k}$. In general this last property is false, we only have $r_x(T) \le \limsup_{k\to\infty} \|T^k x\|^{1/k}$ (see [4]).

If T is an operator of the class $\mathcal{N}$ defined below, that is, $T = H + iK$ where $H, K \in \mathcal{S}$ and H, K commute, then T also has the uniqueness property for the local resolvent. This result, much more difficult to prove, is due to C. Foiaş [21]. Using completely different methods it was strongly generalized in [4].

The holomorphic functional calculus for local spectrum was first discovered by C. Apostol [2] and also obtained by R. G. Bartle and C. A. Kariotis [6] and P. Vrbová [38] in 1973.

LEMMA 2.1 (Holomorphic functional calculus for local spectrum). *Let $T \in \mathcal{B}(X)$, $x \neq 0$, and let f be holomorphic on a neighbourhood D of $\mathrm{Sp}(T)$. Then $f(\mathrm{Sp}_x(T)) \subseteq \mathrm{Sp}_x(f(T))$. If f is injective on D then $f(\mathrm{Sp}_x(T)) = \mathrm{Sp}_x(f(T))$. Moreover, if T has the uniqueness property for the local resolvent then equality holds for any f holomorphic on D.*

Proof. The proof of the inclusion is exactly the first part of the proof of Theorem 1.6, page 6 in [20], where the single-valued extension property is not used at all. The injective case is obtained by applying this inclusion to the holomorphic function f^{-1} and to $f(T)$. The last part is exactly the last part of the proof of Theorem 1.6, page 7 in [20]. ∎

Extending the work of N. Dunford on spectral operators, C. Foiaş in 1960 introduced a wider class of generalized spectral operators which we denote by $\mathcal{N}$ in this paper. Originally this class was defined with the help of spectral distributions instead of spectral measures. Instead of the original definition we use the definition involving the growth of semigroups which is more tractable.

First we define the class $\mathcal{S}$ of real generalized spectral operators, that is, the set of $T \in \mathcal{B}(X)$ which satisfy $\|e^{itT}\| = O(|t|^\gamma)$ for some $\gamma \ge 0$ and for all real t near infinity. This is equivalent to saying that there exist $C \ge 1$

and $\gamma \geq 0$ such that

$$\|e^{itT}\| \leq C|1+it|^{\gamma}$$

for every $t \in \mathbb{R}$, an inequality which we shall use later in the proof of Theorem 2.5. By definition, the *constant* and the *degree* of T are the smallest C and the smallest integer $\gamma \geq 0$ such that the previous inequality is true for $t \in \mathbb{R}$.

It is easy to verify that if $T \in \mathcal{S}$ then its adjoint T' defined on the topological dual space X' of X is in the corresponding set $\mathcal{S}$. If H is hermitian on $\mathcal{B}(X)$ then it is in $\mathcal{S}$ with constant $C = 1$ and degree zero. In that case the spectrum of H is real. The set of hermitian operators is stable by addition, but unfortunately H^2 is not hermitian in general. This was proved by G. Lumer in 1961, but M. J. Crabb gave a simple example in $\mathbb{C}^3$ with a convenient norm (see [9], pp. 57–58). On the contrary, the class $\mathcal{S}$ has nice properties, in particular we have

LEMMA 2.2. (i) *If H is in $\mathcal{S}$ then the spectrum of H is real.*

(ii) *If H is in $\mathcal{S}$ with degree m then $H^2 \in \mathcal{S}$ with degree less than or equal to $m+1$.*

(iii) *If $H_1, H_2 \in \mathcal{S}$ and $H_1H_2 = H_2H_1$ then $H_1+H_2 \in \mathcal{S}$ and $H_1H_2 \in \mathcal{S}$. If m_1, m_2 denote the degrees of H_1, H_2 then* $\deg(H_1+H_2) \leq m_1+m_2$ *and* $\deg(H_1H_2) \leq 2(m_1+m_2+1)$.

Proof. See [5] for details. In that paper the degrees are not explicitly determined but the estimates follow easily from the calculations. ∎

This implies, in particular, that if H is hermitian on X, then H^2 is dissipative and $H^2 \in \mathcal{S}$ with degree less than or equal to 1.

We now define the class $\mathcal{N}$ of generalized spectral operators on a Banach space X to be the set of $N \in \mathcal{B}(X)$ which can be written as $N = H + iK$ where $H, K \in \mathcal{S}$ and $HK = KH$. This definition is equivalent to the Colojoară–Foiaş definition (see [10], Theorem 4.5, p. 160) using spectral distributions, but much easier to handle. The degree of $N \in \mathcal{N}$ is by definition the smallest $m+n$ where $m = \deg(H)$ and $n = \deg(K)$ for all possible decompositions of $\mathcal{N}$. Obviously this class contains all normal operators on X and even polynomials of normal operators.

We now give some terminology and some fundamental results on entire functions of exponential growth. For more details see [7] and [30].

Let f be an entire function on the complex plane and let $M_f(r) = \max_{|z|=r}|f(z)|$. We say that f is of *finite order* if there exists $k \geq 0$ such that

$$M_f(r) \leq e^{r^k} \quad \text{for } r \text{ large.} \tag{1}$$

The infimum of all k satisfying this inequality is called the *order* of f and is denoted by $\tau(f)$.

It is easy to verify that

$$\tau(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}. \tag{2}$$

Suppose now that f is an entire function of finite order $\tau(f)$. We define the *type* of f, denoted by $\sigma(f)$, to be the infimum of all nonnegative numbers A such that

$$M_f(r) \leq e^{Ar^{\tau(f)}}. \tag{3}$$

Then we have

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\tau(f)}}. \tag{4}$$

In the proof of Theorem 2.5 below we only consider entire functions of order at most one. In this case we have by [30], p. 84,

$$\sigma(f) = \limsup_{k\to\infty} |f^{(k)}(0)|^{1/k}. \tag{5}$$

The next two results are fundamental tools needed in the proof of Theorem 2.5.

