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DEPARTMENT OF MATHEMATICS
KANSAS STATE UNIVERSITY
MANHATTAN, KANSAS 66506
U.S.A.
E-mail: FARMER@KSUVM.KSU.EDU

Current address:
DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RHODE ISLAND 02912
U.S.A.

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Oscillatory kernels in certain Hardy-type spaces

by

LUNG-KEE CHEN (Corvallis, Oreg.) and
DASHAN FAN (Milwaukee, Wis.)

Abstract. We consider a convolution operator $Tf = \text{p.v.}\,\Omega * f$ with $\Omega(x) = K(x)e^{ih(x)}$, where $K(x)$ is an (n,β) kernel near the origin and an (α,β), $\alpha \ge n$, kernel away from the origin; $h(x)$ is a real-valued C^∞ function on $\mathbb{R}^n \setminus \{0\}$. We give a criterion for such an operator to be bounded from the space $H_0^p(\mathbb{R}^n)$ into itself.

1. Introduction and notations. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $h(x)$ be a real-valued function. Consider the oscillatory kernel $\Omega(x) = K(x)e^{ih(x)}$ with $K(x)$ being an (n,β) kernel near the origin of $\mathbb{R}^n$ and an (α,β) kernel away from the origin. An (α,β) *kernel* K is a function on $\mathbb{R}^n \setminus \{0\}$ satisfying

$$|D^J K(x)| \le C_J |x|^{-\alpha-|J|} \tag{1.1}$$

with $|J| \le \beta$, $x \ne 0$. The *phase function* $h(x)$ is a C^∞ function on $\mathbb{R}^n \setminus \{0\}$ satisfying (1.2) and (1.3):

$$|D^J h(x)| \le C_J |x|^{b-|J|} \tag{1.2}$$

for all multi-indices J with $|J| \le M$, $x \ne 0$, where M and b are positive integers, and

$$|\nabla h(x)| \ge C|x|^{b-1}, \tag{1.3}$$

where $\nabla = (\partial_{x_1}, \ldots, \partial_{x_n})$ is the gradient operator.

For the above defined kernel $\Omega(x)$, the associated oscillatory singular integral T is defined by

$$Tf(y) = \text{p.v.} \int_{\mathbb{R}^n} e^{ih(y-x)} K(y-x) f(x)\,dx, \tag{1.4}$$

where $K(x)$ satisfies (1.1) and in addition, there exists an $\varepsilon > 0$ such that

$$\text{p.v.} \int_{0<|x|\le\varepsilon} K(x)\,dx = 0. \tag{1.5}$$

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A typical example of such an operator is the oscillating integral

$$f \to \text{p.v.} \int_{\mathbb{R}^n} e^{i|x-y|^a} |x-y|^{-\alpha} \psi(x-y) f(y)\, dy,$$

$\alpha \geq n$, where ψ is a C^∞ function satisfying $\psi(x) = 0$ if $|x| \leq 1$ and $\psi(x) = 1$ if $|x| \geq 2$. This operator was studied extensively and the boundedness properties in H^p $(p > 0)$ spaces were established in Sjölin [11], Jurkat-Sampson [9] and Chanillo *et al.* [1].

We shall consider operators with phase function $h(x)$ satisfying (1.2) and (1.3). The main result we obtain is the boundedness of such operators in certain Hardy-type spaces $H_0^p(\mathbb{R}^n)$, in analogy with Sjölin's result on the Hardy spaces $H^p(\mathbb{R}^n)$ [11]. A similar problem in the context of the Besov space $\dot{B}_0^{1,0}(\mathbb{R}^n) = H_0^1(\mathbb{R}^n)$ was studied earlier by one of the authors [3].

