The separable case of 5.9 reduces via a result of [DT] to the verification that G is an absolute neighborhood retract; it is, however, unclear whether such a G must be even locally connected in dimension 1.

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Almost everywhere convergence of Laguerre series

by

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Abstract. Let $a \in \mathbb{Z}^+$ and $f \in L^p(\mathbb{R}^+), 1 \le p \le \infty$. Denote by c_j the inner product of f and the Laguerre function $\mathcal{L}_j^a$. We prove that if $\{c_j\}$ satisfies

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{n<j\le\lambda n} |\Delta^k c_j| j^{k/2-1/4} = 0 \quad \text{and} \quad |c_j| j^{k/2-1/4} = o(1) \quad \text{as } j \to \infty$$

for some $k \in \mathbb{N}$, then the Laguerre series $\sum c_j \mathcal{L}_j^a$ converges to f almost everywhere.

1. Introduction. Let $L_n^a(t)$ denote the nth Laguerre polynomial of order a on $\mathbb{R}$,

$$L_n^a(t) = \frac{1}{n!} t^{-a} e^t \frac{d^n}{dt^n}(t^{n+a} e^{-t}), \quad a > -1,\ n = 0, 1, 2, \ldots,$$

or, equivalently,

$$L_n^a(t) = \sum_{k=0}^n \frac{(-1)^k}{k!} \binom{n+a}{n-k} t^k, \quad a > -1,\ n = 0, 1, 2, \ldots$$

The Laguerre polynomials form a complete orthogonal system in $L^2(\mathbb{R}^+, t^a e^{-t} dt)$ and satisfy the summation formula [13, p. 102]

$$\sum_{k=0}^n L_k^a(t) = L_n^{a+1}(t). \tag{1}$$

It is well known (cf. [9, p. 348]) that

$$|L_n^a(t)| = O(e^{t/2} t^{-a/2-1/4} n^{a/2-1/4}).$$

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Let

$$\mathcal{L}_n^a(t) = \sqrt{\frac{n!}{(n+a)!}}\, e^{-t/2} t^{a/2} L_n^a(t)$$

be the corresponding Laguerre functions. Then $\{\mathcal{L}_n^a(t)\}_{n=0}^\infty$ is an orthonormal basis in $L^2(\mathbb{R}^+, dt)$; moreover, $|\mathcal{L}_n^a(t)| = O(n^{-1/4} t^{-1/4})$.

The problem of the mean convergence of Laguerre expansions has been studied by many authors in the last four decades. The most celebrated result in this direction was given by Askey and Wainger [1]. They proved that for $a \ge 0$, the partial sums of the Laguerre expansion with respect to the system $\{\mathcal{L}_n^a(t)\}$ of a function $f \in L^p(\mathbb{R}^+)$ converge to f in the L^p norm if and only if $4/3 < p < 4$. Muckenhoupt [6–8] extended this result to all $a > -1$ and weighted L^p; he also proved the almost everywhere convergence of Abel's means for expansion with respect to the system $\{L_n^a(t)\}$ of a function $f \in L^2(\mathbb{R}^+, t^a e^{-t} dt)$ for $a > -1$.

Recently, Długosz [4] investigated mean convergence as well as almost everywhere convergence of Riesz summability method for Laguerre series. Her result is

THEOREM 1. *Let $a \in \mathbb{Z}^+ \equiv \mathbb{N} \cup \{0\}$. For $f \in L^p(\mathbb{R}^+)$ we have*

$$\lim_{w\to\infty} \sum_{j\le w} \left(1 - \frac{j}{w}\right)^N c_j \mathcal{L}_j^a(t) = f(t)$$

almost everywhere if $1 \le p \le \infty$ and $N \ge 10$, and in L^p norm if $1 \le p < \infty$ and $N \ge 4$, where $c_j = (f, \mathcal{L}_j^a) = \int_0^\infty f(t)\mathcal{L}_j^a(t)\,dt$.

The proof of Theorem 1 is based on the fact that the Laguerre functions of an integral order $a = 0, 1, 2, \ldots$, appear (nearly) as eigenfunctions of the sublaplacian acting on the space of functions on the Heisenberg group. Later on, basing on Długosz's idea, Stempak [10–12] not only filled out the gap from $a \in \mathbb{Z}^+$ to $a \in \mathbb{R}^+$ by using a construction of generalized twisted convolution, but also extended it to the Laguerre systems $\{\widetilde{L}_n^a(t)\}_{n=0}^\infty$ and $\{l_n^a(t)\}_{n=0}^\infty$.