LEMMA 2.3. *Let f and g be entire functions.*

(i) *If $\tau(f) > \tau(g)$ then $\tau(fg) = \tau(f)$ and $\sigma(fg) = \sigma(f)$.*

(ii) *If $\tau(f) = \tau(g)$, $0 < \sigma(f) < \infty$ and $\sigma(g) = 0$ then $\tau(fg) = \tau(f)$ and $\sigma(fg) = \sigma(f)$.*

(iii) *If $\tau(f) = \tau(g)$, $\sigma(f) = \infty$ and $0 < \sigma(g) < \infty$ then $\tau(fg) = \tau(f)$ and $\sigma(fg) = \infty$.*

Proof. See [30], Theorem 1.12, pp. 22–24. ∎

LEMMA 2.4 (B. Ya. Levin). *Let g be an entire function of finite type such that*

(i) *$g(z) \neq 0$ for $\mathrm{Im}(z) < 0$,*

(ii) *$h(\alpha) \leq h(-\alpha)$ for some α such that $0 < \alpha < \pi$, where*

$$h(\alpha) = \limsup_{r\to\infty} \frac{\log|g(re^{i\alpha})|}{r}.$$

Suppose moreover that f is an entire function such that $\sigma(f) \leq \sigma(g)$ and $|f(t)| \leq |g(t)|$ for t real. Then $|f^{(k)}(t)| \leq |g^{(k)}(t)|$ for all integers $k \geq 0$ and all real t.

Proof. See [30], Chapter 9, Theorem 11, p. 363. ∎

We are now ready to prove the fundamental inequalities improving Allan Sinclair's theorem.

THEOREM 2.5. *Let $H \in \mathcal{S}$ with constant C and degree n, $x \in X$ and let $k \geq n+1$. Then for $\lambda \in \mathbb{C}$ we have*

$$\|(\lambda + H)^k x\| \leq C\|x\| p_k(\tau)\,,$$

where $\tau = r_x(\lambda + H)$ and

$$p_k(\tau) = \tau^k + kn\tau^{k-1} + \frac{k(k-1)n(n-1)}{2!}\tau^{k-2} + \ldots + \binom{k}{n}\tau^{k-n}\,.$$

Proof. These inequalities are obviously true for $x = 0$ so suppose $x \neq 0$, and let u be a bounded linear functional on X of norm one. If λ is real, then $\lambda + H$ is in $\mathcal{S}$ with the same constant and degree so we may suppose without restriction that $\lambda = i\alpha$ with $\alpha > 0$.

Let $N = i\alpha + H$ and set

$$f(z) = u(e^{izN}x), \qquad g(z) = C\|x\|(1+iz)^n e^{-z\beta}\,,$$

where β is some complex number. These two functions are entire. Because $|f(z)| \leq \|x\| e^{|z|\|N\|}$, the order of f is at most 1. In this case the type of f is $\limsup_{n\to\infty} |f^{(n)}(0)|^{1/n}$; but $|f^{(n)}(0)| \leq \|N^n x\|$ so this type does not exceed $\limsup_{n\to\infty} \|N^n x\|^{1/n}$ which is $\tau = r_x(N)$, because, by Lemma 2.2(i), the spectrum of N has empty interior, so N has the uniqueness property for the local resolvent. We apply Lemma 2.1 to N to get $\mathrm{Sp}_x(i\alpha + H) = i\alpha + \mathrm{Sp}_x(H)$, with $\mathrm{Sp}_x(H)$ real, so that $r_x(\alpha) = |\alpha| \leq r_x(N)$.

We now choose $\gamma \geq 0$ such that $\beta = \alpha - i\gamma$ satisfies $|\beta| = r_x(N)$. With this γ the function g is the product of $C\|x\|(1+iz)^n$, which is of order 0, and $e^{-z\beta}$, which is of order 1 and type $r_x(N)$. Thus by Lemma 2.3, g is of order 1 and type $r_x(N)$. Consequently, the type of f is less than or equal to the type of g. It is obvious that $g(z) \neq 0$ for $\mathrm{Im}(z) < 0$. Taking $\alpha = \pi/2$, we have

$$-\gamma = \lim_{r\to\infty} \frac{\log|g(re^{i\pi/2})|}{r} \leq \lim_{r\to\infty} \frac{\log|g(re^{-i\pi/2})|}{r} = \gamma\,.$$

Moreover, on the real line we have the inequalities

$$\begin{aligned} |f(t)| = |u(e^{itN}x)| &= e^{-\alpha t}|u(e^{itH}x)| \leq e^{-\alpha t}\|e^{itH}x\| \\ &\leq C\|x\|e^{-\alpha t}|1+it|^n = |g(t)|\,. \end{aligned}$$

Now applying Levin's theorem (Lemma 2.4) we conclude that $|f^{(k)}(t)| \leq |g^{(k)}(t)|$ for every real t and $k \geq 1$. Hence, in particular,

$$|f^{(k)}(0)| = |u(N^k x)| \leq |g^{(k)}(0)|\,.$$

But by Leibniz's formula, for $k \geq n+1$ we have

$$g^{(k)}(0) = C\|x\| \sum_{r=0}^{n} r! \binom{k}{r}\binom{n}{r} i^r (-\beta)^{k-r}\,,$$

because the derivatives of $(1+iz)^n$ of order $k \geq n+1$ are zero. For given $k \geq n+1$, there exists a bounded linear functional u such that $|u(N^k x)| = \|N^k x\|$, so we get the result. ∎

COROLLARY 2.6. *Let $H \in \mathcal{S}$ with constant C and degree 0, $x \in X$ and $k \geq 0$. Then for $\lambda \in \mathbb{C}$ we have*

$$\|(\lambda + H)^k x\| \leq C\|x\| r_x(\lambda + H)^k\,,$$

so in particular $\|(\lambda + H)^k\| \leq C r(\lambda + H)^k$.