Suppose $\Phi \in \mathcal{S}$ satisfies $\int \Phi(x)\, dx = 0$. Then the Lusin function $S_\Phi^b(f)$ for any $f \in \mathcal{S}'$ is defined by

$$S_\Phi^b(f)(x) = \Big(\int_{\Gamma(x)} |\Phi_t * f(y)|^{b+1} \, dy \frac{dt}{t^{n+1}} \Big)^{1/(b+1)}, \tag{1.6}$$

where $\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x-y| < t\}$ and $\Phi_t(y) = t^{-n}\Phi(y/t)$ for $t > 0$. Suppose that $f \in \mathcal{S}'$; we say that f *vanishes weakly at infinity* if for any $\Phi \in \mathcal{S}$, $f * \Phi_t \to 0$ in $\mathcal{S}'$ as $t \to \infty$.

Suppose that $0 < p \leq 1$. Let s be an integer greater than or equal to $[n(1/p-1)]$ and $\Phi \in \mathcal{S}$ with

$$\int_0^\infty |\widehat{\Phi}(xt)|^2 \frac{dt}{t} \neq 0$$

for any $x \neq 0$. Suppose, moreover, that $\operatorname{supp} \Phi \subset \{x \in \mathbb{R}^n : |x| < 1\}$ and $\int x^J \Phi(x)\, dx = 0$ for all multi-indices J with $0 \leq |J| \leq s$. The *Hardy-type spaces* are defined by

$$H_b^p(\mathbb{R}^n) = \{f \in \mathcal{S}' : \|S_\Phi^b f\|_p < \infty\}.$$

This type of space is of interest since a well-known fact is that $H_1^p = H^p(\mathbb{R}^n)$ (see [5]) and $H_0^1 = B_0^{1,0}(\mathbb{R}^n)$ (see [6]). In this paper, we are particularly interested in studying the oscillating integral (1.4) on the space H_0^p. For this purpose, we need an atomic characterization of $H_0^p(\mathbb{R}^n)$. Suppose $0 < p \leq 1 \leq q \leq \infty$, and s is an integer at least $[n(1/p-1)]$. A (p,q,s)*-atom* centered at x_0 is a function $a \in L^q(\mathbb{R}^n)$ supported in the ball $B(x_0, \varrho)$ of $\mathbb{R}^n$ with center at x_0 and radius ϱ such that

$$\|a\|_q \leq \varrho^{n(1/q-1/p)} \tag{1.7}$$

and

$$\int a(x) x^J\, dx = 0 \quad \text{where } 0 \leq |J| \leq s. \tag{1.8}$$

A $(p,1,q,s)$*-atom* centered at x_0 is a (p,q,s)-atom satisfying

$$\|\nabla a\|_\infty \leq \varrho^{n(1/q-1/p)-1}. \tag{1.9}$$

A *particular* (p,q,s)*-atom* centered at x_0 is a (p,q,s)-atom $a(x)$ supported in a ball $B(x_0,\varrho)$ satisfying

$$a(x) = \sum_{i=1}^\infty \mu_i a_i,$$

where each a_i is a $(1,1,q,s)$-atom with $\operatorname{supp} a_i \subset B(y_i, r_i) \subset B(x_0, \varrho)$ and

$$\sum_{i=1}^\infty |\mu_i| \leq \varrho^{n-n/p}.$$

DEFINITION. Suppose that $0 < p \leq 1$. The *atomic Hardy-type space* $H_0^{p,q,s}$ is the collection of all tempered distributions $f \in \mathcal{S}'$ of the form $f = \sum c_k a_k$ where $\sum |c_k|^p < \infty$, the a_k's are particular (p,q,s)-atoms and the series converges in the distributional sense. Also the "norm" $\|f\|_{H_0^{p,q,s}}$ is defined to be the infimum of the expressions

$$\Big(\sum_k |c_k|^p\Big)^{1/p}$$

for all such representations of f. It is easy to see that the space $H_0^{p,q,s}$, if $s \geq [n(1/p-1)]$, is a subspace of the Hardy space $H^p(\mathbb{R}^n)$ which was studied by many authors [2], [10].

The following theorem can be found in [7] (or [8]).

