[5] R. Kadison, *Local derivations*, J. Algebra 130 (1990), 494–509.
[6] J. Kraus and D. R. Larson, *Reflexivity and distance formulae*, Proc. London Math. Soc. 53 (1986), 340–356.
[7] A. J. Loginov and V. S. Shul'man, *Hereditary and intermediate reflexivity of W^*-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 1260–1273 (in Russian).
[8] V. S. Shul'man, *On the geometry of some pairs of subspaces in C^*-algebras*, in: Spectral Theory of Operators and its Applications, No. 6, Elm, Baku, 1985, 196–216 (in Russian).

DEPARTMENT OF MATHEMATICS
VOLOGDA POLYTECHNICAL INSTITUTE
15 LENIN ST.
160008 VOLOGDA, RUSSIA

Received April 27, 1993 (3100)
Revised version September 19, 1993

STUDIA MATHEMATICA 109 (1) (1994)

Compactness of Hardy-type integral operators in weighted Banach function spaces

by

DAVID E. EDMUNDS (Sussex), PETR GURKA (Praha) and LUBOŠ PICK (Cardiff and Praha)

Abstract. We consider a generalized Hardy operator $Tf(x) = \phi(x)\int_0^x \psi f v$. For T to be bounded from a weighted Banach function space (X, v) into another, (Y, w), it is always necessary that the Muckenhoupt-type condition $\mathcal{B} = \sup_{R>0} \|\phi\chi_{(R,\infty)}\|_Y \|\psi\chi_{(0,R)}\|_{X'} < \infty$ be satisfied. We say that (X, Y) belongs to the category $\mathcal{M}(T)$ if this Muckenhoupt condition is also sufficient. We prove a general criterion for compactness of T from X to Y when $(X, Y) \in \mathcal{M}(T)$ and give an estimate for the distance of T from the finite rank operators. We apply the results to Lorentz spaces and characterize pairs of Lorentz spaces which fall into $\mathcal{M}(T)$.

1. Introduction. Given two weighted Banach function spaces $X = (X, v)$, $Y = (Y, w)$, and an extra pair of weights (ϕ, ψ), we study boundedness and compactness of the generalized Hardy operator $T_{\phi\psi}f(x) = \phi(x)\int_0^x \psi(t)f(t)v(t)\,dt$ considered as an operator from X to Y. If X and Y are weighted Lebesgue spaces, say, $X = L^r(v)$ and $Y = L^p(w)$, it is enough to consider only the usual Hardy operator $Hf(x) = \int_0^x f(t)\,dt$. For this case, the theory is complete. For example, if $1 < p \le r < \infty$, we have the result of Tomaselli [TO], Talenti [T], Muckenhoupt [MU], Bradley [B], Kokilashvili [K] and Maz'ya [M] which states that there is a constant C such that

$$\|Hf\|_{p,w} \le C\|f\|_{r,v} \quad \text{for all } f \in X \tag{1.1}$$

if and only if

$$\sup_{R>0} B(R) = \sup_{R>0} \Big(\int_R^\infty w\Big)^{1/p} \Big(\int_0^R v^{1-r'}\Big)^{1/r'} = B < \infty \tag{1.2}$$

$(r' = r/(r-1))$.

1991 *Mathematics Subject Classification*: 46E30, 47B38, 47G60.
Key words and phrases: weighted Banach function space, Hardy-type operator, compact operator, Lorentz space.
Research of the second author was supported by the Royal Society of London.

It is easy to observe that (1.2) is always necessary for (1.1), but it is not always sufficient. More precisely, (1.2) implies (1.1) if and only if $1 \le p \le r \le \infty$ (cf. [M] and [OK]).

These results were extended by Sawyer [S] to the context of Lorentz spaces. He proved that an analogue of the condition (1.2) is sufficient for boundedness of H from $L^{r,s}(v)$ to $L^{p,q}(w)$ if and only if $q \ge \max(r,s)$.

It is easy to formulate an analogue of (1.2) suitable for a couple of Banach function spaces X, Y and the operator $T_{\phi\psi}$, and to observe that this condition is again always necessary for $T_{\phi\psi} : (X,v) \to (Y,w)$ (cf. [BER]). We introduce a category $\mathcal{M}(T_{\phi\psi})$ of couples of spaces for which that condition is sufficient. In our main result we establish a general criterion for $T_{\phi\psi}$ to be compact from X to Y when $(X,Y) \in \mathcal{M}(T_{\phi\psi})$, and in the non-compact case give upper and lower bounds for the distance of $T_{\phi\psi}$ from the subspace of finite rank maps from X to Y.

We apply these general results to the particular case of Lorentz spaces and characterize those pairs of Lorentz spaces which fall into $\mathcal{M}(T_{\phi\psi})$. It turns out that this class is essentially smaller than $\mathcal{M}(H)$; in particular, the parameter p, which played no role in Sawyer's above-mentioned result, becomes important.

Section 2 contains the statement of the main results in a general Banach function space setting. The material on Lorentz spaces is presented in Section 3, all the proofs being given in Section 4.

We are very grateful to W. D. Evans and D. J. Harris for interesting discussions of this material. In particular, the proof of the sufficiency part of Lemma 2 is based on an idea due to D. J. Harris.

2. Compactness of operators in Banach function spaces. Let v, w, ϕ, ψ be weights, that is, Lebesgue-measurable functions, positive and finite a.e. on $(0,\infty)$. Our concept of Banach function spaces (BFS) follows Luxemburg ([LUX]) but we restrict ourselves to weighted BFS only. We say that a real normed linear space $X = (X,v)$ is a BFS if

(P1) the norm $\|f\|_X = \|f\|_{X,v}$ is defined for every Lebesgue-measurable function f, and $f \in X$ if and only if $\|f\|_X < \infty$; $\|f\|_X = 0$ if and only if $f = 0$ a.e.;

(P2) $\|f\|_X = \|\,|f|\,\|_X$ for all $f \in X$;

(P3) if $0 \le f \le g$ a.e., then $\|f\|_X \le \|g\|_X$;

(P4) if $0 \le f_n \uparrow f$ a.e., then $\|f_n\|_X \uparrow \|f\|_X$;

(P5) if E is a measurable subset of $(0,\infty)$ such that $v(E) := \int_E v < \infty$, then $\|\chi_E\|_X < \infty$ (where χ_E is the characteristic function of the set E);

(P6) for all measurable $E \subset (0,\infty)$ with $v(E) < \infty$ there is a constant $C_E > 0$ such that $\int_E fv \le C_E\|f\|_X$ for all $f \in X$.

Given a BFS $X = (X,v)$, its *associate space* $X' = (X',v)$ given by

$$X' = (X',v) = \Big\{ f : f \text{ is measurable and } \int_0^\infty fgv < \infty \text{ for every } g \in X \Big\}$$

and endowed with the *associate norm*

$$\|f\|_{X'} = \sup\Big\{ \int_0^\infty fgv : \|g\|_X \le 1 \Big\}$$

is also a Banach function space in the sense of (P1)–(P6).

The spaces X, X' are complete normed linear spaces and $X'' = X$. The Hölder inequality

$$\int_0^\infty fgv \le \|f\|_X \|g\|_{X'}$$

holds for all $f \in X$ and $g \in X'$ and is sharp. (For more details we refer the reader to [LUX] or [BS].)