For $C = 1$ this corollary gives immediately A. Sinclair's theorem [34]. For $\lambda = 0$ it gives Theorem 2 of [12].

COROLLARY 2.7. *Let $H \in \mathcal{S}$ with constant C and degree 0 and let $\lambda \neq \mu$ be two distinct complex numbers. Suppose that $x \in \mathrm{Ker}(\lambda - H)$ and $y \in \mathrm{Ker}(\mu - H)$. Then*

$$\|x\| \leq C\|x + y\|\,.$$

Proof. We apply Corollary 2.6 to $\mu - H$ and $x + y$ with $k = 1$ to get

$$|\lambda - \mu|\|x\| = \|(\mu - H)x\| \leq C\|x + y\| r_{x+y}(\mu - H)\,.$$

But $(\mu - H)^n(x + y) = (\mu - H)^n x = (\mu - \lambda)^n x$ so $r_{x+y}(\mu - H) = |\lambda - \mu|$, which gives the result. ∎

This corollary says nothing if $\lambda, \mu \notin \mathrm{Sp}(H) \subset \mathbb{R}$, but if λ, μ are eigenvalues of H, then the corresponding kernels are orthogonal for the abstract orthogonality defined by the relation $\|x\| \leq C\|x + y\|$. This result is certainly related to the Sinclair–Crabb theorem which says that $\mathrm{Ker}(H - \lambda)$ is orthogonal to $R(H - \lambda)$ if H is hermitian on X (see [10], pp. 24–34); this has been generalized to normal operators, using a more elementary argument which avoids Kakutani's fixed point theorem, by C. K. Fong [22]. Unfortunately, we have not been able to prove this last theorem using the fundamental inequalities and consequently to extend it to the class $\mathcal{S}$.

THEOREM 2.8. *Let $N \in \mathcal{N}$, that is, $N = H + iK$ with $HK = KH$ and $H, K \in \mathcal{S}$. Then $r_x(e^N) = r_x(e^H)$ for all x in X. Moreover, $r_x(H) \leq r_x(N)$ and $r_x(K) \leq r_x(N)$.*

Proof. Suppose without restriction that $\|x\| = 1$. Because H and K commute we have $e^{tH} = e^{tN}e^{-itK}$ for t real. Denoting by C the constant of K and by n its degree we get

$$\frac{1}{(C(1+t^2)^{n/2})^{1/t}}\|e^{tN}x\|^{1/t} \leq \|e^{tH}x\|^{1/t} \leq (C(1+t^2)^{n/2})^{1/t}\|e^{tN}x\|^{1/t}\,.$$

Because the spectrum of e^H is real this operator has the uniqueness property for the local resolvent, so $r_x(e^H) = \limsup_{t\to\infty} \|e^{tH}x\|^{1/t}$. By Foiaş's result mentioned before, N also has the uniqueness property for the local resolvent

so $r_x(e^N) = \limsup_{t\to\infty} \|e^{tN}x\|^{1/t}$. But the above inequalities imply that the two upper limits are equal, so $r_x(e^N) = r_x(e^H)$.

We know that $\mathrm{Sp}(H)$ is real. Replacing H by $-H$ if necessary we may suppose that $\max\{\lambda : \lambda \in \mathrm{Sp}_x(H)\} = r_x(H)$. By Lemma 2.1, $\mathrm{Sp}_x(e^H) = \{e^t : t \in \mathrm{Sp}_x(H)\}$ so $r_x(e^H) = e^{r_x(H)}$. Applying again Lemma 2.1 to N we obtain $r_x(e^N) \le e^{r_x(N)}$. Consequently,

$$e^{r_x(H)} = r_x(e^H) = r_x(e^N) \le e^{r_x(N)},$$

hence $r_x(H) \le r_x(N)$. Replacing N by iN which satisfies the hypothesis of the theorem we get the last inequality $r_x(K) \le r_x(iN) = r_x(N)$. ∎

Lemma 3.5, pp. 106–107 of [15], says that if $N \in \mathcal{N}$ and is quasi-nilpotent then it is nilpotent. This result can be extended locally.

COROLLARY 2.9. *Let $N \in \mathcal{N}$ of degree r and let $x \in X$. Then $r_x(N) = 0$ implies $N^{r+1}x = 0$.*

Proof. By Theorem 2.8 we have $r_x(H) = r_x(K) = 0$. If m, n denote the respective degrees of H and K then by Theorem 2.5 we have $H^{m+1}x = K^{n+1}x = 0$, so $N^{n+m+1}x = 0$. ∎

3. Applications. Let H be a hermitian element of a Banach algebra, which means that its numerical range $V(H)$ is real. I. Vidav [37] proved that $\max V(H) = \max \mathrm{Sp}(H)$. This property is in fact equivalent to saying that $\|e^H\| = r(e^H)$, for H hermitian (see [9], Theorem 5.10, Theorem 3.4 and p. 54), and it is, in some sense, weaker than A. Sinclair's theorem.

We now extend Vidav's result locally to the class $\mathcal{S}$.