For the sake of convenience, we use c_j to denote the inner product of f and $\mathcal{L}_j^a$,

$$c_j = (f, \mathcal{L}_j^a) = \int_0^\infty f(t)\mathcal{L}_j^a(t)\,dt,$$

and

$$\Delta^0 c_j = c_j, \quad \Delta^k c_j = \Delta^{k-1} c_j - \Delta^{k-1} c_{j+1} \quad \text{for } k \in \mathbb{N}.$$

The purpose of this paper is to establish the following result.

THEOREM 2. *Let $a \in \mathbb{Z}^+$ and $f \in L^p(\mathbb{R}^+)$, $1 \le p \le \infty$. If $\{c_j\}$ satisfies*

$$\begin{cases} \lim_{\lambda\downarrow 1} \overline{\lim}_{n\to\infty} \sum_{n<j\le\lambda n} |\Delta^k c_j| j^{k/2-1/4} = 0, \\ |c_j| j^{k/2-1/4} = o(1) \quad \text{as } j \to \infty \end{cases} \tag{2}$$

for some $k \in \mathbb{N}$, then $\sum_{j=0}^\infty c_j \mathcal{L}_j^a(t) = f(t)$ almost everywhere.

The approach we use is rather straightforward, and relies only on elementary calculation, which was first given by the first author [2, 3] for trigonometric series. In the next section we review some properties of Cesàro means and Riesz means, and in Section 3 we give the proof of Theorem 2. Finally, we mention that C will be used to denote a constant, $C_{a,\mu,t}$ the constant dependent on a, μ, t, and so on. All of these constants are not necessarily the same at each occurrence.

2. Preliminaries.

For $\alpha > -1$, define the nth (C, α) *mean* of the sequence $\{s_n(t)\}$ by

$$\sigma_n^\alpha(t) = \sum_{j=0}^n \frac{A_{n-j}^{\alpha-1}}{A_n^\alpha} s_j(t) = \sum_{j=0}^n \frac{A_{n-j}^\alpha}{A_n^\alpha} c_j \mathcal{L}_j^a(t),$$

where

$$A_n^\alpha = \binom{n+\alpha}{n} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \approx \frac{n^\alpha}{\Gamma(\alpha+1)}$$

and

$$s_n(t) = \sum_{j=0}^n c_j \mathcal{L}_j^a(t).$$

The sum $\sigma_n^\alpha(t)$ is called the nth *Cesàro mean* of $\{s_n(t)\}$ of order α. Whenever α is a positive integer, $\sigma_n^\alpha(t)$ can be rewritten as

$$\sigma_n^\alpha(t) = \sum_{j=0}^n \left(1 - \frac{j}{n+1}\right)\left(1 - \frac{j}{n+2}\right)\cdots\left(1 - \frac{j}{n+\alpha}\right) c_j \mathcal{L}_j^a(t).$$

We say that $\sum_{j=0}^\infty c_j \mathcal{L}_j^a(t)$ is *summable* (C, α) to $s(t)$ if $\lim_{n\to\infty} \sigma_n^\alpha(t) = s(t)$.

For each α and n, σ_n^α can be regarded as a linear operator acting on the sequence $\{s_m(t)\}$. In this sense, we write $\sigma_n^\alpha(\{s_m(t)\})$ instead of $\sigma_n^\alpha(t)$. It is well known that

$$\sigma_n^\alpha(\{\sigma_m^\beta(t)\}) \to A, \quad \sigma_n^\beta(\{\sigma_m^\alpha(t)\}) \to A, \quad \text{and} \quad \sigma_n^{\alpha+\beta}(t) \to A$$

are equivalent (cf. [5, Chapter 5]).