Henceforth we shall work with two weighted BFS $X = (X,v)$ and $Y = (Y,w)$.

Define

$$Tf(x) = T_{\phi\psi}f(x) = \phi(x) \int_0^x \psi(t)f(t)v(t)\,dt\,.$$

We shall write T rather than $T_{\phi\psi}$ when no confusion can arise.

The operator $T_{\phi\psi}$ is a generalization of the usual Hardy operator $Hf(x) = \int_0^x f$ (note that $H = T_{1,1/v}$). The weight v appears in the definition of $T_{\phi\psi}$ for convenience only; the main reason is that then the associate operator $T'_{\phi\psi}$, given by

$$T'g(x) = T'_{\phi\psi}g(x) = \psi(x) \int_x^\infty \phi(t)g(t)w(t)\,dt\,,$$

satisfies

$$\int_0^\infty Tf(x)g(x)w(x)\,dx = \int_0^\infty T'g(x)f(x)v(x)\,dx\,.$$

By a simple exercise, this shows that

(i) T is bounded from X to Y if and only if T' is bounded from Y' to X' (with the same norm as T),

(ii) T is compact from X to Y if and only if T' is compact from Y' to X',

(iii) $\frac{1}{2}\alpha(T) \le \alpha(T') \le 2\alpha(T)$ where

$$\alpha(T) = \inf\{\|T - F\| : F \text{ has finite rank}\}$$

(cf. [EE]).

We start with proving a universal necessary condition.

Suppose that the operator T is bounded from X to Y, i.e. there exists a constant $C > 0$ such that for all $f \in X$ we have

$$\|Tf\|_Y \le C\|f\|_X . \tag{2.1}$$

If $\|f\|_X \le 1$ and $R \in (0,\infty)$ we obtain

$$C \ge C\|f\|_X \ge \|Tf\|_Y \ge \|Tf \cdot \chi_{(R,\infty)}\|_Y \ge \int_0^R \psi f v \cdot \|\phi\chi_{(R,\infty)}\|_Y .$$

Taking the supremum over all such f and R we obtain

$$\mathcal{B} = \sup_{R>0} \mathcal{B}(R) = \sup_{R>0} \|\phi\chi_{(R,\infty)}\|_Y \|\psi\chi_{(0,R)}\|_{X'} \le C < \infty . \tag{2.2}$$

We observe that similar results hold when we replace ϕ and ψ by $\phi\chi_I$ and $\psi\chi_I$, respectively, where I is any subinterval of $(0,\infty)$.

With later applications to compactness in mind (see (4.8) and (4.9) below) this leads us to formulate

LEMMA 1. *Let I be any subinterval of $(0,\infty)$. If the operator*

$$T_I f(x) = \phi(x)\chi_I(x) \int_0^x \psi(t)\chi_I(t) f(t) v(t)\, dt$$

is bounded from (X,v) to (Y,w), then

$$\mathcal{B}_I = \sup_{R\in I} \mathcal{B}_I(R) = \sup_{R\in I} \|\phi\chi_I\chi_{(R,\infty)}\|_Y \|\psi\chi_I\chi_{(0,R)}\|_{X'} < \infty . \tag{2.2*}$$

Moreover, $\mathcal{B}_I \le \|T_I\|_{X\to Y}$.

The condition (2.2)* is thus always necessary for (2.1), but it need not always be sufficient (for example if $X = L^r(v)$, $Y = L^p(w)$, and $r > p$, see [OK]). The family of pairs of Banach function spaces (X,Y) therefore naturally splits into two parts according to whether or not (2.2)* suffices for (2.1).

DEFINITION. We say that a pair of Banach function spaces (X,Y) belongs to the category $\mathcal{M}(T_{\phi\psi})$, and write $(X,Y) \in \mathcal{M}(T_{\phi\psi})$, if for each subinterval I of $(0,\infty)$ the condition (2.2)* guarantees the boundedness of $T_I : (X,v) \to (Y,w)$ and

$$\mathcal{B}_I \le \|T_I\|_{X\to Y} \le K\mathcal{B}_I , \tag{2.3}$$

where $K \ge 1$ is a constant independent of v, w, ϕ, ψ, and I.

EXAMPLES. (i) ([OK]) $(L^p(v), L^q(w)) \in \mathcal{M}(H)$ if and only if $1 \le p \le q \le \infty$;

(ii) ([S]) $(L^{r,s}(v), L^{p,q}(w)) \in \mathcal{M}(H)$ if and only if $q \ge \max(r,s)$;

(iii) ([LP]) $(X, L^\infty) \in \mathcal{M}(H)$ for every X.

Note that in [S] and [LP] the verification of the above conditions is carried out only when $I = (0,\infty)$. However, the methods of proof work equally well for arbitrary intervals I.

In our main result we characterize the compactness of $T_{\phi\psi}$ from X to Y provided that $(X,Y) \in \mathcal{M}(T_{\phi\psi})$.

THEOREM 1. *Let $(X,Y) \in \mathcal{M}(T_{\phi\psi})$. Then $T_{\phi\psi}$ is compact from X to Y if and only if the following two statements are satisfied:*

(i) *we have both*

$$\lim_{a\to 0^+} \sup_{0<R<a} \|\phi\chi_{(R,a)}\|_Y \|\psi\chi_{(0,R)}\|_{X'} = 0 \tag{2.4}$$

and

$$\lim_{b\to\infty} \sup_{b<R<\infty} \|\phi\chi_{(R,\infty)}\|_Y \|\psi\chi_{(b,R)}\|_{X'} = 0 ; \tag{2.5}$$

(ii) *for every $\alpha \in (0,\infty)$ the following two alternatives hold*:

$$\lim_{x\to\alpha^+} \|\phi\chi_{(\alpha,x)}\|_Y = 0 \quad or \quad \lim_{x\to\alpha^+} \|\psi\chi_{(\alpha,x)}\|_{X'} = 0 , \tag{2.6}$$

and

$$\lim_{x\to\alpha^-} \|\phi\chi_{(x,\alpha)}\|_Y = 0 \quad or \quad \lim_{x\to\alpha^-} \|\psi\chi_{(x,\alpha)}\|_{X'} = 0 . \tag{2.7}$$

Remarks. (a) The statement (i) is quite natural and appears in the literature in various modifications (see e.g. [EEH]). On the other hand, it is often replaced by the stronger condition

$$\lim_{R\to 0^+} \mathcal{B}(R) = \lim_{R\to\infty} \mathcal{B}(R) = 0 . \tag{2.8}$$

It is easy to see that this replacement does not affect the theorem as long as X' and Y are spaces with absolutely continuous norms (f *has absolutely continuous* (AC) *norm* in X if $\|f\chi_{E_n}\|_X \to 0$ for each sequence of measurable sets such that $E_n \downarrow \emptyset$, and we say that X is a *space with AC norm* if every $f \in X$ has AC norm in X). However, in a general context, our method fails to prove necessity of (2.8) and we do not know whether or not (2.8) is equivalent to (2.4) and (2.5).

