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Operators preserving ideals in C^*-algebras

by

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Abstract. The aim of this paper is to prove that derivations of a C^*-algebra $\mathcal{A}$ can be characterized in the space of all linear continuous operators $T : \mathcal{A} \to \mathcal{A}$ by the conditions $T(1) = 0$, $T(L \cap R) \subset L + R$ for any closed left ideal L and right ideal R. As a corollary we get an extension of the result of Kadison [5] on local derivations in W^*-algebras. Stronger results of this kind are proved under some additional conditions on the cohomologies of $\mathcal{A}$.

Notations. As usual, $\mathcal{X}^*$ denotes the dual space of a Banach space $\mathcal{X}$; $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ is the space of all linear bounded operators from $\mathcal{X}_1$ to $\mathcal{X}_2$; $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$; $\mathcal{Y}^\circ$ is the annihilator in $\mathcal{X}^*$ of a subspace $\mathcal{Y} \subset \mathcal{X}$; and $\operatorname{dist}(x, \mathcal{Y})$ is the distance from $x \in \mathcal{X}$ to $\mathcal{Y}$. Subspaces $\mathcal{Y}_1$, $\mathcal{Y}_2$ constitute an *M-pair* if

$$\operatorname{dist}(x, \mathcal{Y}_1 \cap \mathcal{Y}_2) = \max\{\operatorname{dist}(x, \mathcal{Y}_1), \operatorname{dist}(x, \mathcal{Y}_2)\}$$

for any $x \in \mathcal{X}$; they constitute an *L-pair* if

$$\|x + y\| = \inf\{\|x - z\| + \|y + z\| : z \in \mathcal{Y}_1 \cap \mathcal{Y}_2\} \quad \text{for } x \in \mathcal{Y}_1,\ y \in \mathcal{Y}_2.$$

In both cases $\mathcal{Y}_1 + \mathcal{Y}_2$ is closed. It is known ([8], Proposition 7) that $(\mathcal{Y}_1, \mathcal{Y}_2)$ is an M-pair (L-pair) iff $(\mathcal{Y}_1^\circ, \mathcal{Y}_2^\circ)$ is an L-pair (M-pair).

The set of all left (right) closed ideals of a C^*-algebra $\mathcal{A}$ will be denoted by $\operatorname{left}\mathcal{A}$ ($\operatorname{right}\mathcal{A}$). It was proved in [8] that (L, R) is an M-pair for any $L \in \operatorname{left}\mathcal{A}$, $R \in \operatorname{right}\mathcal{A}$. The bidual space $\mathcal{A}^{**}$ of $\mathcal{A}$ is identified with the universal enveloping von Neumann algebra. A projection $p \in \mathcal{A}^{**}$ is called *open* if it equals the supremum of an increasing net of positive elements in $\mathcal{A}$. It is known (see [1]) that openness of p is equivalent to the conditions $\mathcal{A}^{**}p = L^{\circ\circ}$ for some $L \in \operatorname{left}\mathcal{A}$ (or $p\mathcal{A}^{**} = R^{\circ\circ}$ for $R \in \operatorname{right}\mathcal{A}$). We write $p^\perp$ instead of $1 - p$.

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I am indebted to Bojan Magajna for drawing my attention to the results of [5].

An operator $\mathcal{D} \in \mathcal{L}(\mathcal{A})$ is called a *derivation* of $\mathcal{A}$ if

$$\mathcal{D}(xy) = x\mathcal{D}(y) + \mathcal{D}(x)y \tag{1}$$

for any $x, y \in \mathcal{A}$. The space of all derivations will be denoted by $\operatorname{der}\mathcal{A}$. More generally, if $\mathcal{M}$ is an $\mathcal{A}$-bimodule then $\operatorname{der}(\mathcal{A},\mathcal{M})$ is the space of all $\mathcal{M}$-valued derivations on $\mathcal{A}$, that is, operators $\mathcal{D} \in \mathcal{L}(\mathcal{A},\mathcal{M})$ satisfying (1). For $\xi \in \mathcal{M}$ the left multiplication operator $l_\xi : \mathcal{A} \to \mathcal{M}$ acts by the rule $l_\xi(a) = a\xi$.