THEOREM 3.1. *Let $T \in \mathcal{S}$ with degree n and constant C and let $x \in X$. Then*

$$\|e^T x\| \le 2^n C \|x\| r_x(e^T).$$

Proof. We apply Theorem 2.5 to $k+T$ where $k \ge n+1$, to get

$$\|(k+T)^k x\| \le C\|x\|(\tau^k + kn\tau^{k-1} + \ldots + k(k-1)\ldots(k-n+1)\tau^{k-n}),$$

where $\tau = r_x(k+T)$. Dividing by k^k and setting $\sigma = r_x(1+T/k)$ we obtain

$$\left\|\left(1+\frac{T}{k}\right)^k x\right\| \le C\|x\|\left(\sigma^k + n\sigma^{k-1} + \ldots + \frac{k(k-1)\ldots(k-n+1)}{k^n}\sigma^{k-n}\right).$$

Let $k-n \le l \le k$. Using Lemma 2.1 we conclude that $\mathrm{Sp}_x((1+T/k)^l) = \{(1+z/k)^l : z \in \mathrm{Sp}_x(T)\}$, hence

$$\limsup_{k\to\infty} r_x\left(\left(1+\frac{T}{k}\right)^l\right) \le \max\{|e^z| : z \in \mathrm{Sp}_x(T)\} = r_x(e^T).$$

So

$$\|e^T x\| \le C\|x\| r_x(e^T)\left(1 + n + \frac{n(n-1)}{2} + \ldots + 1\right). \blacksquare$$

COROLLARY 3.2 (I. Vidav). *If H is hermitian then $r(e^H) = \|e^H\|$.*

Proof. We have $C = 1$ and the degree of H is zero. So $\|e^H x\| \le \|x\| r_x(e^H)$, and consequently,

$$\|e^H\| = \sup_{x\ne 0} \frac{\|e^H x\|}{\|x\|} \le \sup_{x\ne 0} r_x(e^H) \le r(e^H) \le \|e^H\|. \blacksquare$$

From Vidav's result it is easy to conclude that the same property is true for N normal, that is, $N = H + iK$ with H, K hermitian and commuting. But if we take $N \in \mathcal{N}$ with the constants and degrees of H, K denoted by C_H, C_K, m, n respectively, then by Theorems 3.1 and 2.8 we have

$$\|e^N x\| = \|e^H u\| \le 2^m C_H \|u\| r_u(e^N),$$

where $u = e^{iK}x$. Because H and K commute we have $r_u(e^N) \le r_x(e^N)$. Moreover, $\|e^{iK}x\| \le C_K 2^n \|x\|$ by the growth condition, and consequently,

$$\|e^N x\| \le 2^{m+n} C_H C_K \|x\| r_x(e^N).$$

The argument used in the proof of Theorem 3.1 can be used to prove the following theorem.

THEOREM 3.3. *The following properties are equivalent:*

(i) *T is in the class $\mathcal{S}$ with degree n.*

(ii) *T has real spectrum and satisfies the fundamental inequality of Theorem 2.5 for degree n.*

Proof. From Lemma 2.2 and Theorem 2.5 we see that (i) implies (ii). Suppose now that (ii) is true. We apply Theorem 2.5 to $\left(\frac{k}{it}+T\right)^k$ where t is real and k is an integer, $k \ge n+1$. So we get

$$\left\|\left(\frac{k}{it}+T\right)^k x\right\| \le C\|x\|(\tau^k + kn\tau^{k-1} + \ldots + k(k-1)\ldots(k-n+1)\tau^{k-n}),$$

where $\tau = r_x\left(\frac{k}{it}+T\right)$. Dividing by $\left(\frac{k}{it}\right)^k$ we obtain

$$\left\|\left(1+\frac{itT}{k}\right)^k x\right\| \le C\|x\|\left(\sigma^k + |t|\sigma^{k-1} + \ldots + \frac{k(k-1)\ldots(k-n+1)}{k^n}|t|^n\sigma^{k-n}\right),$$

where $\sigma = r_x(1+itT/k)$. As previously, when k goes to infinity, $\sigma^l = r_x((1+itT/k)^l)$ has upper limit at most $r_x(e^{iT})$ if $k-n \le l \le k$. But

because the spectrum of T is real, the same is true for its local spectrum, so $r_x(e^{iT}) = 1$. Hence

$$\|e^{itT}x\| \le C\|x\|(1 + |t| + \ldots + |t|^n) \le C\|x\|(1 + |t|)^n . \blacksquare$$

In 1941, I. M. Gelfand [23] proved that if an element T of a Banach algebra satisfies $\mathrm{Sp}(T) = \{1\}$ and $\sup_{k\in\mathbb{Z}} \|T^k\| < \infty$, then $T = I$. This result was generalized by E. Hille (see [24] or [25], Theorem 4.10.1) who proved that if $\mathrm{Sp}(T) = \{1\}$ and $\|T^k\| = o(|k|)$ for $k \in \mathbb{Z}$, then $T = I$. Nobody noticed that these results come from the weak form of Sinclair's theorem. This is the reason why the proof of Gelfand's result given in [1], Theorem 1.1, is almost identical with the elementary proof of Sinclair's result given by F. F. Bonsall (see [10], pp. 56–57). Just looking at Gelfand's result, the condition $\mathrm{Sp}(T) = \{1\}$ implies, by using the holomorphic functional calculus, that $T = e^{iS}$ where $r(S) = 0$. The extra hypothesis implies that $\sup_{t\in\mathbb{R}} \|e^{itS}\| < \infty$, so S is hermitian for an equivalent norm. Hence $r(S) = \|S\| = 0$, so $T = I$.

All this suggests that these results can be extended locally.