(b) If X' or Y is a space with absolutely continuous norm (for example, if X or Y is reflexive), then the statement (ii) of Theorem 1 becomes redundant as it is automatically satisfied. For example, if X and Y are Lebesgue spaces, we obtain known results as a particular case of Theorem 1.

(c) Another corollary of Theorem 1 is the result of Lai and Pick [LP] which states that H is compact from a BFS (X, v) to L^∞ if and only if the function $1/v$ has a continuous norm in (X', v) (f *has continuous norm* in X if $\lim_{n\to\infty} \|f\chi_{(\alpha,x_n)}\|_X = \lim_{n\to\infty} \|f\chi_{(y_n,\beta)}\|_X = 0$ for every $\alpha \in [0,\infty)$, $\beta \in (0,\infty]$, $x_n \downarrow \alpha$ and $y_n \uparrow \beta$). To see this observe that since $Y = L^\infty$, a space in which no function except that which is identically zero has continuous norm, the second alternatives in (2.6) and (2.7) should be satisfied. This fact together with (2.4) and (2.5), which amount to continuity of the norm of $1/v$ at the endpoints of $(0,\infty)$, are equivalent to the condition of Lai and Pick.

(d) The alternatives in (ii) of Theorem 1 involve certain statements about the continuity of the norm of the weights in question. It has been known for years that compactness of the Hardy-type operators in Banach function spaces is somehow connected with (absolute) continuity of the norm. We refer to the classical paper [LZ], and also to the recent result [LP]. In [LZ] the assumption is made that the image of T lies in that part of the target space which has absolutely continuous norm, and the condition is expressed in terms of uniform absolute continuity. We need less here and also we do not make that assumption, but on the other hand our result does not apply to a class of integral operators as general as in [LZ], so our results overlap somehow.

(e) The connection between compactness and absolute continuity of the norm was also pointed out in [BER]. The author states (without a proof) that if X is an ℓ-concave BFS, and Y is a proper ℓ-convex BFS for some sequence space ℓ, then H is compact from X to Y if and only if (2.4) and (2.5) hold. This statement is, however, wrong: to see this, take X and Y normed by

$$\|f\|_X = \|f\chi_{(0,1)\cup(2,\infty)}\|_{L^2} + \|f\chi_{(1,2)}\|_{L^1},$$
$$\|f\|_Y = \|f\chi_{(0,1)\cup(2,\infty)}\|_{L^2} + \|f\chi_{(1,2)}\|_{L^\infty},$$

and take any pair of weights v, w for which H is compact from $L^2(v)$ to $L^2(w)$. Then X is an ℓ^2-concave BFS, Y is a proper and ℓ^2-convex BFS, but H is not compact as for $\alpha \in [1,2]$ neither (2.6) nor (2.7) is satisfied.

The proof of Theorem 1 has quite a standard form: the operator $T_{\phi\psi}$ is written as a sum of several operators, two of which act near the endpoints and are thus controlled by (2.4) and (2.5), and an "inner" remainder operator, which is compact if and only if (2.6) and (2.7) hold. This result concerning the remainder operator is new and a little surprising. In fact, it is a key to Theorem 1 and is formulated separately, being of independent interest.

LEMMA 2. *Let $a, b \in (0,\infty)$, with $a < b$, be such that $\|\phi\chi_{(a,b)}\|_Y = C_1 < \infty$ and $\|\psi\chi_{(a,b)}\|_{X'} = C_2 < \infty$. The operator T_{ab} defined by*

$$T_{ab}f(x) = \chi_{(a,b)}(x)\phi(x)\int_a^x \psi(t)f(t)v(t)\,dt$$

is compact from X to Y if and only if (2.6) holds for every $\alpha \in [a,b)$ and (2.7) holds for every $\alpha \in (a,b]$.

In the non-compact case it is helpful to estimate the distance of an operator from the compact (or finite rank) operators. We have

THEOREM 2. *Suppose that the set of measurable functions with compact support in $(0,\infty)$ is dense in Y or in X'. Let $(X,Y) \in \mathcal{M}(T_{\phi\psi})$, and let the conditions (2.6) and (2.7) be satisfied. Put*

$$J_L = \lim_{a\to0+} \sup_{0<R<a} \|\phi\chi_{(R,a)}\|_Y \|\psi\chi_{(0,R)}\|_{X'},$$
$$J_R = \lim_{b\to\infty} \sup_{b<R<\infty} \|\phi\chi_{(R,\infty)}\|_Y \|\psi\chi_{(b,R)}\|_{X'},$$

and $J = J_L + J_R$. Then

$$\frac{1}{4}J \le \alpha(T_{\phi\psi}) \le KJ,$$

where K is the constant from (2.3).

Remark. The set of measurable functions with compact support is dense in Y (or X') for example if Y (or X') is a space with absolutely continuous norm (in particular, if Y or X is reflexive). In this case (2.6) and (2.7) are automatically satisfied. (See [BS] for details.)

3. Lorentz spaces—examples. We begin with the definition of the weighted Lorentz space $L^{p,q}(w)$.

If f is a measurable function defined on a measure space $((0,\infty), w\,dx)$, the non-increasing rearrangement f_w^* of f with respect to $w\,dx$ is given by

$$f_w^*(t) = \inf\{\lambda > 0 : w(\{x > 0 : |f(x)| > \lambda\}) \le t\}$$

(we recall that $w(E) = \int_E w\,dx$).

For $p \in (0,\infty)$ and $q \in (0,\infty]$ the Lorentz space $L^{p,q}(w)$ consists of all functions f satisfying $\|f\|_{p,q,w} < \infty$, where

$$\|f\|_{p,q,w} = \begin{cases} \left[\int_0^\infty \frac{q}{p} t^{q/p-1} f_w^*(t)^q\,dt\right]^{1/q} & \text{for } q \in (0,\infty), \\ \sup_{t>0} t^{1/p} f_w^*(t) & \text{for } q = \infty. \end{cases} \tag{3.1}$$

Note that $\|f\|_{p,p,w} = \|f\|_{p,w} = (\int_0^\infty |f|^p w\,dx)^{1/p}$, so we can also consider the space $L^{\infty,\infty}(w)$ as the space $L^\infty(w) = L^\infty$ ($\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in(0,\infty)} |f(x)|$).

The Lorentz space $L^{p,q}(w)$ is a BFS if and only if $p = q = 1$, or $p = q = \infty$, or $p \in (1,\infty)$ and $q \in [1,p]$. If $1 < p < q \le \infty$ then (3.1) is only a quasinorm, but $L^{p,q}(w)$ is then a BFS with respect to another norm, equivalent to (3.1) (see [H] or [BS] for details).