THEOREM 1. *Let $\mathcal{A}$ be a unital C^*-algebra and $T \in \mathcal{L}(\mathcal{A})$. If*

$$T(L \cap R) \subset L + R \tag{2}$$

for any $L \in \operatorname{left}\mathcal{A}$ and $R \in \operatorname{right}\mathcal{A}$, then $T = \mathcal{D} + l_a$ where $a = T(1)$ and $\mathcal{D} \in \operatorname{der}\mathcal{A}$.

Proof. We may suppose that $T(1) = 0$ (otherwise replace T by $T - l_{T(1)}$). First of all let us prove that

$$T^*(L^\circ \cap R^\circ) \subset L^\circ + R^\circ \tag{3}$$

and

$$T^{**}(L^{\circ\circ} \cap R^{\circ\circ}) \subset L^{\circ\circ} + R^{\circ\circ}\,. \tag{4}$$

Indeed, (L°, R°) is an L-pair, so $L^\circ + R^\circ$ is closed and (by the Banach theorem) weak* closed. So to prove (3) it is sufficient to show that

$$(L^\circ + R^\circ)^\circ \cap \mathcal{A} \subset (T^*(L^\circ \cap R^\circ))^\circ\,. \tag{5}$$

But since $(L^\circ + R^\circ)^\circ \cap \mathcal{A} = L \cap R$ and $L^\circ \cap R^\circ = (L+R)^\circ$, (5) is an immediate corollary of (2). The inclusion (4) may be proved in a similar way.

It follows from (4) that $T^*(p\mathcal{A}^{**}q) \subset p\mathcal{A}^{**} + \mathcal{A}^{**}q$ or equivalently

$$p^\perp T^{**}(pxq)q^\perp = 0 \tag{6}$$

for any open projections $p, q \in \mathcal{A}^{**}$ and any $x \in \mathcal{A}^{**}$. But every closed projection is the weak (and hence strong) limit of a net of open projections ([1], Proposition 2.3), so (6) is also true for closed p, q.

Now use the obvious identities

$$p^\perp S(px) - pS(p^\perp x) = S(px) - pS(x)\,, \tag{7}$$

$$V(xq)q^\perp - V(xq^\perp)q = V(xq) - V(x)q\,, \tag{8}$$

where S and V are arbitrary linear operators in $\mathcal{A}^{**}$. For $S(x) = T^{**}(xq)q^\perp$, (7) and (6) imply

$$T^{**}(pxq)q^\perp - pT^{**}(xq)q^\perp = 0\,.$$

So for $V(x) = T^{**}(px) - pT^{**}(x)$ we have $V(xq^\perp)q = 0$, $V(xq)q^\perp = 0$ and, by (8), $V(xq) - V(x)q = 0$. Therefore

$$T^{**}(pxq) - pT^{**}(xq) - T^{**}(px)q + pT^{**}(x)q = 0\,.$$

For $x = 1$ we get $T^{**}(pq) = pT^{**}(q) + T^{**}(p)q$ and hence $T^{**}(xy) = xT^{**}(y) + T^{**}(x)y$ for all x, y in the closed linear hull $\mathcal{E}$ of the set of all open or closed projections. But the spectral projections of hermitian elements of $\mathcal{A}$ corresponding to open parts of spectra are open; therefore $\mathcal{A} \subset \mathcal{E}$ and $T \in \operatorname{der}\mathcal{A}$.

Following [5] we call an operator $T \in \mathcal{L}(\mathcal{A})$ a *local derivation* if for any $x \in \mathcal{A}$ there exists $\mathcal{D}_x \in \operatorname{der}\mathcal{A}$ such that $T(x) = \mathcal{D}_x(x)$.