THEOREM 3.4. *Let $T \in \mathcal{B}(X)$ and $x \in X$. Suppose that $r_x(T) = 0$ and that -1 is not in the polynomially convex hull of the spectrum of T. Then $T^r x = 0$ for some $r \ge 1$ if and only if $\|(I + T)^k x\| = o(|k|^r)$ as $|k| \to \infty$.*

Proof. It is obvious that in general $T^r x = 0$ implies $\|(I + T)^k x\| = o(|k|^r)$. The converse is obviously true if $x = 0$, so suppose $x \ne 0$. By Lemma 2.1 we have $I + T = e^{iS}$, where $r_x(S) = 0$. For every $t \in \mathbb{R}$ we have $t = k + s$, where $k \in \mathbb{Z}$ and $0 \le s < 1$. Because $\|e^{isS}x\|$ is bounded when $0 \le s < 1$ and because $|k| \le 1 + |t|$ we conclude from the hypothesis that

$$\|e^{itS}x\| = o((1 + |t|)^r) .$$

So there exists $C > 0$ such that

$$\|e^{itS}x\| \le C\|x\|(1 + |t|)^r \quad \text{for } t \in \mathbb{R} .$$

By Corollary 2.9 we have $S^{r+1}x = 0$, so $T^{r+1}x = 0$ because $T = e^{iS} - 1$. Thus for $k \ge r + 1$ we have

$$\|(I + T)^k x\|$$
$$= \left\| x + kTx + \frac{k(k-1)}{2}T^2x + \ldots + \frac{k(k-1)\ldots(k-r+1)}{r!}T^r x \right\| = o(|k|^r) ,$$

and consequently $T^r x = 0$. $\blacksquare$

If we want to get Hille's result in a Banach algebra using the previous theorem it is enough to represent the algebra A in $\mathcal{B}(A)$ using the representation $T \to R_T$ where $R_T X = XT$.

B. Bollobás [8] proved that if an invertible hermitian element H satisfies $\|H\| = \|H^{-1}\| = 1$ then $H = H^{-1}$, in which case there exists an idempotent P such that $H = I - 2P$. This result was generalized by H. R. Dowson [17] who assumed $\mathrm{Sp}(T)$ is real and $\|T^k\| = o(|k|)$ as $|k| \to \infty$. We now extend this result locally.

THEOREM 3.5. *Let $T \in \mathcal{B}(X)$, T invertible, and $x \in X$. Suppose $\mathrm{Sp}(T)$ is real and $\|T^k x\| = o(|k|^r)$ as $|k| \to \infty$, for some integer r. Then $(T^2 - I)^r x = 0$.*

Proof. We may suppose that $x \ne 0$. From the hypothesis we have $r_x(T) \le 1$, $r_x(T^{-1}) \le 1$, so $\mathrm{Sp}_x(T) \subset \Gamma \cap \mathbb{R}$, where Γ is the unit circle. Hence $\mathrm{Sp}_x(T) \subset \{-1, 1\}$, which implies that $T^2 = I + Q$, where $r_x(Q) = \{0\}$. So -1 is not in the spectrum of Q and $\mathrm{Sp}(Q) \subset \mathbb{R}$, so -1 is not in a hole of $\mathrm{Sp}(Q)$. By Theorem 3.4 applied to Q, we have $Q^r x = 0$ because $\|T^{2k}x\| = o(|k|^r)$. $\blacksquare$

In 1958, M. Rosenblum gave an ingenious proof of the following result due to B. Fuglede and C. R. Putnam: if N is a normal operator on a Hilbert space $\mathcal{H}$ commuting with T then N^* commutes with T (see for instance [33], Theorem 12.16). This result was extended in 1977 by M. J. Crabb and P. G. Spain [16] in the following way. Let N, Q be two commuting operators on a Banach space X such that N is normal, that is, $N = H + iK$ where H, K are two commuting hermitian operators, and Q is quasi-nilpotent. Suppose moreover that $(N + Q)^2 x = 0$ for some $x \in X$. Then $Hx = Kx = 0$. It is easy to see that the Fuglede–Putnam–Rosenblum theorem derives from the previous one just by taking $X = \mathcal{B}(\mathcal{H})$, the inner derivation $\Delta_N : S \mapsto NS - SN$ which is normal on X because N is normal on $\mathcal{H}$ in the traditional sense, and $Q = 0$. By hypothesis we have $\Delta_N(T) = 0$, so $\Delta_N^2(T) = 0$, and consequently $\Delta_H(T) = \Delta_K(T) = 0$, which implies the result. Incidently we remark that the Crabb–Spain theorem generalizes Theorem 1 of [18].

We now give an extension to the class $\mathcal{S}$ of the Crabb–Spain theorem.

LEMMA 3.6. *Let $N, Q \in \mathcal{B}(X)$ and $x \in X$. Suppose that N and Q commute and that Q is quasi-nilpotent. Then $\mathrm{Sp}_x(N + Q) = \mathrm{Sp}_x(N)$, and consequently $r_x(N + Q) = r_x(N)$.*

Proof. If $x = 0$ there is nothing to prove, so suppose $x \ne 0$. Let $\lambda \notin \mathrm{Sp}_x(N + Q)$. Then there exists a disk V_λ centred at λ and u analytic on V_λ such that

$$(z - T)u(z) = x \quad \text{for } z \in V_\lambda, \quad \text{where} \quad T = N + Q .$$

Taking the kth derivative of this expression we get

$$(z - T)u^{(k)}(z) = -ku^{(k-1)}(z) \quad \text{for } z \in V_\lambda,\ k \ge 1 . \tag{1}$$