Applying the change of variable $t = w(\{x > 0 : |f(x)| > \lambda\})$ to the right-hand side of (3.1) and integrating by parts we obtain ([H])

(3.2)
$$\|f\|_{p,q,w} = \begin{cases} \Big[\int\limits_0^\infty q\lambda^{q-1} w(\{x > 0 : |f(x)| > \lambda\})^{q/p}\, d\lambda \Big]^{1/q} & \text{for } q \in (0,\infty)\,, \\ \sup_{\lambda>0} \lambda w(\{x > 0 : |f(x)| > \lambda\})^{1/p} & \text{for } q = \infty\,. \end{cases}$$

First we determine which pairs of Lorentz spaces $X = L^{r,s}(v)$ and $Y = L^{p,q}(w)$ fall into the category $\mathcal{M}(T_{\phi\psi})$, where

$$(3.3)\qquad \begin{cases} r = s = 1, \text{ or } r = s = \infty, \text{ or } r \in (1,\infty) \text{ and } s \in [1,\infty]\,, \\ \text{and} \\ p = q = 1, \text{ or } p = q = \infty, \text{ or } p \in (1,\infty) \text{ and } q \in [1,\infty]\,. \end{cases}$$

Let us recall again Sawyer's result mentioned above as Example (ii). It shows that as far as H is concerned, the parameter p plays no part in the problem whether or not $(L^{r,s}(v), L^{p,q}(w))$ falls into $\mathcal{M}(H)$. This is a bit surprising since p realizes the Boyd index of $L^{p,q}(w)$. However, our next result shows that it is no longer true when we switch to general operators $T_{\phi\psi}$.

The result may be summarized as follows.

THEOREM 3. *Let p, q, r, s satisfy the condition* (3.3). *Put $X = L^{r,s}(v)$, $Y = L^{p,q}(w)$ and suppose that one of the following four alternatives is satisfied*:

$$(3.4)\qquad \begin{cases} \text{(i) } \max(r,s) \le \min(p,q), \text{ or} \\ \text{(ii) } \phi = 1 \text{ and } \max(r,s) \le q, \text{ or} \\ \text{(iii) } \psi = 1 \text{ and } s \le \min(p,q), \text{ or} \\ \text{(iv) } \phi = \psi = 1 \text{ and } s \le q\,. \end{cases}$$

Then $(X,Y) = \mathcal{M}(T_{\phi\psi})$ with $K = 4$. Conversely, if (3.4) *is not valid, then there exist functions ϕ, ψ, v and w so that $(X,Y) \notin \mathcal{M}(T_{\phi\psi})$.*

As a consequence of Theorems 1–3 we obtain

THEOREM 4. *Suppose that the parameters p, q, r, s and functions ϕ, ψ satisfy the conditions* (3.3) *and* (3.4). *Then the operator $T = T_{\phi\psi}$ is compact from $L^{r,s}(v)$ to $L^{p,q}(w)$ if and only if*

$$(3.5)\qquad \mathcal{B} = \sup_{R>0} \mathcal{B}(R) = \sup_{R>0} \|\phi\chi_{(R,\infty)}\|_{p,q,w} \|\psi\chi_{(0,R)}\|_{r',s',v} < \infty\,,$$

and one of the following alternatives holds:

(i) *$q < \infty$ or $s > 1$, and*

$$(3.6)\qquad \begin{aligned} &\lim_{a\to 0^+} \sup_{0<R<a} \|\phi\chi_{(R,a)}\|_{p,q,w} \|\psi\chi_{(0,R)}\|_{r',s',v} = 0\,, \\ &\lim_{b\to\infty} \sup_{b<R<\infty} \|\phi\chi_{(R,\infty)}\|_{p,q,w} \|\psi\chi_{(b,R)}\|_{r',s',v} = 0\,; \end{aligned}$$

or

(ii) *$q = \infty$, $s = 1$ and either $p \ne \infty$ or $r \ne 1$; moreover,* (3.6) *holds and for each $\alpha \in (0,\infty)$ we have*

$$(3.7)\qquad \lim_{x\to\alpha^+} \|\phi\chi_{(\alpha,x)}\|_{p,\infty,w} = 0 \quad \textit{or} \quad \lim_{x\to\alpha^+} \|\psi\chi_{(\alpha,x)}\|_{r',\infty,v} = 0\,,$$

and

$$(3.8)\qquad \lim_{x\to\alpha^-} \|\phi\chi_{(x,\alpha)}\|_{p,\infty,w} = 0 \quad \textit{or} \quad \lim_{x\to\alpha^-} \|\psi\chi_{(x,\alpha)}\|_{r',\infty,v} = 0\,.$$

Remarks. (a) If $p = q = 1$ or $p \in (1,\infty)$ and $q \in [1,\infty)$, then as $L^{p,q}(w)$ has absolutely continuous norm (see [BS]) the conditions (2.6), (2.7) are satisfied if $q < \infty$ or $s > 1$.

(b) As L^∞ has no non-trivial function with continuous norm, the operator $T_{\phi\psi} : L^1(v) \to L^\infty(w)$ cannot be compact for any choice of weights.

(c) Using the inequalities

$$\|\cdot\|_{p,\infty,w} \le \|\cdot\|_{p,p,w} = \|\cdot\|_{p,w}$$

and $\|\cdot\|_{r',\infty,v} \le \|\cdot\|_{r',v}$ for $p < \infty$, $r > 1$, and the fact that $L^p(w)$ and $L^{r'}(v)$ have absolutely continuous norm, we see that (3.7) and (3.8) are satisfied if $\phi \in L^p_{\mathrm{loc}}(w)$, $p < \infty$, or $\psi \in L^{r'}_{\mathrm{loc}}(v)$, $r > 1$.

(d) The condition (3.7) or (3.8) is not always satisfied. To see this take $\alpha = 1$, put $v = w = 1$, and $\phi(x) = |1-x|^{-1/p}$, $\psi(x) = |1-x|^{-1/r'}$ in a neighbourhood of 1.

Using the fact that a BFS which has absolutely continuous norm contains a dense subset of functions with compact support we find, as a consequence of (i), the previous Remarks and Theorem 2,

THEOREM 5. *Suppose that the parameters p, q, r, s and functions ϕ, ψ, v, w satisfy the conditions* (3.3)–(3.5), *and $q < \infty$ or $s > 1$. Then*

$$\frac{1}{4} J \le \alpha(T_{\phi\psi}) \le 4J\,,$$

with J from Theorem 2 and with $X = L^{r,s}(v)$ and $Y = L^{p,q}(w)$.

4. Proofs

Proof of Lemma 2. *Sufficiency.* Given $\varepsilon > 0$, we associate every $\alpha \in [a,b]$ with c, d, $c < \alpha < d$, such that

$$(4.1)\qquad \|\phi\chi_{(c,\alpha)}\|_Y < \varepsilon \quad \text{or} \quad \|\psi\chi_{(c,\alpha)}\|_{X'} < \varepsilon$$

and

$$\|\phi\chi_{(\alpha,d)}\|_Y < \varepsilon \quad \text{or} \quad \|\psi\chi_{(\alpha,d)}\|_{X'} < \varepsilon \tag{4.2}$$

with the usual modification if $\alpha = a$ or $\alpha = b$. Then the union of the (c,d)'s taken over all $\alpha \in [a,b]$ is an open covering of $[a,b]$. Choose a finite subcovering $\{(c_i,d_i)\}$ with corresponding interior points α_i. Obviously the points c_i, α_i and d_i divide $[a,b]$ into a finite number of closed intervals, say I_j, $j = 1,\ldots,N$, with pairwise disjoint interiors and on each I_j we have

$$\|\phi\chi_{I_j}\|_Y < \varepsilon \quad \text{or} \quad \|\psi\chi_{I_j}\|_{X'} < \varepsilon. \tag{4.3}$$