THEOREM 1.10. *Suppose $s \geq [n(1/p-1)]$ and assume $S_\Phi(f) \equiv S_\Phi^0(f)$. Then*

(1) *For any particular (p,q,s)-atom $a(x)$, there exists a constant C independent of $a(x)$ such that $\|S_\Phi(a)\|_p \leq C$;*

(2) *$H_0^p(\mathbb{R}^n) = H_0^{p,q,s}(\mathbb{R}^n)$ for $q \geq 1$, and moreover,*

$$\|f\|_{H_0^{p,q,s}} \approx \|f\|_{H_0^p}. \tag{1.11}$$

Thus a linear operator T defined on $\mathcal{S}(\mathbb{R}^n) \cap H_0^p(\mathbb{R}^n)$ extends to a bounded operator in H_0^p if there is a constant C independent of f such that

$$\|S_\Phi(Tf)\|_p \leq C\|f\|_{H_0^{p,q,s}}. \tag{1.12}$$

The main result of this paper is the following

THEOREM A. *For $k = 1,2,3,\ldots$ and $p_k = n/(n+k)$, let $\alpha \geq kb+n$ and let K be a kernel of type $(n, k+1)$ near the origin and of type $(\alpha, k+1)$ away from the origin. In addition, suppose K satisfies (1.5). Then the operator*

$Tf = \text{p.v.}\, \Omega * f$ *is bounded in* $H_0^p(\mathbb{R}^n)$ *for* $p_k \le p \le 1$ *provided* $h(x)$ *satisfies* (1.2) *and* (1.3) *with* $b > 0$, $b \neq 1$ *and* $M \ge k+1$.

Clearly, to prove the theorem, by a standard argument (see [7] or [8]), it suffices to show that for any particular (p, ∞, s)-atom $a(x)$,

$$\|S_\Phi(Ta)\|_p \le C \quad (p < 1) \tag{1.13}$$

with a constant C independent of $a(x)$.

Note. The case $p = 1$ has been proved in [3].

We will prove (1.13) in the third section and give some necessary lemmas in the second section.

Throughout this paper, the letter C will denote (possibly different) constants that are independent of the essential variables.

2. Some lemmas. In this section, we will prove some lemmas which are necessary for proving the main theorem. We first observe a simple fact that if $s \ge [n(1/p - 1)]$ and $a(x)$ is a $(1,1,\infty,s)$-atom supported in $B(x_0, \varrho)$ then $\varrho^{n-n/p}a(x)$ is a $(p,1,\infty,s)$-atom.

LEMMA 2.1. *Let* $Tf = K * f$ *be any convolution operator. If*

$$\|Ta\|_{H_0^p} \le C \tag{2.2}$$

with a constant independent of any $(p,1,\infty,s)$*-atom* $a(x)$, *then*

$$\|Tf\|_{H_0^p} \le C\|f\|_{H_0^p}, \quad 0 < p < 1.$$

Proof. We need to prove (1.13) for any particular (p, ∞, s)-atom $a(x)$ with support in $B(x_0, \varrho)$. By definition, a (p,∞,s)-atom $a(x)$ has a decomposition $a(x) = \sum \mu_i a_i(x)$ with a_i being a $(1,1,\infty,s)$-atom supported in $B(x_i, r_i) \subset B(x_0, \varrho)$ and $\sum |\mu_i| \le \varrho^{n-n/p}$. Note that $r_i^{n-n/p} a_i$ is a $(p,1,\infty,s)$-atom. So by (2.2), one has

$$\|Ta\|_{H_0^p} \le C \sum |\mu_i| r_i^{-n+n/p} \le \varrho^{n-n/p} r_i^{-n+n/p}.$$

Since $n/p > n$ and $r_i \le \varrho$ we have $\|Ta\|_{H_0^p} \le C$. Lemma 2.1 is proved.

Since any convolution operator commutes with shift operators, without loss of generality, we can assume that the atom involved in our argument has support in $B(0, \varrho)$.

Let Ψ be a C^∞ non-negative radial function with $\operatorname{supp} \Psi \subset \{1/2 \le |x| \le 2\}$ and $\sum \Psi(2^j|y|) = 1$ for $y \neq 0$. Let

$$\eta(x) = 1 - \sum_{j=N}^{\infty} \Psi(2^{-j-2}\varrho^{-1}|x|)$$

where N is the integer that appears in the following lemma.