We consider the series

$$\sum_{k=0}^{\infty} Q^k \frac{u^{(k)}(z)}{k!} .$$

First we prove that it converges and defines an analytic function on V_λ. Let $0 < r < s$ be such that the closed disk $\overline{D}(\lambda, s)$ is included in V_λ. Taking z in $\overline{D}(\lambda, r)$ we have by Cauchy's inequalities

$$\left\| \frac{u^{(k)}(z)}{k!} \right\| = \left\| \frac{1}{2\pi i} \int\limits_{|\xi-\lambda|=s} \frac{u(\xi)}{(\xi - z)^{k+1}}\, d\xi \right\| \leq \frac{Mr}{(s-r)^{k+1}},$$

where $M = \sup\{\|u(\xi)\| : |\xi - \lambda| = s\}$. Given $\varepsilon > 0$, because Q is quasi-nilpotent there exists k_0 such that $k \geq k_0$ implies $\|Q^k\| \leq \varepsilon^k$. If we take $\varepsilon \leq (s-r)/2$ we conclude that the series converges uniformly on $\overline{D}(\lambda, r)$, and hence on any compact subset of V_λ, so it defines an analytic function $v(z)$ on V_λ. Moreover, we have

$$\begin{aligned}(z-N)v(z) &= \sum_{k=0}^{\infty} (z-N)Q^k \frac{u^{(k)}(z)}{k!} \\ &= \sum_{k=0}^{\infty} (z-T)Q^k \frac{u^{(k)}(z)}{k!} + \sum_{k=0}^{\infty} Q^{k+1} \frac{u^{(k)}(z)}{k!} \\ &= x - \sum_{k=1}^{\infty} Q^k \frac{u^{(k-1)}(z)}{(k-1)!} + \sum_{k=0}^{\infty} Q^{k+1} \frac{u^{(k)}(z)}{k!} = x,\end{aligned}$$

because $QT = TQ$ and relation (1) holds. So $\lambda \notin \mathrm{Sp}_x(N)$. Replacing N by $N - Q$, we get the converse inclusion. ∎

THEOREM 3.7. *Let $N, Q \in \mathcal{B}(X)$ and $x \in X$. Suppose that Q is quasi-nilpotent and that N and Q commute. Suppose moreover that $N \in \mathcal{N}$, that is, $N = H + iK$ where $H, K \in \mathcal{S}$ and $HK = KH$, and that $r_x(N+Q) = 0$. Then $H^{m+1}x = K^{n+1}x = 0$ where m denotes the degree of H and n the degree of K.*

Proof. Because Q is quasi-nilpotent and commutes with N, by Lemma 3.6 we have $r_x(N) = 0$. We finish as in the proof of Corollary 2.9. ∎

If H is a self-adjoint operator on a Hilbert space, then for every complex number λ, we have $\mathrm{Ker}(\lambda - H)^2 = \mathrm{Ker}(\lambda - H)$. This is also true for any hermitian operator on a Banach space. This results comes from the Nirschl–Schneider theorem (see [9], Corollary 10.11). It can also be proved elementarily (see [26]). B. A. Barnes [5] proved in §3 of his paper that the kernels $\mathrm{Ker}(\lambda - T)^k$ stabilize for $k \geq m$, for some integer m, if $T \in \mathcal{S}$, the same being true for the closures of the ranges of the powers $(\lambda - T)^k$. But he was not able to determine m.

THEOREM 3.8. *Let $N \in \mathcal{N}$ of degree r. Given $\lambda \in \mathbb{C}$ the kernels $\mathrm{Ker}(\lambda - N)^k$ and the closures of the ranges $R((\lambda - T)^k)$ stabilize for $k \geq r+1$.*

Proof. In §2 we have seen that if $T \in \mathcal{S}(X)$ then T', the adjoint of T, is in $\mathcal{S}(X')$, where X' is the topological dual of X, and moreover the closure of the range of T is the annihilator of the kernel of T'. So the second part of the theorem follows from the first.

Suppose that $(\lambda - N)^k x = 0$ for $k \geq r+1$. Then $r_x(\lambda - N) = 0$. Let $\lambda = \alpha + i\beta$ with α, β real. Then by Theorem 2.8, we have $r_x(\alpha - H) = r_x(\beta - K) = 0$. By Theorem 2.5, $(\alpha - H)^{m+1}x = (\beta - K)^{n+1}x = 0$ where m, n are the respective degrees of H and K. By commutativity, we get $(\lambda - N)^{m+n+1}x = 0$, so the kernel of $(\lambda - N)^k$ stabilizes after $r+1 = n+m+1$. ∎

By definition the *ascent* of T is the smallest integer k such that $\mathrm{Ker}(T^k) = \mathrm{Ker}(T^{k+1})$. So the previous theorem says that for $N \in \mathcal{S}$ and $\lambda \in \mathbb{C}$, the ascent of $\lambda - N$ is less than or equal to $r+1$.

For $T \in \mathcal{B}(X)$, in 1909, H. Weyl introduced what is now called the *Weyl spectrum* $\omega(T)$ of T, which is by definition the intersection of $\mathrm{Sp}(T+K)$ for all compact operators K. If λ_0 is an isolated point in the spectrum of T, then we can write the Laurent expansion of the resolvent,

$$(\lambda - T)^{-1} = \sum_{n=-\infty}^{\infty} (\lambda - \lambda_0)^n a_n \quad \text{for } 0 < |\lambda - \lambda_0| < \delta,$$

where $a_{-k} = (T - \lambda_0)^{k-1} P$ $(k \geq 1)$ and P is the spectral projection associated with λ_0. We say that λ_0 is a *pole of order* $k \geq 1$ of the resolvent if $a_{-k} \neq 0$ and $a_{-n} = 0$ for $n > k$. Using results of M. Schechter about defect and nullity, K. K. Warner [39] proved that if the ascent of $\lambda - T$ is finite on a dense subset of $\mathbb{C}$, then $\omega(T)$ is $\mathrm{Sp}(T)$ minus the poles of the resolvent of finite order. I. Istrăţescu and V. Istrătescu [26] gave a very elementary proof of the fact that for a hermitian operator $T \in \mathcal{B}(X)$, $\omega(T)$ has the same property, and asked if that is true for normal operators. This last question can be answered positively using elementary arguments, but all of this is a particular case of the following result.