Put $I_j = [\beta_{j-1},\beta_j]$ where $\beta_0 = a$ and $\beta_N = b$. Define

$$S_jf(x) = \chi_{I_j}(x)\phi(x)\int_a^{\beta_{j-1}} \psi f v, \quad j = 1,\ldots,N, \quad Sf(x) = \sum_{j=1}^N S_jf(x).$$

Obviously, S is a finite rank operator. We have

$$T_{ab}f(x) - Sf(x) = \sum_{j=1}^N \chi_{I_j}(x)\phi(x)\int_{\beta_{j-1}}^x \psi(t)f(t)v(t)\,dt. \tag{4.4}$$

Hence, by definition of the operator norm,

$$\|T_{ab}-S\|_{X\to Y} \le \sup_{\|f\|_X\le 1}\sup_{\|g\|_{Y'}\le 1}\sum_{j=1}^N \int_{I_j}\phi(x)g(x)w(x)\,dx\cdot\int_{I_j}\psi(t)f(t)v(t)\,dt.$$

Now, let $A_1 = \{j : \|\phi\chi_{I_j}\|_Y < \varepsilon\}$ and $A_2 = \{j : \|\psi\chi_{I_j}\|_{X'} < \varepsilon\}$. Then

$$\begin{aligned}\sup_{\|f\|_X\le 1}\sup_{\|g\|_{Y'}\le 1}\sum_{j\in A_1}\int_{I_j}\phi g w\cdot\int_{I_j}\psi f v &\le \varepsilon\sup_{\|f\|_X\le 1}\sum_{j\in A_1}\int_{I_j}\psi f v \\ &\le \varepsilon\|\psi\chi_{(a,b)}\|_{X'} \le \varepsilon C_2.\end{aligned} \tag{4.5}$$

Similarly,

$$\sup_{\|f\|_X\le 1}\sup_{\|g\|_{Y'}\le 1}\sum_{j\in A_2}\int_{I_j}\phi g w\cdot\int_{I_j}\psi f v \le \varepsilon\|\psi\chi_{(a,b)}\|_Y \le \varepsilon C_1. \tag{4.6}$$

Since $A_1 \cup A_2 = \{1,\ldots,N\}$, we obtain

$$\|T_{ab} - S\|_{X\to Y} \le \varepsilon(C_1 + C_2).$$

As ε was independent of a, b, we see that T_{ab} is a limit of finite rank operators, and therefore compact.

Necessity. Assume that (2.6) does not hold, i.e. there are $\alpha \in [a,b)$ and $\varepsilon > 0$ such that

$$\|\phi\chi_{(\alpha,R_n)}\|_Y \ge \varepsilon \quad \text{and} \quad \|\psi\chi_{(\alpha,R_n)}\|_{X'} \ge \varepsilon$$

for a sequence $R_n \downarrow \alpha$ as $n \to \infty$. By the converse of Hölder's inequality, given $\gamma \in (0,1)$, there exist functions f_n, g_n, supported on $[\alpha, R_n]$, such that $\|f_n\|_X \le 1$, $\|g_n\|_{Y'} \le 1$, and

$$\int_\alpha^{R_n}\psi f_n v \ge \gamma\|\psi\chi_{(\alpha,R_n)}\|_{X'}, \qquad \int_\alpha^{R_n}\phi g_n w \ge \gamma\|\phi\chi_{(\alpha,R_n)}\|_Y.$$

By the continuity of the integral, there exist $\beta_n \in (\alpha, R_n)$ such that

$$\int_{\beta_n}^{R_n}\psi f_n v \ge \gamma^2\|\psi\chi_{(\alpha,R_n)}\|_{X'}, \qquad \int_{\beta_n}^{R_n}\phi g_n w \ge \gamma^2\|\phi\chi_{(\alpha,R_n)}\|_Y.$$

Put $F_n = f_n\chi_{(\beta_n,R_n)}$. Now let m, k, n be such that $R_m < \beta_k < R_k < \beta_n$. Then

$$\begin{aligned}\|T_{ab}F_m - T_{ab}F_n\|_Y &\ge \|\chi_{(\beta_k,R_k)}(T_{ab}F_m - T_{ab}F_n)\|_Y \\ &= \Big\|\chi_{(\beta_k,R_k)}(x)\phi(x)\int_{\beta_m}^x \psi(t)\chi_{(\beta_m,R_m)}(t)f_m(t)v(t)\,dt\Big\|_Y \\ &= \int_{\beta_m}^{R_m}\psi f_m v\cdot\|\phi\chi_{(\beta_k,R_k)}\|_Y \ge \Big(\int_{\beta_m}^{R_m}\psi f_m v\Big)\Big(\int_{\beta_k}^{R_k}\phi g_k w\Big) \\ &\ge \gamma^4\varepsilon^2 > 0.\end{aligned}$$

Hence, T_{ab} takes the bounded set $\{F_n\}$ to a set from which no convergent subsequence can be chosen. Therefore, T_{ab} is not compact.

We have proved the necessity of (2.6). The necessity of (2.7) can be obtained in a similar way using the fact that T_{ab} is compact if and only if $(T_{ab})'$ is compact. It is left to the reader. ∎

Proof of Theorem 1. *Sufficiency.* For $0 < a < b < \infty$ write $\phi_a = \phi\chi_{(0,a)}$, $\phi_{ab} = \phi\chi_{(a,b)}$, $\phi_b = \phi\chi_{(b,\infty)}$, and do the same for ψ. Put $T_a = T_{\phi_a\psi_a}$, $T_b = T_{\phi_b\psi_b}$ and (as above) $T_{ab} = T_{\phi_{ab}\psi_{ab}}$. Then

$$T = T_a + T_b + T_{ab} + (T_{\phi_{ab}\psi_a} + T_{\phi_b\psi_a} + T_{\phi_b\psi_{ab}}). \tag{4.7}$$

Each of the three operators in brackets is one-dimensional and therefore has no role for the purposes of Theorem 1. By (2.2) we have $\|\phi\chi_{(a,b)}\|_Y < \infty$ and $\|\psi\chi_{(a,b)}\|_{X'} < \infty$. By Lemma 2, (ii) is equivalent to the compactness of T_{ab}.

Let $\varepsilon > 0$. Using (2.3) we obtain

$$\|T_a\| \le K\sup_{R\in(0,a)}\|\phi\chi_{(R,a)}\|_Y\|\psi\chi_{(0,R)}\|_{X'}, \tag{4.8}$$

and

$$\|T_b\| \le K\sup_{R\in(b,\infty)}\|\phi\chi_{(R,\infty)}\|_Y\|\psi\chi_{(b,R)}\|_{X'}. \tag{4.9}$$

By (2.4) and (2.5) there exist a, b, with $0 < a < b < \infty$, such that $\|T_a\| < \varepsilon$ and $\|T_b\| < \varepsilon$. By Lemma 2 and (ii), the operator T_{ab} is compact. Hence T is compact, as it is a limit of compact operators.