LEMMA 2.3. *Let* $a(x)$ *be a* $(1,1,\infty,s)$*-atom supported in* $B(0,\varrho)$ *and* $0 < t \le 1$. *Let*

$$A_j(\varrho, a) = \sup_{2^j\varrho < |y| \le 2^{j+4}\varrho} \Big| \int_{\mathbb{R}^n} e^{ih(y-tx)} a(x) x^\beta \, dx \Big|,$$

where β *is any multi-index with* $|\beta| = s$ *and* $h(x)$ *is the phase function satisfying* (1.2) *and* (1.3) *for* $M \ge 2$ *and* $b \neq 1$. *Then for* $j = 1, 2, \ldots,$ *we have*

$$A_j(\varrho, a) \le C\varrho^{b+s} 2^{j(b-1)} t \tag{2.4}$$

and there exists an $N > 0$ *independent of* ϱ *such that if* $j \ge N$ *then*

$$A_j(\varrho, a) \le C(2^j\varrho)^{-b} \varrho^s 2^j t^{-1}. \tag{2.5}$$

Proof. Lemma 2.3 is an easy modification of Lemma (2.1) in [3].

Let $\Omega(x)$ be the kernel in Theorem A. We have

$$\Omega(x) = \eta(x)\Omega(x) + \sum_{j=N}^{\infty} \Omega(x)\Psi(2^{-j-2}\varrho^{-1}|x|) = \Omega_0(x) + \sum_{j=N}^{\infty} \Omega_j(x).$$

Suppose $a(x)$ denotes a $(p,1,\infty,s)$-atom with support in $B(0,\varrho)$. It is clear that $\operatorname{supp}(\Omega_0 * a) \subset B(0, 2^{N+5}\varrho)$ and $\operatorname{supp}(\Omega_j * a) \subset B(0, 2^{j+4}\varrho)$ for $j = N+1, N+2, \ldots$ Also by the cancellation condition on $a(x)$, one easily sees that

$$\int_{\mathbb{R}^n} x^J (\Omega_0 * a)(x)\, dx = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} x^J (\Omega_j * a)(x)\, dx = 0 \tag{2.6}$$

for all $j = N+1,\ N+2, \ldots,$ and all multi-indices J with $|J| \le s$. This implies that, up to the size conditions, $\Omega_0 * a$ and $\Omega_j * a$ are atoms. Hence we need to check the size conditions. To estimate $\Omega_0 * a$, if $2^N\varrho \le |y| \le 2^{N+5}\varrho$, by the hypothesis on $K(x)$, one can see that

$$\begin{aligned} |\Omega_0 * a| &= \Big| \int e^{ih(y-x)} K(y-x)\eta(y-x) a(x)\, dx \Big| \\ &\le \int_{B(0,\varrho)} |K(y-x)||a(x)|\, dx \le C\varrho^{-n/p}. \end{aligned}$$

If $|y| < 2^N\varrho$, then $\eta(x-y) \equiv 1$ for all x in $B(0,\varrho)$. Therefore,

$$\begin{aligned} \|\Omega_0 * a\|_2 &\le C\varrho^{-n/p+n/2} + C\Big\{ \int_{|y| \le 2^N\varrho} \Big| \int_{\mathbb{R}^n} a(x) e^{ih(y-x)} K(y-x)\, dx \Big|^2 dy \Big\}^{1/2} \\ &\le C\varrho^{-n/p+n/2} + C\|\Omega * a\|_2. \end{aligned}$$

By Theorem 1 of [4], we have $\|\Omega * a\|_2 \le C\|a\|_2 \le C\varrho^{-n/p+n/2}$. This shows $\|\Omega_0 * a\|_2 \le C\varrho^{-n/p+n/2}$. Following the same ideas, we can prove that

$$\|\nabla(\Omega_0 * a)\|_2 = \|\Omega_0 * \nabla a\|_2 \le C\varrho^{-n/p+n/2-1}.$$

Thus, up to a constant, $\varrho^{n-n/p}\Omega_0 * a$ is a $(1,1,2,s)$-atom and clearly $\Omega_0 * a$ is a particular $(p,2,s)$-atom.