COROLLARY 1. *Every local derivation of a C^*-algebra is a derivation.*

Proof. Let $T \in \mathcal{L}(\mathcal{A})$ be a local derivation. For any $L \in \operatorname{left}\mathcal{A}$, $R \in \operatorname{right}\mathcal{A}$, $x \in R$, and $y \in L$ we have

$$T(xy) = \mathcal{D}_{xy}(xy) = \mathcal{D}_{xy}(x)y + x\mathcal{D}_{xy}(y) \in L + R\,,$$

so $T(RL) \subset L + R$ and $T(R \cap L) \subset \overline{T(RL)} \subset L + R$. It follows from Theorem 1 and the obvious equality $T(1) = 0$ that $T \in \operatorname{der}\mathcal{A}$.

Recall that a subspace $\mathcal{E} \subset \mathcal{L}(\mathcal{X},\mathcal{Y})$ is called *reflexive* [7] if it contains every operator $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ such that $Tx \in \overline{\mathcal{E}x}$ for each $x \in \mathcal{X}$. If $\mathcal{E}$ contains all operators satisfying the more restrictive condition $Tx \in \mathcal{E}x$, then $\mathcal{E}$ is called *algebraically reflexive*. Corollary 1 may be reformulated as follows: for any C^*-algebra $\mathcal{A}$ the space $\operatorname{der}\mathcal{A}$ is algebraically reflexive. But the proof shows the stronger property:

COROLLARY 2. *The space* $\operatorname{der}\mathcal{A}$ *is reflexive for any C^*-algebra $\mathcal{A}$.*

For any subspace $\mathcal{E} \subset \mathcal{L}(\mathcal{X},\mathcal{Y})$ and any $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, set

$$\delta(T,\mathcal{E}) = \sup_{\|x\|\le 1} \operatorname{dist}(Tx, \overline{\mathcal{E}x})\,.$$

It is clear that $\delta(T,\mathcal{E}) \le \operatorname{dist}(T,\mathcal{E})$; reflexivity of $\mathcal{E}$ means that the conditions $\delta(T,\mathcal{E}) = 0$ and $\operatorname{dist}(T,\mathcal{E}) = 0$ are equivalent. If $\operatorname{dist}(T,\mathcal{E}) \le C\delta(T,\mathcal{E})$ for some $C > 0$ and all $T \in \mathcal{L}(\mathcal{X}_1,\mathcal{X}_2)$, then $\mathcal{E}$ is called *hyperreflexive* [6].

THEOREM 2. *For any C^*-algebra $\mathcal{A}$ with $H^2(\mathcal{A},\mathcal{A}) = 0$ the space* $\operatorname{der}\mathcal{A}$ *is hyperreflexive.*

Proof. Let

$$0 \to \mathcal{A} \xrightarrow{d^1} \mathcal{L}(\mathcal{A}) \xrightarrow{d^2} \mathcal{L}_2(\mathcal{A}) \to \ldots$$

be the usual cochain complex, that is, $\mathcal{L}_n(\mathcal{A})$ is the space of all n-linear maps from $\mathcal{A}\times\ldots\times\mathcal{A}$ to $\mathcal{A}$ and for $S\in\mathcal{L}_n(\mathcal{A})$,

$$\begin{aligned}d^{n+1}S(a_0,\ldots,a_n)&=a_0S(a_1,\ldots,a_n)\\&\quad+\sum_{i=1}^{n}(-1)^iS(a_0,\ldots,a_{i-2},a_{i-1}a_i,\ldots,a_n)\\&\quad+(-1)^{n+1}S(a_0,\ldots,a_{n-1})a_n\,.\end{aligned}$$

The condition $H^2(\mathcal{A},\mathcal{A})=0$ implies that the image of d^2 is a closed subspace of $\mathcal{L}_2(\mathcal{A})$; since $\operatorname{Ker} d^2=\operatorname{der}\mathcal{A}$ we get

$$\operatorname{dist}(T,\operatorname{der}\mathcal{A})\le C\|d^2T\| \tag{9}$$

for all $t\in\mathcal{L}(\mathcal{A})$ and some $C>0$.