Necessity. As for necessity of (2.4), suppose the contrary. Then, given $\gamma \in (0,1)$, there are $a_n \to 0_+$, $R_n \in (0, a_n)$, some $\varepsilon > 0$, and functions f_n, with $\|f_n\|_X \le 1$, such that

$$\int_0^{R_n} \psi f_n v \ge \gamma \|\psi\chi_{(0,R_n)}\|_{X'} \quad \text{and} \quad \|\phi\chi_{(R_n,a_n)}\|_Y \|\psi\chi_{(0,R_n)}\|_{X'} \ge \varepsilon .$$

By continuity of the integral, there are $\beta_n \in (0, R_n)$ such that

$$\int_{\beta_n}^{R_n} \psi f_n v \ge \gamma^2 \|\psi\chi_{(0,R_n)}\|_{X'} .$$

Set $F_n = f_n\chi_{(\beta_n,R_n)}$. Then for m and n such that $a_m < \beta_n$ we have

$$\begin{aligned} \|TF_m - TF_n\|_Y &\ge \|\chi_{(R_m,a_m)}(TF_m - TF_n)\|_Y \\ &= \|\chi_{(R_m,a_m)}TF_m\|_Y = \int_{\beta_m}^{R_m} \psi f_m v \cdot \|\phi\chi_{(R_m,a_m)}\|_Y \\ &\ge \gamma^2 \|\psi\chi_{(0,R_m)}\|_{X'} \|\phi\chi_{(R_m,a_m)}\|_Y \ge \gamma^2\varepsilon > 0 , \end{aligned}$$

and thus T is not compact. Necessity of (2.5) can be established in a similar way, obtaining a contradiction with the compactness of T', and is left to the reader. The necessity of (ii) follows from Lemma 2. ∎

Proof of Theorem 2. First, obviously (with the notation of the proof of Theorem 1)

$$\alpha(T) \le \alpha(T_a + T_b) \le \|T_a\| + \|T_b\| ,$$

since T_{ab} is a limit of finite rank operators. Therefore, the upper bound for $\alpha(T)$ follows from (4.8) and (4.9).

As for the lower bound, we shall employ the method from [EH], Lemma 2.2. Suppose first that Y has a dense subset of functions with compact support in $(0,\infty)$. Let $\lambda > \alpha(T)$. Then there is a finite rank operator, say $F : X \to Y$,

$$Ff(x) = \sum_{i=1}^{N} \alpha_i(f) g_i(x) ,$$

with $\alpha_i(f) \in \mathbb{R}$ and $g_i \in Y$, such that $\alpha(T) \le \|T - F\| < \lambda$. Applying the density assumption to the functions $g_1, \ldots, g_N$ we can find a finite rank operator F_0 such that $\operatorname{supp} F_0 f \subset [a_0, b_0]$ for some a_0, b_0, with $0 < a_0 < b_0 < \infty$ and all $f \in X$, and $\|T - F_0\| < \lambda$. Take $f \in X$ such that $\|f\|_X \le 1$.

Then, for $a \in (0, a_0)$,

$$\begin{aligned} \lambda \ge \lambda\|f\chi_{(0,a)}\|_X &\ge \|[T(f\chi_{(0,a)}) - F_0(f\chi_{(0,a)})]\chi_{(0,a)}\|_Y \\ &= \|T_a f\|_Y \ge \int_0^R \psi f v \cdot \|\phi\chi_{(R,a)}\|_Y \end{aligned}$$

for $R \in (0,a)$. Taking the supremum over all such f and R we obtain

$$\lambda \ge \sup_{R\in(0,a)} \|\phi\chi_{(R,a)}\|_Y \|\psi\chi_{(0,R)}\|_{X'} .$$

As the left hand side does not depend on a we have $\lambda \ge J_L$. Similarly we obtain $\lambda \ge J_R$ and so $\lambda \ge \frac{1}{2}(J_L + J_R)$, which implies $\alpha(T) \ge \frac{1}{2}(J_L + J_R)$. In the case when X' has a dense subset of functions with compact support in $(0,\infty)$, we proceed in a similar way with the associate operator T', using the fact that $\alpha(T') \le 2\alpha(T)$. ∎

For the proof of Theorem 3 we shall need some preliminary work. First, if $q < s$, then

$$\|f\|_{p,s,w} \le \|f\|_{p,q,w}, \quad f \in L^{p,q}(w) . \tag{4.10}$$

Further, for the norm of the characteristic function we have $\|\chi_E\|_{p,q,w} = w(E)^{1/p}$ ($p \in [1,\infty)$). A key to the sufficiency in Theorem 3 is the following lemma, basically due to Chung, Hunt and Kurtz [CHK] (see also [S]).

LEMMA 3. *Let $0 < p < \infty$, $0 < q \le \infty$, $(0,\infty) = \bigcup E_k$, E_k disjoint and measurable.*

(i) *Let $\sigma \ge \max(p,q)$. Then*

$$\sum_k \|\chi_{E_k} f\|_{p,q,w}^{\sigma} \le \|f\|_{p,q,w}^{\sigma} . \tag{4.11}$$

(ii) *Let $\sigma \le \min(p,q)$. Then*

$$\sum_k \|\chi_{E_k} f\|_{p,q,w}^{\sigma} \ge \|f\|_{p,q,w}^{\sigma} . \tag{4.12}$$

In the necessity part of Theorem 3 we shall make use of functions $f_{\alpha,\beta}$ introduced by Sawyer [S]. Put

$$f_{\alpha,\beta}(x) = x^{-\alpha}(1 + |\log x|)^{-\beta} .$$

The following properties of $f_{\alpha,\beta}$ can be easily verified. We write $A \approx B$ if A/B is bounded from above and below by positive constants independent of x. We have

$$\int_0^\infty f_{\alpha,\beta} < \infty \Leftrightarrow \alpha = 1 \text{ and } \beta > 1\,; \tag{4.13}$$

$$\int_x^\infty f_{\alpha,\beta} \approx f_{\alpha-1,\beta}(x) \quad \text{if } \alpha > 1\,; \tag{4.14}$$

$$\int_0^x f_{\alpha,\beta} \approx f_{\alpha-1,\beta}(x) \quad \text{if } \alpha < 1\,. \tag{4.15}$$

Proof of Theorem 3. *Sufficiency.* We shall only prove that (i) of (3.4) implies that $(X,Y) \in \mathcal{M}(T_{\phi\psi})$. Other cases can be handled in the same way as in [S] (proof of Theorem 1) applied either to T or to T'. Let us recall that in the case (ii) Sawyer's proof is based on the fact that $\phi = 1$, which implies that the function $\phi(x)\int_0^x \psi f v$ is non-decreasing in x, and $\|\phi\chi_{(R,\infty)}\|_{p,q,w} = (\int_R^\infty w)^{1/p}$.

We give the proof for the interval $(0,\infty)$, that for intervals $I \subset (0,\infty)$ being similar.