LEMMA 2.7. *For $j = N+1, N+2, \ldots$, and any $(p,1,\infty,s)$ atom $a(x)$ with support in $B(0,\varrho)$, if $2^j\varrho \ge 1$, then*

(a)
$$\begin{aligned}\|\Omega_j * a\|_\infty &\le C2^{-j}(2^j\varrho)^{-\alpha}\varrho^{s-n/p+n}(2^j\varrho)^{s(b-1)}\\ &\quad + C(2^j\varrho)^{-\alpha}\varrho^{s-n/p+n}(2^j\varrho)^{s(b-1)}\\ &\quad\times\int_0^1 \min\{2^{-j}(2^j\varrho)^b t, 2^j(2^j\varrho)^{-b}t^{-1}\}\,dt,\end{aligned}$$

(b)
$$\begin{aligned}\|\nabla(\Omega_j * a)\|_\infty &\le C2^{-j}(2^j\varrho)^{-\alpha}\varrho^{s-n/p+n-1}(2^j\varrho)^{s(b-1)}\\ &\quad + C(2^j\varrho)^{-\alpha}\varrho^{s-n/p+n-1}(2^j\varrho)^{s(b-1)}\\ &\quad\times\int_0^1 \min\{2^{-j}(2^j\varrho)^b t, 2^j(2^j\varrho)^{-b}t^{-1}\}\,dt.\end{aligned}$$

Proof. By the proof in [8], one easily sees that we can assume that $\varrho\nabla a$ is also a $(p,1,\infty,s)$-atom. So noting that $\|\nabla(\Omega_j * a)\|_\infty = \varrho^{-1}\|\Omega_j * \varrho\nabla a\|_\infty$, we only need to prove (a). It suffices to assume that $s = [n(1/p-1)] \ge 1$. By the cancellation property of $a(x)$,

$$\begin{aligned}(2.8)\quad |\Omega_j * a(y)| &= \Big|\sum_{|\beta|=s-1}\int_{\mathbb{R}^n}\int_0^1 \frac{(1-t)^\beta}{\beta!}D^{\beta+1}\Omega_j(y-tx)\,dt\,x^{\beta+1}a(x)\,dx\Big|\\ &\le C\sum_{|\beta|=s-1}\int_0^1\Big|\int_{\mathbb{R}^n} D^{\beta+1}\Omega_j(y-tx)x^{\beta+1}a(x)\,dx\Big|\,dt.\end{aligned}$$

Here

$$\begin{aligned}&D^{\beta+1}\Omega_j(y)\\ &\quad= \sum_{|\beta_1|+|\beta_2|+|\beta_3|=|\beta|+1} C_{\beta_1\beta_2\beta_3}D_y^{\beta_1}K(y)D_y^{\beta_2}\Psi(2^{-j-2}\varrho^{-1}y)D_y^{\beta_3}e^{ih(y)},\\ &\qquad |D_y^{\beta_1}K(y)| \le C|y|^{-\alpha-|\beta_1|},\\ &\qquad |D_y^{\beta_2}\Psi(2^{-j-2}\varrho^{-1}y)| \le (2^{-j}\varrho^{-1})^{|\beta_2|}\widetilde{\Psi}(2^{-j-2}\varrho^{-1}y),\end{aligned}$$

for some nice function $\widetilde{\Psi}(y)$ supported in $\{1/2 \le |y| \le 2\}$. Using the assumption (1.2), we have no difficulty in showing that

$$D_y^{\beta_3}e^{ih(y)} \le (2^j\varrho)^{|\beta_3|(b-1)}Q(y)e^{ih(y)},$$

since $2^j\varrho \le y \le 2^{j+4}\varrho$ and $2^j\varrho \ge 1$, where $Q(y)$ is a function satisfying $|D_yQ(y)| \le C|y|^{-1}$. Hence the right hand side of inequality (2.8) is bounded by