The *Riesz typical means* with respect to $\mathcal{L}_j^a$ of order $k \in \mathbb{Z}^+$ are defined by

$$R_w^k(t) = \sum_{j\le w} \left(1 - \frac{j}{w}\right)^k c_j \mathcal{L}_j^a(t),$$

where $w>0$. In this notation, the first part of Theorem 1 can be written as

$$(3)\qquad \lim_{w\to\infty} R_w^N(t)=f(t)\quad \text{almost everywhere if } 1\le p\le\infty \text{ and } N\ge 10\,.$$

It is known that the almost everywhere convergence of the Riesz means of an order $k\ge 0$ is equivalent to the almost everywhere convergence of the Cesàro means of the same order (cf. [5, Chapter 5]). Furthermore, we have the following result.

LEMMA 3. *Let $k\in\mathbb{Z}^+$. If $c_j=o(j^{1/4})$, then*

$$\lim_{n\to\infty}(\sigma_n^k(t)-R_n^k(t))=0\quad \textit{for all } t\in\mathbb{R}^+.$$

Proof. Fix $t\in\mathbb{R}^+$. Since $|\mathcal{L}_j^a(t)|=O(j^{-1/4}t^{-1/4})$, we have $c_j\mathcal{L}_j^a(t)=o(1)$ as $j\to\infty$. It follows from the definitions of σ_n^k and R_n^k that

$$\begin{aligned}
&|\sigma_n^k(t)-R_n^k(t)|\\
&\le \sum_{j=0}^{n}\left|\left(\left(1-\frac{j}{n+1}\right)\cdots\left(1-\frac{j}{n+k}\right)-\left(1-\frac{j}{n}\right)^k\right)c_j\mathcal{L}_j^a(t)\right|\\
&\le \sum_{j=0}^{n}\left|\left(\left(1-\frac{j}{n+1}\right)\cdots\left(1-\frac{j}{n+k-1}\right)-\left(1-\frac{j}{n}\right)^{k-1}\right)c_j\mathcal{L}_j^a(t)\right|\\
&\quad+\frac{k}{n}\sum_{j=0}^{n}|c_j\mathcal{L}_j^a(t)|\,.
\end{aligned}$$

Using induction on k, we get

$$\begin{aligned}
|\sigma_n^k(t)-R_n^k(t)|&\le\left(\frac{k}{n}+\frac{k-1}{n}+\ldots+\frac{1}{n}\right)\sum_{j=0}^{n}|c_j\mathcal{L}_j^a(t)|\\
&=\frac{k(k+1)}{2}\left(\frac{1}{n}\sum_{j=0}^{n}|c_j\mathcal{L}_j^a(t)|\right)\\
&\to 0\quad \text{as } n\to\infty\,.
\end{aligned}$$

COROLLARY 4. *If $c_j=o(j^{1/4})$, then*

$$\lim_{n\to\infty}\sigma_n^\alpha(\{R_m^\beta(t)\})=A\quad \textit{and}\quad \lim_{w\to\infty}R_w^{\alpha+\beta}(t)=A$$

are equivalent for all nonnegative integers α,β.

3. Proof of the main result. Throughout this section, we assume $a\in\mathbb{Z}^+$. For $y>w>0$,

$$\frac{y^N}{(y-w)^N}\left(1-\frac{j}{y}\right)^N=\sum_{s=0}^{N}\binom{N}{s}\left(\frac{y}{w}-1\right)^{-s}\left(1-\frac{j}{w}\right)^s.$$

Multiplying by $c_j\mathcal{L}_j^a(t)$ on both sides and summing over $0\le j\le w$, we get

$$\begin{aligned}
&\frac{y^N}{(y-w)^N}\sum_{j\le w}\left(1-\frac{j}{y}\right)^N c_j\mathcal{L}_j^a(t)\\
&\qquad=\sum_{s=0}^{N}\binom{N}{s}\left(\frac{y}{w}-1\right)^{-s}\sum_{j\le w}\left(1-\frac{j}{w}\right)^s c_j\mathcal{L}_j^a(t)\\
&\qquad=\sum_{s=0}^{N}\binom{N}{s}\left(\frac{y}{w}-1\right)^{-s}R_w^s(t)\\
&\qquad=\sum_{j\le w}c_j\mathcal{L}_j^a(t)+\sum_{s=1}^{N}\binom{N}{s}\left(\frac{y}{w}-1\right)^{-s}R_w^s(t)\,,
\end{aligned}$$

and hence

$$\begin{aligned}
(4)\qquad &\frac{y^N}{(y-w)^N}\Big(\sum_{j\le y}-\sum_{w<j\le y}\Big)\left(1-\frac{j}{y}\right)^N c_j\mathcal{L}_j^a(t)\\
&\qquad=\sum_{j\le w}c_j\mathcal{L}_j^a(t)+\sum_{s=1}^{N}\binom{N}{s}\left(\frac{y}{w}-1\right)^{-s}R_w^s(t)\,.
\end{aligned}$$