THEOREM 3.9. *Let $N \in \mathcal{N}$ of degree r. Then any pole of the resolvent of N has order at most $r+1$. In particular, $\omega(N)$ is $\mathrm{Sp}(N)$ minus the poles of order less than or equal to $r+1$.*

Proof. Let λ_0 be a pole of the resolvent of N, P its spectral projection and Y the range of P which is N-invariant. We have $N = H + iK$, where $H, K \in \mathcal{S}$ and $HK = KH$. Using the elementary argument on the inner derivation Δ_N given just before Lemma 3.5 we conclude that P commutes with H and K so Y is invariant by H and K. Because H, K are in $\mathcal{S}(X)$ their restrictions to Y, denoted by H_1, K_1, are in $\mathcal{S}(Y)$ with the same constants and no greater degrees. So $H_1 + iK_1 = N_1$, the restriction of N to Y,

and $N_1 \in \mathcal{N}(Y)$. But $\mathrm{Sp}_{\mathcal{B}(Y)}(N_1) = \{\lambda_0\}$, so $r(\lambda_0 - N_1) = 0$. Hence by Corollary 2.9, $(H_1 - \alpha_0)^{m+1} = (K_1 - \beta_0)^{n+1} = 0$ on Y, where $\lambda_0 = \alpha_0 + i\beta_0$, and $m = \deg(H), n = \deg(K)$. So $(N_1 - \lambda_0)^{r+1} = 0$ on Y and this means that $a_{-n} = 0$ for $n \geq r + 2$. Then we apply Theorem 3.8 and Warner's theorem. ∎

It is also possible to improve Corollary 1 of [31], which is wrongly stated because if we have $\|T^k\| = O(|k|^r)$ as $|k| \to \infty$, for some integer $r \geq 1$, then any isolated point of the spectrum of T is a pole of order less than or equal to $r + 1$ and not r, as easily seen with $T = e^{iH}$, where H is hermitian, for which $r = 0$.

COROLLARY 3.10. *If $\|T^k\| = o(|k|^r)$ as $|k| \to \infty$, for some integer $r \geq 1$, then any isolated point of the spectrum of T is a pole of order at most r.*

Proof. If T has no isolated point in its spectrum there is nothing to prove. The hypothesis implies that the spectrum of T is included in the unit circle. Without loss of generality suppose that 1 is an isolated point of the spectrum of T. Let P be the spectral idempotent associated with 1. Then the range of P is T-invariant and $r_x(T - I) = 0$ for any x in this range. So by Theorem 3.4 applied to $T - I$, whose spectrum has no holes, we get $(T - I)^r x = 0$ for x in the range of P or equivalently $(T - I)^r P = 0$. This says that $a_{-n} = 0$ for $n \geq r + 1$ in the Laurent expansion, so the order of the pole 1 is at most r. ∎

References

[1] G. R. Allan and T. J. Ransford, *Power-dominated elements in a Banach algebra*, Studia Math. 94 (1989), 63–79.

[2] C. Apostol, *Teorie spectrală şi calcul functional*, Stud. Cerc. Mat. 20 (1968), 635–668.

[3] B. Aupetit, *A Primer on Spectral Theory*, Springer, 1991.

[4] B. Aupetit and D. Drissi, *Local spectrum theory revisited*, to appear.

[5] B. A. Barnes, *Operators which satisfy polynomial growth conditions*, Pacific J. Math. 138 (1987), 209–219.

[6] R. G. Bartle and C. A. Kariotis, *Some localizations of the spectral mapping theorem*, Duke Math. J. 40 (1973), 651–660.

[7] R. P. Boas, *Entire Functions*, Academic Press, 1954.

[8] B. Bollobás, *A property of hermitian elements*, J. London Math. Soc. 4 (1971), 379–380.

[9] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, London Math. Soc. Lecture Note Ser. 2, Cambridge University Press, 1971.

[10] —, —, *Numerical Ranges II*, London Math. Soc. Lecture Note Ser. 10, Cambridge University Press, 1973.

[11] —, —, *Complete Normed Algebras*, Springer, 1973.

[12] H. N. Bojadjiev [K. N. Boyadzhiev], *New applications of Bernstein inequality to the theory of operators: a local Sinclair lemma and a generalization of the Fuglede–Putnam theorem*, in: Complex Analysis and Applications 85, Sofia, 1986, 97–104.

[13] —, *Sinclair type inequalities for the local spectral radius and related topics*, Israel J. Math. 57 (1987), 272–284.

[14] A. Browder, *On Bernstein's inequality and the norm of hermitian operators*, Amer. Math. Monthly 78 (1971), 871–873.

[15] I. Colojoară and C. Foiaş, *Theory of Generalized Spectral Operators*, Gordon and Breach, 1968.

[16] M. J. Crabb and P. G. Spain, *Commutators and normal operators*, Glasgow Math. J. 18 (1977), 197–198.

[17] H. R. Dowson, *Some properties of prespectral operators*, Proc. Roy. Irish Acad. 74 (1974), 207–221.

[18] H. R. Dowson, T. A. Gillespie and P. G. Spain, *A commutativity theorem for hermitian operators*, Math. Ann. 220 (1976), 215–217.

[19] D. Drissi, *Quelques inégalités spectrales pour les opérateurs scalaires généralisés*, Ph.D. thesis, Université Laval, 1993.

[20] I. Erdelyi and R. Lange, *Spectral Decompositions on Banach Spaces*, Lecture Notes in Math. 623, Springer, 1977.

[21] C. Foiaş, *Une application des distributions vectorielles à la théorie spectrale*, Bull. Sci. Math. 84 (1960), 147–158.

[22] C. K. Fong, *Normal operators on Banach spaces*, Glasgow Math. J. 20 (1979), 163–168.