$$\begin{aligned}&C\sum_{|\beta|=s-1}\int_0^1\sum_{|\beta_1|+|\beta_2|+|\beta_3|=|\beta|}(2^j\varrho)^{-|\beta_2|+|\beta_3|(b-1)}\\ &\times\Big|\int_{\mathbb{R}^n} D_y^{\beta_1}K(y-tx)Q(y-tx)\Psi(2^{-j-2}\varrho^{-1}(y-tx))e^{ih(y-tx)}x^{\beta+1}a(x)\,dx\Big|\,dt,\end{aligned}$$

where $\beta+1$ denotes a multi-index and $|\beta+1| = s$. The inner integral in the above formula is estimated by

$$\begin{aligned}&\Big|\int_{\mathbb{R}^n}\{D_y^{\beta_1}K(y-tx)Q(y-tx) - D_y^{\beta_1}K(y)Q(y)\}\\ &\qquad\qquad\times\Psi(2^{-j-2}\varrho^{-1}(y-tx))e^{ih(y-tx)}x^{\beta+1}a(x)\,dx\Big|\\ &\quad + |D_y^{\beta_1}K(y)Q(y)|\\ &\quad\times\Big|\int_{\mathbb{R}^n}\{\Psi(2^{-j-2}\varrho^{-1}(y-tx)) - \Psi(2^{-j-2}\varrho^{-1}y)\}e^{ih(y-tx)}x^{\beta+1}a(x)\,dx\Big|\\ &\quad + |D_y^{\beta_1}K(y)Q(y)|\Big|\int_{\mathbb{R}^n} e^{ih(y-tx)}x^{\beta+1}a(x)\,dx\Big|\\ &\equiv I + II + III.\end{aligned}$$

It is clear that

$$\begin{aligned}I &\le (2^j\varrho)^{-\alpha-|\beta_1|-1}\varrho^{s-n/p+n+1},\\ II &\le (2^j\varrho)^{-\alpha-|\beta_1|-1}\varrho^{s-n/p+n+1},\\ III &\le (2^j\varrho)^{-\alpha-|\beta_1|}\varrho^{s}\Big|\int_{\mathbb{R}^n} e^{ih(y-tx)}\varrho^{-s}x^{\beta+1}a(x)\,dx\Big|.\end{aligned}$$

Thus by Lemma 2.3,

$$III \le C(2^j\varrho)^{-\alpha-|\beta_1|}\varrho^{-n/p+n}\Big|\int_{\mathbb{R}^n} e^{ih(y-tx)}x^{\beta+1}\varrho^{-n+n/p}a(x)\,dx\Big|.$$

Since $\varrho^{-n+n/p}a(x)$ is a $(1,1,\infty,s)$-atom, by Lemma 2.3 we have

$$III \le C(2^j\varrho)^{-\alpha-|\beta_1|}\varrho^{s-n/p+n}\min\{2^j(2^j\varrho)^{-b}t^{-1}, 2^{-j}(2^j\varrho)^b t\}.$$

From the hypothesis $2^j\varrho \geq 1$, one has

$$|\Omega_j * a(y)| \leq C(2^j\varrho)^{-\alpha}(2^j\varrho)^{s(b-1)}2^{-j}\varrho^{s-n/p+n} + C(2^j\varrho)^{-\alpha}(2^j\varrho)^{s(b-1)}\varrho^{s-n/p+n} \times \int_0^1 \min\{2^j(2^j\varrho)^{-b}t^{-1}, 2^{-j}(2^j\varrho)^b t\}\,dt.$$

This completes the proof of Lemma 2.7.

LEMMA 2.9. *Suppose $2^j\varrho \leq 1$. Then there exists an $\varepsilon > 0$ such that*

$$|\Omega_j * a(y)| \leq C(2^j\varrho)^{-n/p}2^{-j\varepsilon} \tag{2.10}$$

and

$$|\nabla(\Omega_j * a)(y)| \leq C\varrho^{-1}(2^j\varrho)^{-n/p}2^{-j\varepsilon}. \tag{2.11}$$

Proof. We note that

$$\operatorname{supp} \Omega_j * a(y) = \operatorname{supp} \int_{\mathbb{R}^n} K(y-x)\Psi(2^{-j-2}\varrho^{-1}(y-x))e^{ih(y-x)}a(x)\,dx \subset \{y : 2^j\varrho \leq |y| \leq 2^{j+4}\varrho\}.$$