On the other hand, we multiply the equality

$$\frac{y^N}{(y-w)^N}=\sum_{s=0}^{N}\binom{N}{s}\left(\frac{y}{w}-1\right)^{-s}=1+\sum_{s=1}^{N}\binom{N}{s}\left(\frac{y}{w}-1\right)^{-s}$$

by $f(t)$ to obtain

$$(5)\qquad \frac{y^N}{(y-w)^N}f(t)=f(t)+\sum_{s=1}^{N}\binom{N}{s}\left(\frac{y}{w}-1\right)^{-s}f(t)\,.$$

The equation (4) minus (5) leaves

$$\begin{aligned}
&\frac{y^N}{(y-w)^N}\Big(\{R_y^N(t)-f(t)\}-\sum_{w<j\le y}\left(1-\frac{j}{y}\right)^N c_j\mathcal{L}_j^a(t)\Big)\\
&\quad=\Big(\sum_{j\le w}c_j\mathcal{L}_j^a(t)-f(t)\Big)+\Big(\sum_{s=1}^{N}\binom{N}{s}\left(\frac{y}{w}-1\right)^{-s}\{R_w^s(t)-f(t)\}\Big).
\end{aligned}$$

Let $\lambda>1$. We set $w=m\in\mathbb{N}$, $y=\lambda m$, and take the operator σ_n^9 on both sides of the above equality to get

(6)
$$\left(\frac{\lambda}{\lambda-1}\right)^N\left[\sigma_n^9(\{R_{\lambda m}^N(t)-f(t)\})-\sigma_n^9\left(\left\{\sum_{m<j\le\lambda m}\left(1-\frac{j}{\lambda m}\right)^N c_j\mathcal{L}_j^a(t)\right\}\right)\right]$$
$$=\sigma_n^9(t)-f(t)+\sum_{s=1}^N\binom{N}{s}(\lambda-1)^{-s}\sigma_n^9(\{R_m^s(t)-f(t)\}).$$

Consider $f\in L^p(\mathbb{R}^+)$, $1\le p\le\infty$, and $N\ge 10$. By (3),
$$\lim_{m\to\infty}\{R_{\lambda m}^N(t)-f(t)\}=0\quad\text{almost everywhere,}$$
and hence for almost all $t\in\mathbb{R}^+$,
$$\sigma_n^9(\{R_{\lambda m}^N(t)-f(t)\})\to 0\quad\text{as } n\to\infty.$$
For $1\le s\le N$, we have $9+s\ge 10$. Thus, for almost all $t\in\mathbb{R}^+$, Corollary 4 and (3) yield
$$\sigma_n^9(\{R_m^s(t)-f(t)\})=\sigma_n^9(\{R_m^s(t)\})-f(t)\to 0\quad\text{as } n\to\infty.$$
Taking limit superior on both sides of (6), we obtain

LEMMA 5. *Let $f\in L^p(\mathbb{R}^+)$, $1\le p\le\infty$, and $N\ge 10$. Then, if $c_j=o(j^{1/4})$, for $\lambda>1$,*

(7)
$$\varlimsup_{n\to\infty}|\sigma_n^9(t)-f(t)|$$
$$=\left(\frac{\lambda}{\lambda-1}\right)^N\varlimsup_{n\to\infty}\left|\sigma_n^9\left(\left\{\sum_{m<j\le\lambda m}\left(1-\frac{j}{\lambda m}\right)^N c_j\mathcal{L}_j^a(t)\right\}\right)\right|$$
holds almost everywhere.