[23] I. Gelfand, *Zur theorie der Charaktere der abelschen topologischen Gruppen*, Rec. Math. N.S. (Mat. Sb.) 9 (51) (1941), 49–50.

[24] E. Hille, *On the theory of characters of groups and semi-groups in normed vector rings*, Proc. Nat. Acad. Sci. 30 (1944), 58–60.

[25] E. Hille and R. S. Phillips, *Functional Analysis and Semi-groups*, Amer. Math. Soc. Colloq. Publ. 31, 1957.

[26] I. Istrăţescu and V. Istrăţescu, *A note on the Weyl's spectrum of an operator*, Rev. Roumaine Math. Pures Appl. 15 (1970), 1445–1447.

[27] V. È. Kacnel'son [V. È. Katsnel'son], *A conservative operator has norm equal to its spectral radius*, Mat. Issled. 5 (3) (17) (1970), 186–189 (in Russian).

[28] S. Kantorovitz, *Classification of operators by means of their operational calculus*, Trans. Amer. Math. Soc. 115 (1965), 194–224.

[29] G. K. Leaf, *A spectral theory for a class of linear operators*, Pacific J. Math. 13 (1963), 141–155.

[30] B. Ja. Levin, *Distribution of Zeros of Entire Functions*, Amer. Math. Soc., 1964.

[31] T. Pytlik, *Analytic semigroups in Banach algebras and a theorem of Hille*, Colloq. Math. 51 (1987), 287–294.

[32] F. Riesz et B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, Acad. Sci. Hongrie, Szeged, 1955.

[33] W. Rudin, *Functional Analysis*, McGraw-Hill, 1973.

[34] A. M. Sinclair, *The norm of a hermitian element in a Banach algebra*, Proc. Amer. Math. Soc. 28 (1971), 446–450.

[35] D. R. Smart, *Conditionally convergent expansions*, J. Austral. Math. Soc. 1 (1960), 319–333.

[36] B. G. Tillman, *Vector-valued distributions and the spectral theorem for self-adjoint operators in Hilbert space*, Bull. Amer. Math. Soc. 69 (1963), 67–71.

[37] I. Vidav, *Eine metrische Kennzeichnung der selbstadjungierten Operatoren*, Math. Z. 66 (1956), 121–128.
[38] P. Vrbová, *On local spectral properties of operators in Banach spaces*, Czechoslovak Math. J. 23 (1973), 483–492.
[39] K. K. Warner, *A note on a theorem of Weyl*, Proc. Amer. Math. Soc. 23 (1969), 469–471.
[40] F. Wolf, *Operators in Banach space which admit a generalized spectral decomposition*, Nederl. Akad. Wetensch. Indag. Math. 19 (1957), 302–311.

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Operators preserving ideals in C^*-algebras

by

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Abstract. The aim of this paper is to prove that derivations of a C^*-algebra $\mathcal{A}$ can be characterized in the space of all linear continuous operators $T : \mathcal{A} \to \mathcal{A}$ by the conditions $T(1) = 0$, $T(L \cap R) \subset L + R$ for any closed left ideal L and right ideal R. As a corollary we get an extension of the result of Kadison [5] on local derivations in W^*-algebras. Stronger results of this kind are proved under some additional conditions on the cohomologies of $\mathcal{A}$.

Notations. As usual, $\mathcal{X}^*$ denotes the dual space of a Banach space $\mathcal{X}$; $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ is the space of all linear bounded operators from $\mathcal{X}_1$ to $\mathcal{X}_2$; $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$; $\mathcal{Y}^\circ$ is the annihilator in $\mathcal{X}^*$ of a subspace $\mathcal{Y} \subset \mathcal{X}$; and $\operatorname{dist}(x, \mathcal{Y})$ is the distance from $x \in \mathcal{X}$ to $\mathcal{Y}$. Subspaces $\mathcal{Y}_1$, $\mathcal{Y}_2$ constitute an *M-pair* if

$$\operatorname{dist}(x, \mathcal{Y}_1 \cap \mathcal{Y}_2) = \max\{\operatorname{dist}(x, \mathcal{Y}_1), \operatorname{dist}(x, \mathcal{Y}_2)\}$$

for any $x \in \mathcal{X}$; they constitute an *L-pair* if

$$\|x + y\| = \inf\{\|x - z\| + \|y + z\| : z \in \mathcal{Y}_1 \cap \mathcal{Y}_2\} \quad \text{for } x \in \mathcal{Y}_1,\ y \in \mathcal{Y}_2.$$

In both cases $\mathcal{Y}_1 + \mathcal{Y}_2$ is closed. It is known ([8], Proposition 7) that $(\mathcal{Y}_1, \mathcal{Y}_2)$ is an M-pair (L-pair) iff $(\mathcal{Y}_1^\circ, \mathcal{Y}_2^\circ)$ is an L-pair (M-pair).

The set of all left (right) closed ideals of a C^*-algebra $\mathcal{A}$ will be denoted by $\operatorname{left}\mathcal{A}$ ($\operatorname{right}\mathcal{A}$). It was proved in [8] that (L, R) is an M-pair for any $L \in \operatorname{left}\mathcal{A}$, $R \in \operatorname{right}\mathcal{A}$. The bidual space $\mathcal{A}^{**}$ of $\mathcal{A}$ is identified with the universal enveloping von Neumann algebra. A projection $p \in \mathcal{A}^{**}$ is called *open* if it equals the supremum of an increasing net of positive elements in $\mathcal{A}$. It is known (see [1]) that openness of p is equivalent to the conditions $\mathcal{A}^{**}p = L^{\circ\circ}$ for some $L \in \operatorname{left}\mathcal{A}$ (or $p\mathcal{A}^{**} = R^{\circ\circ}$ for $R \in \operatorname{right}\mathcal{A}$). We write $p^\perp$ instead of $1 - p$.

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