Let now $T\in\mathcal{L}(\mathcal{A})$ and $\alpha=\delta(T,\operatorname{der}\mathcal{A})$. Suppose that $T(1)=0$. For any $L\in\operatorname{left}\mathcal{A}$, $R\in\operatorname{right}\mathcal{A}$, $x\in L\cap R$ with $\|x\|=1$, and $\mathcal{D}\in\operatorname{der}\mathcal{A}$ we have

$$\|Tx-\mathcal{D}x\|\ge\operatorname{dist}(Tx,L+R)=\sup\{|\langle Tx,f\rangle|:f\in Q\}$$

where $Q=\{f\in(L+R)^\circ:\|f\|\le1\}$. In other words,

$$\|Tx-\mathcal{D}x\|\ge\sup\{|\langle Tx,p^\perp fq^\perp\rangle|:f\in\mathcal{A}^*;\ \|f\|\le1\}$$

where p and q are projections in $\mathcal{A}^{**}$ with $p\mathcal{A}^{**}=R^{\circ\circ}$ and $\mathcal{A}^{**}q=L^{\circ\circ}$. Hence $|\langle Tx,p^\perp fq^\perp\rangle|\le\alpha$ for any $x\in\mathcal{A}$, $f\in\mathcal{A}^*$ with $\|x\|\le1$, $\|f\|\le1$. It follows that

$$\|p^\perp T^{**}(pxq)q^\perp\|\le\alpha\|x\| \tag{10}$$

for any $x\in\mathcal{A}$ and any open projections p, q. Using (7), (8) as in the proof of Theorem 1 we get

$$\|T^{**}(pq)-pT^{**}(q)-T^{**}(p)q\|\le4\alpha$$

and as a consequence,

$$\|T(xy)-xT(y)-T(x)y\|\le4\alpha\|x\|\|y\|$$

for any $x\in\mathcal{A}_+$ and $y\in\mathcal{A}_+$. Hence for all $x,y\in\mathcal{A}$,

$$\|T(xy)-xT(y)-T(x)y\|\le16\alpha\|x\|\|y\|\,.$$

In other words, $\|d^2T\|\le16\alpha$; by (9) this gives

$$\operatorname{dist}(T,\operatorname{der}\mathcal{A})\le16\alpha C\,.$$

To remove the restriction $T(1)=0$ it is enough to notice that

$$T(1)\le\delta(T,\operatorname{der}\mathcal{A})=\alpha\,,$$

and

$$\begin{aligned}\operatorname{dist}(T,\operatorname{der}\mathcal{A})&\le\operatorname{dist}(T-l_{T(1)},\operatorname{der}\mathcal{A})+\|T(1)\|\\&\le16C\delta(T-l_{T(1)},\operatorname{der}\mathcal{A})+\alpha\le(32C+1)\alpha\,.\end{aligned}$$

The result may be stated in a form similar to Arveson's "distance formula" [2]. Let p and q be projections in a W^*-algebra $\mathfrak{A}$; for $x\in\mathfrak{A}$ put $p\otimes q(x)=pxq$.

COROLLARY 3. *Under the conditions of Theorem* 2, *there exists* $C>0$ *such that*

$$\operatorname{dist}(T,\operatorname{der}\mathcal{A})\le C\sup_{p,q}\|p^\perp\otimes q^\perp T^{**}p\otimes q\| \tag{11}$$

for each $T\in\mathcal{L}(\mathcal{A})$ *with* $T(1)=0$.

It was proved in [3] that for any Banach $\mathcal{A}$-bimodule $\mathcal{M}$ with $\mathcal{M}^*$ weakly sequentially complete (wsc) the usual $\mathcal{A}$-bimodule structure in $\mathcal{M}^{**}$ may be extended to an $\mathcal{A}^{**}$-bimodule structure in such a way that $\mathcal{M}^{**}$ is a dual normal $\mathcal{A}^{**}$-bimodule. This permits extending the previous results to operators from $\mathcal{A}$ to $\mathcal{M}$.

THEOREM 3. *Let* $\mathcal{A}$ *be a* C^*-*algebra, and* $\mathcal{M}$ *a Banach* $\mathcal{A}$-*bimodule with* M^* *wsc. For* $T\in\mathcal{L}(\mathcal{A},\mathcal{M})$ *the following conditions are equivalent*:

(i) $T(L\cap R)\subset\overline{\mathcal{M}L+R\mathcal{M}}$ *for any* $L\in\operatorname{left}\mathcal{A}$ *and* $R\in\operatorname{right}\mathcal{A}$.