Next we are going to estimate the right hand side of (7). Set
$$b_j=\sqrt{\frac{j!}{(j+a)!}}\,c_j.$$
Then, for $y>\omega>0$,

(8)
$$\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N c_j\mathcal{L}_j^a(t)=e^{-t/2}t^{a/2}\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N b_jL_j^a(t).$$

Formula (1) and summation by parts yield

(9)
$$\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N b_jL_j^a$$
$$=\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N b_j(L_j^{a+1}-L_{j-1}^{a+1})$$
$$=\sum_{\omega<j\le y}\Delta\left\{\left(1-\frac{j}{y}\right)^N b_j\right\}L_j^{a+1}+\left(1-\frac{[y]+1}{y}\right)^N b_{[y]+1}L_{[y]}^{a+1}$$
$$-\left(1-\frac{[\omega]+1}{y}\right)^N b_{[\omega]+1}L_{[\omega]}^{a+1}$$
$$=\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N\Delta b_jL_j^{a+1}+\sum_{\omega<j\le y}\Delta\left(1-\frac{j}{y}\right)^N b_{j+1}L_j^{a+1}$$
$$+\left(1-\frac{[y]+1}{y}\right)^N b_{[y]+1}L_{[y]}^{a+1}-\left(1-\frac{[\omega]+1}{y}\right)^N b_{[\omega]+1}L_{[\omega]}^{a+1}.$$

Applying the mean value theorem to $f(x)=(1-x)^N$, we obtain
$$\left|\Delta\left(1-\frac{j}{y}\right)^N\right|=\left|\left(1-\frac{j}{y}\right)^N-\left(1-\frac{j+1}{y}\right)^N\right|\le\frac{N}{y}.$$
Plug in (9) and get
$$\left|\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N b_jL_j^a\right|$$
$$\le\left|\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N\Delta b_jL_j^{a+1}\right|+\frac{N}{y}\sum_{\omega<j\le y}|b_{j+1}L_j^{a+1}|$$
$$+|b_{[y]+1}L_{[y]}^{a+1}|+|b_{[\omega]+1}L_{[\omega]}^{a+1}|$$
$$\le\left|\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N\Delta b_jL_j^{a+1}\right|+(N+2)\max_{[\omega]\le j\le[y]}|b_{j+1}L_j^{a+1}|.$$

Replace b_j by $\Delta b_j,\Delta^2b_j,\ldots,\Delta^\mu b_j$, etc. Finally, we get

(10)
$$\left|\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N b_jL_j^a\right|\le\left|\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N\Delta^\mu b_jL_j^{a+\mu}\right|+\Phi_{\omega,y}(t),$$
where
$$\Phi_{\omega,y}(t)=(N+2)\sum_{i=0}^{\mu-1}\max_{[\omega]\le j\le[y]}|\Delta^ib_{j+1}L_j^{a+i+1}(t)|.$$

There exists a constant C, independent of i and j, such that

$$|\Delta^i b_{j+1} L_j^{a+i+1}(t)| \le C \max_{j+1\le k\le i+j+1} |b_k L_j^{a+i+1}(t)|$$
$$\le C_{\mu,t} \max_{j+1\le k\le i+j+1} \sqrt{\frac{k!}{(k+a)!}}|c_k| j^{(a+i+1)/2-1/4}$$
$$\le C_{\mu,t} \max_{j+1\le k\le i+j+1} |c_k| k^{i/2+1/4}$$

for all $0\le i\le \mu-1$, which implies

$$|\Phi_{\omega,y}(t)| \le C_{\mu,t} \max_{k\ge[\omega]+1} (|c_k| k^{\mu/2-1/4}). \tag{11}$$

Leibniz's rule gives

$$|\Delta^\mu b_j| = \left|\Delta^\mu\left(\sqrt{\frac{j!}{(j+a)!}}c_j\right)\right| \tag{12}$$
$$\le \sqrt{\frac{(j+\mu)!}{(j+\mu+a)!}}|\Delta^\mu c_j|$$
$$+\sum_{i=0}^{\mu-1}\binom{\mu}{i}\left|\Delta^{\mu-i}\sqrt{\frac{(j+i)!}{(j+i+a)!}}\right|\cdot|\Delta^i c_j|.$$