Suppose that (i) of (3.4) is satisfied and fix $f \ge 0$. Let m be an integer such that $\int_0^\infty \psi f v \in (2^m, 2^{m+1}]$. Then there is an increasing sequence $\{x_k\}_{k=-\infty}^m$ such that

$$2^k = \int_0^{x_k} \psi f v = \int_{x_k}^{x_{k+1}} \psi f v \quad \text{for } k \le m-1\,, \tag{4.16}$$

and

$$2^m = \int_0^{x_m} \psi f v\,. \tag{4.17}$$

If we put $E_k = [x_k, x_{k+1})$, $k \le m-1$, and $x_{m+1} = \infty$, then

$$\bigcup_{k\le m} E_k = (0,\infty), \qquad E_k \text{ disjoint.} \tag{4.18}$$

If $\int_0^\infty \psi f v = \infty$, then (4.16) holds for all integers k and (4.18) remains valid. (In this case put $m = \infty$.) By the assumption, (4.16), and (4.17), we have

$$Tf(x) \le \phi(x)\cdot 2^{k+1} \quad \text{for } x \in E_k,\ k \le m\,. \tag{4.19}$$

Choose σ so that $\max(r,s) \le \sigma \le \min(p,q)$. Then by (4.18), (4.12), (4.19), (4.16), Hölder's inequality and (4.11), we have, with $\mathcal{B}$ given by (2.2),

$$\|Tf\|_{p,q,w}^\sigma = \Big\|\sum_{k\le m} Tf\cdot\chi_{E_k}\Big\|_{p,q,w}^\sigma \le \sum_{k\le m} \|Tf\chi_{E_k}\|_{p,q,w}^\sigma$$

$$\le \sum_{k\le m} 2^{\sigma(k+1)}\|\phi\chi_{E_k}\|_{p,q,w}^\sigma = 4^\sigma \sum_{k\le m} 2^{\sigma(k-1)}\|\phi\chi_{E_k}\|_{p,q,w}^\sigma$$

$$\le 4^\sigma \sum_{k\le m}\Big(\int_{x_{k-1}}^{x_k} \psi f v\Big)^\sigma \|\phi\chi_{(x_k,\infty)}\|_{p,q,w}^\sigma$$

$$\le 4^\sigma \sum_{k\le m} \|f\chi_{E_{k-1}}\|_{r,s,v}^\sigma \|\psi\chi_{(0,x_k)}\|_{r',s',v}^\sigma \|\phi\chi_{(x_k,\infty)}\|_{p,q,w}^\sigma$$

$$\le (4\mathcal{B})^\sigma \|f\|_{r,s,v}^\sigma\,,$$

proving the sufficiency part.

Necessity. For the converse, assume that (3.4) is not valid. We consider several cases separately and give a list of corresponding counterexamples. The first two cases were treated by Sawyer [S]. Let us recall them for the sake of completeness.

(a) If $q < s$, put $v = \phi = \psi \equiv 1$ on $(0,\infty)$, $w(x) = (p/r')x^{-p/r'-1}$, $f(x) = f_{\alpha,\beta}(x)$, where $\alpha = 1/r$, $q < 1/\beta < s$.

(b) If $s \le q < r$, put $\phi = f \equiv 1$ on $(0,\infty)$, $v(x) = f_{1,\beta}(x)$, $\psi(x) = f_{-1,-\beta}(x)$, where $\beta \in (1, r/q)$, $w(x) = f_{\gamma,\delta}(x)$, where $\gamma = p+1$, $\delta = \beta p/r$.

Sawyer proved that in both cases (a) and (b), $\|f\|_{r,s,v} < \infty$, $\mathcal{B} < \infty$, and $\|Tf\|_{p,q,w} = \infty$. (Note that obviously $\mathcal{B}_I \le \mathcal{B}$ for all I.)

Now, the parameter p will come into the picture.

(c) If $p < s \le q$, put

$$\psi \equiv 1 \text{ on } (0,\infty)\,,$$

$$\phi(x) = f_{r-1,-\delta}(x), \quad v(x) = f_{1-r,\delta}(x), \quad \text{where} \quad \delta = r\frac{s+3p}{4sp}\,,$$

$$w(x) = f_{1,\beta}(x), \quad \text{where} \quad \beta = \frac{p}{2}\Big(\frac{1}{s}+\frac{1}{p}\Big)\,,$$

$$f(x) = f_{1,0}(x)\,.$$

We shall show that $\|f\|_{r,s,v} < \infty$, $\mathcal{B} < \infty$, and $\|Tf\|_{p,q,w} = \infty$. Note that we can suppose that $r > 1$.

Using (4.10) and $p < q$ we have

$$\|\phi\chi_{(R,\infty)}\|_{p,q,w} \le \|\phi\chi_{(R,\infty)}\|_{p,w} = \Big(\int_R^\infty f_{p(r-1)+1,\beta-\delta p}\Big)^{1/p} \approx f_{r-1,\beta/p-\delta}(R)$$

by (4.14). Further,

$$\|\psi\chi_{(0,R)}\|_{r',s',v} = \|\chi_{(0,R)}\|_{r',s',v} = \Big(\int_0^R v\Big)^{1/r'} \approx f_{1-r,\delta/r'}(R)$$

by (4.15). Then

$$\sup_{R>0} \mathcal{B}(R) = \sup_{R>0} \|\phi\chi_{(R,\infty)}\|_{p,q,w}\|\psi\chi_{(0,R)}\|_{r',s',v} \leq C \sup_{R>0}(1+|\log R|)^{(p-s)/(4sp)} < \infty .$$

Let us calculate the norm of f. By (4.15) we obtain

$$\|f\|_{r,s,v} = \Big(\int_0^\infty st^{s-1}v(\{x>0: 1/x > t\})^{s/r}dt\Big)^{1/s} = \Big(\int_0^\infty st^{s-1}\Big(\int_0^{1/t} v(\tau)\,d\tau\Big)^{s/r}dt\Big)^{1/s} \approx \Big(\int_0^\infty st^{s-1}(f_{r,\delta}(t))^{s/r}dt\Big)^{1/s} = \Big(\int_0^\infty f_{1,\delta s/r}(t)\,dt\Big)^{1/s} < \infty,$$

by (4.13), as $\delta s/r = (s+3p)/(4p) > 1$. Now, we calculate $\|Tf\|_{p,q,w}$. Using (4.15) we have

$$Tf(x) = f_{r-1,-\delta}(x)\int_0^x f_{1,0}(t)f_{1-r,\delta}(t)\,dt \approx f_{r-1,-\delta}(x)f_{1-r,\delta}(x) = 1.$$

Then

$$\|Tf\|_{p,q,w} = \Big(\int_0^\infty f_{1,\beta}(x)\,dx\Big)^{1/p} = \infty,$$

as $\beta = (s+p)/(2s) < 1$.

The remaining case is when $p < r$.

(d) If $p < r$, we put $\phi(x) = v(x) = w(x) = 1/x$, $\psi(x) = x$, $f(x) = \chi_{(a,b)}(x)$, $0 < a < b < \infty$.