(ii) $T=\mathcal{D}+l_\xi$ *for some* $\mathcal{D}\in\operatorname{der}(\mathcal{A},\mathcal{M})$ *and* $\xi\in\mathcal{M}$.

Proof. In a way similar to the proof of Theorem 1 one can show that

$$T^{**}(L^{\circ\circ}\cap R^{\circ\circ})\subset\overline{(\mathcal{M}L)^{\circ\circ}+(R\mathcal{M})^{\circ\circ}}$$

where the closure is taken in the $\sigma(\mathcal{M}^{**},\mathcal{M}^*)$-topology. Now for p, q as before we have $(\mathcal{M}L)^{\circ\circ}q^\perp=0$ and $p^\perp(R\mathcal{M})^{\circ\circ}=0$, hence $p^\perp T^{**}(pxq)q^\perp=0$. The end of the proof is the same as in Theorem 1.

COROLLARY 4. *For* $\mathcal{M}$ *as in Theorem* 3, *every local derivation* $T\in\mathcal{L}(\mathcal{A},\mathcal{M})$ *is a derivation. Moreover*, $\operatorname{der}(\mathcal{A},\mathcal{M})$ *is reflexive.*

COROLLARY 5. *If* $\mathcal{M}^*$ *is wsc and* $H^2(\mathcal{A},\mathcal{M})=0$ *then* $\operatorname{der}(\mathcal{A},\mathcal{M})$ *is hyperreflexive.*

The results of [4] imply that the last conclusion is true for nuclear $\mathcal{A}$ and dual $\mathcal{M}$ with $\mathcal{M}^*$ wsc. The author does not know if the condition on $\mathcal{M}^*$ can be removed.

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Compactness of Hardy-type integral operators in weighted Banach function spaces

by

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Abstract. We consider a generalized Hardy operator $Tf(x) = \phi(x)\int_0^x \psi f v$. For T to be bounded from a weighted Banach function space (X, v) into another, (Y, w), it is always necessary that the Muckenhoupt-type condition $\mathcal{B} = \sup_{R>0} \|\phi\chi_{(R,\infty)}\|_Y \|\psi\chi_{(0,R)}\|_{X'} < \infty$ be satisfied. We say that (X, Y) belongs to the category $\mathcal{M}(T)$ if this Muckenhoupt condition is also sufficient. We prove a general criterion for compactness of T from X to Y when $(X, Y) \in \mathcal{M}(T)$ and give an estimate for the distance of T from the finite rank operators. We apply the results to Lorentz spaces and characterize pairs of Lorentz spaces which fall into $\mathcal{M}(T)$.

1. Introduction. Given two weighted Banach function spaces $X = (X, v)$, $Y = (Y, w)$, and an extra pair of weights (ϕ, ψ), we study boundedness and compactness of the generalized Hardy operator $T_{\phi\psi}f(x) = \phi(x)\int_0^x \psi(t)f(t)v(t)\,dt$ considered as an operator from X to Y. If X and Y are weighted Lebesgue spaces, say, $X = L^r(v)$ and $Y = L^p(w)$, it is enough to consider only the usual Hardy operator $Hf(x) = \int_0^x f(t)\,dt$. For this case, the theory is complete. For example, if $1 < p \le r < \infty$, we have the result of Tomaselli [TO], Talenti [T], Muckenhoupt [MU], Bradley [B], Kokilashvili [K] and Maz'ya [M] which states that there is a constant C such that

$$\|Hf\|_{p,w} \le C\|f\|_{r,v} \quad \text{for all } f \in X \tag{1.1}$$

if and only if

$$\sup_{R>0} B(R) = \sup_{R>0} \Big(\int_R^\infty w\Big)^{1/p} \Big(\int_0^R v^{1-r'}\Big)^{1/r'} = B < \infty \tag{1.2}$$

$(r' = r/(r-1))$.

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