Using the inequality $1-\sqrt{1-y}\le y$ for $y\in[0,1]$, we obtain

$$\left|\Delta\left(\sqrt{\frac{j!}{(j+a)!}}\right)\right| = \sqrt{\frac{j!}{(j+a)!}}\left(1-\sqrt{\frac{j+1}{j+a+1}}\right) \le \sqrt{\frac{j!}{(j+a)!}}\frac{a}{j+a+1},$$

which leads to

$$\left|\Delta^k\left(\sqrt{\frac{j!}{(j+a)!}}\right)\right| \le C_\mu\sqrt{\frac{j!}{(j+a)!}}\frac{a}{j+a+1} \tag{13}$$
$$\le aC_\mu j^{-a/2-1} \quad \text{for } 1\le k\le\mu.$$

Both inequalities (12) and (13) imply

$$|\Delta^\mu b_j| \le \sqrt{\frac{(j+\mu)!}{(j+\mu+a)!}}|\Delta^\mu c_j| + C_{a,\mu} j^{-a/2-1}\sum_{i=0}^{\mu-1}\binom{\mu}{i}|\Delta^i c_j|$$
$$\le C_{a,\mu}(j^{-a/2}|\Delta^\mu c_j| + j^{-a/2-1}\max_{j\le k\le j+\mu-1}|c_k|).$$

This leads to

$$\left|\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N \Delta^\mu b_j L_j^{a+\mu}(t)\right| \tag{14}$$

$$\le \left(\frac{y-\omega}{y}\right)^N \sum_{\omega<j\le y}|\Delta^\mu b_j|\cdot|L_j^{a+\mu}(t)|$$
$$\le C_{a,\mu,t}\left(\frac{y-\omega}{y}\right)^N$$
$$\times\Big[\sum_{\omega<j\le y}|\Delta^\mu c_j| j^{\mu/2-1/4} + \sum_{\omega<j\le y} j^{-1}(\max_{j\le k\le j+\mu-1}|c_k|)j^{\mu/2-1/4}\Big]$$
$$\le C_{a,\mu,t}\left(\frac{y-\omega}{y}\right)^N$$
$$\times\Big[\sum_{\omega<j\le y}|\Delta^\mu c_j| j^{\mu/2-1/4} + \Big(\sum_{\omega<j\le y} j^{-1}\Big)\max_{k\ge\omega}(|c_k|k^{\mu/2-1/4})\Big]$$
$$\le C_{a,\mu,t}\left(\frac{y-\omega}{y}\right)^N$$
$$\times\Big[\sum_{\omega<j\le y}|\Delta^\mu c_j| j^{\mu/2-1/4} + \log\frac{y}{\omega}\cdot\max_{k\ge\omega}(|c_k|k^{\mu/2-1/4})\Big].$$

It follows from (8), (10), (11), and (14) that

$$\left|\sum_{\omega<j\le y}\left(1-\frac{j}{y}\right)^N c_j\mathcal{L}_j^a(t)\right|$$
$$\le C_{a,\mu,t}\left(\frac{y-\omega}{y}\right)^N\Big[\sum_{\omega<j\le y}|\Delta^\mu c_j| j^{\mu/2-1/4} + \log\frac{y}{\omega}\cdot\max_{k\ge[\omega]}(|c_k|k^{\mu/2-1/4})\Big]$$
$$+C_{\mu,t}\max_{k\ge[\omega]+1}(|c_k|k^{\mu/2-1/4}).$$

Let $\omega=m$, $y=\lambda m$, and then take σ_n^α on both sides. Thus we get

$$\left|\sigma_n^\alpha\Big(\Big\{\sum_{m<j\le\lambda m}\left(1-\frac{j}{\lambda m}\right)^N c_j\mathcal{L}_j^a(t)\Big\}\Big)\right|$$
$$\le C_{a,\mu,t}\left(\frac{\lambda-1}{\lambda}\right)^N\sigma_n^\alpha\Big(\Big\{\sum_{m<j\le\lambda m}|\Delta^\mu c_j| j^{\mu/2-1/4}\Big\}\Big)$$
$$+C_{a,\mu,t}\left(\frac{\lambda-1}{\lambda}\right)^N\log\lambda\cdot\sigma_n^\alpha(\{\max_{k\ge m}(|c_k|k^{\mu/2-1/4})\})$$
$$+C_{\mu,t}\sigma_n^\alpha(\{\max_{k\ge m+1}(|c_k|k^{\mu/2-1/4})\}),$$