First, one can easily check that

$$\Big\|\frac{1}{x}\chi_{(R,\infty)}(x)\Big\|_{p,q,1/x} \approx R^{-1} \quad \text{and} \quad \|x\chi_{(0,R)}(x)\|_{r',s',1/x} \approx R,$$

which implies that $\mathcal{B} < \infty$. Further, we see that

$$Tf(x) = \Big(1-\frac{a}{x}\Big)\chi_{(a,b)}(x) + \frac{b-a}{x}\chi_{(b,\infty)}(x),$$

whence

$$\{x>0: Tf(x) > \lambda\} = \Big(\frac{a}{1-\lambda}, \frac{b-a}{\lambda}\Big) \quad \text{for } \lambda < 1-\frac{a}{b}.$$

With the choice $\lambda_0 = (b-a)/(2b)$ we obtain $w(\{Tf > \lambda_0\}) = \log(1+b/a)$, and thus

$$\|Tf\|_{p,q,w} \geq \|Tf\|_{p,\infty,w} \geq \lambda_0 w(\{Tf > \lambda_0\})^{1/p} = \frac{b-a}{2b}\Big(\log\Big(1+\frac{b}{a}\Big)\Big)^{1/p}.$$

On the other hand,

$$\|f\|_{r,s,v} = (\log(b/a))^{1/r}.$$

As $1/r < 1/p$, letting $b/a \to \infty$ we see that (2.3) is satisfied for no K and hence $(X,Y) \notin \mathcal{M}(T_{\phi\psi})$.

Now, the case (iv) of (3.4) is covered by the counterexample (a), the case (iii) is covered by (a) and (c), and (ii) is covered by (a) and (b). For the case (i) of (3.4) we have to use all the counterexamples (a)–(d). ∎

Remark. The counterexamples (b) and (c) also work without the assumption $s \leq q$. The assumption is there only to simplify the calculation.

References

[BS] C. Bennett and R. Sharpley, *Interpolation of Operators*, Pure Appl. Math. 129, Academic Press, 1988.

[BER] E. I. Berezhnoĭ, *Weighted inequalities of Hardy type in general ideal spaces*, Soviet Math. Dokl. 43 (1991), 492–495.

[B] J. S. Bradley, *Hardy inequality with mixed norms*, Canad. Math. Bull. 21 (1978), 405–408.

[CHK] H.-M. Chung, R. A. Hunt and D. S. Kurtz, *The Hardy–Littlewood maximal function on L(p, q) spaces with weights*, Indiana Univ. Math. J. 31 (1982), 109–120.

[EE] D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Oxford Univ. Press, Oxford, 1987.

[EEH] D. E. Edmunds, W. D. Evans and D. J. Harris, *Approximation numbers of certain Volterra integral operators*, J. London Math. Soc. 38 (1988), 471–489.

[EH] W. D. Evans and D. J. Harris, *Sobolev embeddings for generalized ridged domains*, Proc. London Math. Soc. 54 (1987), 141–175.

[H] R. A. Hunt, *On L(p, q) spaces*, Enseign. Math. 12 (1966), 249–276.

[K] V. M. Kokilashvili, *On Hardy's inequalities in weighted spaces*, Soobshch. Akad. Nauk. Gruzin. SSR 96 (1979), 37–40 (in Russian).

[LP] Q. Lai and L. Pick, *The Hardy operator, L_∞, and BMO*, J. London Math. Soc. (2) 48 (1993), 167–177.

[LUX] W. A. J. Luxemburg, *Banach Function Spaces*, thesis, Delft, 1955.

[LZ] W. A. J. Luxemburg and A. C. Zaanen, *Compactness of integral operators in Banach function spaces*, Math. Ann. 149 (1963), 150–180.

[M] V. G. Maz'ya, *Sobolev Spaces*, Springer, Berlin, 1985.

[MU] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. 44 (1972), 31–38.

[OK] B. Opic and A. Kufner, *Hardy-type Inequalities*, Pitman Res. Notes Math. Ser. 219, Longman Sci. & Tech., Harlow, 1990.

[S] E. T. Sawyer, *Weighted Lebesgue and Lorentz norm inequalities for the Hardy operator*, Trans. Amer. Math. Soc. 281 (1984), 329–337.
[T] G. Talenti, *Osservazioni sopra una classe di disuguaglianze*, Rend. Sem. Mat. Fis. Milano 39 (1969), 171–185.
[TO] G. Tomaselli, *A class of inequalities*, Boll. Un. Mat. Ital. 21 (1969), 622–631.

David E. Edmunds
CENTRE FOR MATHEMATICAL ANALYSIS
AND ITS APPLICATIONS
THE UNIVERSITY OF SUSSEX
FALMER
BRIGHTON BN1 9QH, ENGLAND
E-mail: MMFB7@SYMA.SUSSEX.AC.UK

Petr Gurka
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF AGRICULTURE
160 21 PRAHA 6, CZECH REPUBLIC
E-mail: GURKA@CSEARN.BITNET

Luboš Pick
SCHOOL OF MATHEMATICS
UNIVERSITY OF WALES
COLLEGE OF CARDIFF
SENGHENYDD ROAD
CARDIFF CF2 4AG, UK

and

MATHEMATICAL INSTITUTE
ACADEMY OF SCIENCES
OF THE CZECH REPUBLIC
ŽITNÁ 25, 115 67 PRAHA 1
CZECH REPUBLIC
E-mail: PICK@CSEARN.BITNET

Received June 1, 1993 (3111)

STUDIA MATHEMATICA 109 (1) (1994)

Spectrum preserving linear mappings in Banach algebras

by

B. AUPETIT (Québec) and H. du T. MOUTON (Bloemfontein)

Abstract. Let A and B be two unitary Banach algebras. We study linear mappings from A into B which preserve the polynomially convex hull of the spectrum. In particular, we give conditions under which such surjective linear mappings are Jordan morphisms.

1. Introduction. The theory of spectrum preserving linear mappings originates from Hua's theorem on fields which has very interesting geometrical applications. This theorem says that an additive mapping $\sigma : K_1 \to K_2$, where K_1, K_2 are two fields, such that $\sigma(1) = 1$, and $\sigma(x^{-1}) = \sigma(x)^{-1}$ for $x \neq 0$, is an isomorphism or an anti-isomorphism. If ϕ is a linear mapping from a Banach algebra A_1 into another one A_2 such that $\phi(1) = 1$ and $\phi(x)^{-1} = \phi(x^{-1})$ for x invertible, then using exponentials it is easy to prove that ϕ is a Jordan morphism, that is, $\phi(x^2) = \phi(x)^2$ for every x in A. In the situation of Banach algebras the problem was enlarged by I. Kaplansky [5] to the following one: if ϕ is linear, satisfies $\phi(1) = 1$ and ϕ maps invertible elements into invertible elements, is it true that ϕ is a Jordan morphism? By Lemma 4, page 30 of [1], this question is equivalent to the study of linear mappings which preserve the spectrum.

Almost at the same time, in 1967–1968, A. Gleason, J.-P. Kahane and W. Żelazko proved that if A and B are Banach algebras, with B commutative and semisimple and if $\phi : A \to B$ is a linear mapping that satisfies $\phi(1) = 1$ and x invertible in A implies $\phi(x)$ invertible in B, then ϕ is a homomorphism (see [2], pp. 69–70, for the simple and elegant proof given by M. Roitman and Y. Sternfeld).

In the case of matrices the general problem is justified by a result of M. Marcus and R. Purves [6] which says that if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear mapping which preserves eigenvalues and their multiplicity then ϕ is either of the form $\phi(T) = ATA^{-1}$ or $\phi(T) = AT^tA^{-1}$ (incidentally, the same conclusion is true if ϕ preserves only the greatest eigenvalue).

1991 *Mathematics Subject Classification*: 46Hxx, 47A10, 17C65.