which implies

$$\overline{\lim_{n\to\infty}}\left|\sigma_n^\alpha\left(\left\{\sum_{m<j\le\lambda m}\left(1-\frac{j}{\lambda m}\right)^N c_j\mathcal{L}_j^\alpha(t)\right\}\right)\right|$$
$$\le C_{a,\mu,t}\left(\frac{\lambda-1}{\lambda}\right)^N\overline{\lim_{n\to\infty}}\Big(\sum_{n<j\le\lambda n}|\Delta^\mu c_j|j^{\mu/2-1/4}\Big)$$
$$+C_{a,\mu,t}\left(\frac{\lambda-1}{\lambda}\right)^N\log\lambda\cdot\overline{\lim_{n\to\infty}}\,(\max_{k\ge n}|c_k|k^{\mu/2-1/4})$$
$$+C_{\mu,t}\overline{\lim_{n\to\infty}}\,(\max_{k\ge n+1}|c_k|k^{\mu/2-1/4})$$

for all $\mu\in\mathbb{N}$. Hence we have verified the following result.

LEMMA 6. *Let $\mu,N\in\mathbb{N}$. If $|c_j|j^{\mu/2-1/4}=o(1)$ as $j\to\infty$, then*

$$\overline{\lim_{n\to\infty}}\left|\sigma_n^\alpha\left(\left\{\sum_{m<j\le\lambda m}\left(1-\frac{j}{\lambda m}\right)^N c_j\mathcal{L}_j^\alpha(t)\right\}\right)\right|\le C_{a,\mu,t}\left(\frac{\lambda-1}{\lambda}\right)^N\phi_\mu(\lambda),$$

where $t\in\mathbb{R}^+,\alpha>0$, $\lambda>1$, and

$$\phi_\mu(\lambda)=\overline{\lim_{n\to\infty}}\sum_{n<j\le\lambda n}|\Delta^\mu c_j|j^{\mu/2-1/4}.$$

Proof of Theorem 2. Let $\phi_\mu(\lambda)$ be defined as in Lemma 6. Then condition (2) implies $\lim_{\lambda\downarrow1}\phi_k(\lambda)=0$ for some $k\in\mathbb{N}$. For $\lambda>1$, it follows from Lemmas 5 and 6 that for almost all $t\in\mathbb{R}^+$,

$$\overline{\lim_{n\to\infty}}|\sigma_n^9(t)-f(t)|$$
$$=\left(\frac{\lambda}{\lambda-1}\right)^{10}\overline{\lim_{n\to\infty}}\left|\sigma_n^9\left(\left\{\sum_{m<j\le\lambda m}\left(1-\frac{j}{\lambda m}\right)^{10}c_j\mathcal{L}_j^\alpha(t)\right\}\right)\right|$$
$$\le C_{a,k,t}\,\phi_k(\lambda).$$

Taking $\lim_{\lambda\downarrow1}$ on both sides, we obtain

$$\overline{\lim_{n\to\infty}}|\sigma_n^9(t)-f(t)|\le C_{a,k,t}\lim_{\lambda\downarrow1}\phi_k(\lambda)=0.$$

Therefore, $\sigma_n^9(t)\to f(t)$ almost everywhere and hence $R_w^9(t)\to f(t)$ almost everywhere as $w\to\infty$.

Adopting the argument before Lemma 5, we obtain

$$\overline{\lim_{n\to\infty}}|\sigma_n^8(t)-f(t)|$$
$$=\left(\frac{\lambda}{\lambda-1}\right)^9\overline{\lim_{n\to\infty}}\left|\sigma_n^8\left(\left\{\sum_{m<j\le\lambda m}\left(1-\frac{j}{\lambda m}\right)^9 c_j\mathcal{L}_j^\alpha(t)\right\}\right)\right|.$$

Combining this with Lemma 6, we conclude that $\sigma_n^8(t)\to f(t)$ almost everywhere. Consequently, $R_w^8(t)\to f(t)$ almost everywhere. Repeating the above argument, we see that $R_w^7(t)\to f(t)$ almost everywhere, $R_w^6(t)\to f(t)$ almost everywhere, and so on. Finally, we conclude that $\sum_{j=0}^\infty c_j\mathcal{L}_j^\alpha(t)=f(t)$ almost everywhere.

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