In this case, by definition, $K(x)$ is an (n,s) kernel. Thus if we let

$$\widetilde{K}(y) = K(y)\Psi(2^{-j-2}\varrho^{-1}y)e^{ih(y)},$$

then we obtain

$$|D_y^\beta \widetilde{K}(y-x)| \leq C(2^j\varrho)^{-n-|\beta|} \quad \text{for } |\beta| \leq s+1 \text{ and } |x| \leq \varrho.$$

By the cancellation condition on $a(x)$, one has

$$|\Omega_j * a(y)| \leq C\|a\|_\infty \varrho^{s+1+n}(2^j\varrho)^{-n-s-1} = C(2^j\varrho)^{-n/p}(2^j)^{-n-s-1+n/p} \equiv C(2^j\varrho)^{-n/p}2^{-j\varepsilon}$$

with $\varepsilon > 0$. This proves (2.10). Similarly, we can prove (2.11).

3. The proof of the main theorem. By Lemma 2.1, it suffices to show (2.2). For simplicity, we prove (2.2) when $p = p_k$ where $p_k = n/(n+k)$ and $s = [n(1/p-1)] = k$. The proof for $p > p_k$ is similar. By the discussion in the second section, clearly we only need to prove

$$\|\Omega_0 * a\|_{H_0^p}^p + \sum_{j=N}^{\infty} \|\Omega_j * a\|_{H_0^p}^p \leq C \tag{3.1}$$

with a constant C independent of the $(p,1,\infty,s)$-atom $a(x)$. Since we know that $\Omega_0 * a$ is a particular $(p,2,s)$-atom, by Theorem 1.10 we have $\|\Omega_0 * a\|_{H_0^p} \leq C$.

We claim that

$$\sum_{j=N}^{2^j\varrho\leq 1} \|\Omega_j * a\|_{H_0^p}^p \leq C \tag{3.2}$$

and

$$\sum_{2^j\varrho\geq 1}^{\infty} \|\Omega_j * a\|_{H_0^p}^p \leq C. \tag{3.3}$$

We will use Lemma 2.7 in proving (3.3) and Lemma 2.9 in proving (3.2). The two proofs are similar so we only prove (3.3). Let us write

$$\begin{aligned}
\|\Omega_j * a\|_{H_0^p}^p &= \int_{\mathbb{R}^n} \bigg| \int_0^\infty \int_{|x-y|<t} |(\Phi_t * \Omega_j * a)(y)|\,dy\,\frac{dt}{t^{n+1}} \bigg|^p dx \\
&= \int_{|x|\leq 2^{j+6}\varrho} \bigg| \int_0^{2\varrho} \int_{|x-y|<t} (\ldots) \bigg|^p dx \\
&\quad + \int_{|x|\leq 2^{j+6}\varrho} \bigg| \int_{2\varrho}^{2^{j+4}\varrho} \int_{|x-y|<t} (\ldots) \bigg|^p dx \\
&\quad + \int_{|x|\leq 2^{j+6}\varrho} \bigg| \int_{2^{j+4}\varrho}^{\infty} \int_{|x-y|<t} (\ldots) \bigg|^p dx \\
&\quad + \int_{|x|> 2^{j+6}\varrho} \bigg| \int_{2^{j+4}\varrho}^{\infty} \int_{|x-y|<t} (\ldots) \bigg|^p dx \\
&\quad + \int_{|x|> 2^{j+6}\varrho} \bigg| \int_0^{2^{j+4}\varrho} \int_{|x-y|<t} (\ldots) \bigg|^p dx \\
&= I + II + III + IV + V.
\end{aligned}$$

By the support condition on $\Omega_j * a$, we have $V \equiv 0$. By Hölder's inequality and the cancellation of Φ,

$$I \leq (2^{j+6}\varrho)^{n(1-p)} \bigg(\int_{|x|\leq 2^{j+6}\varrho} \int_0^{2\varrho} \int_{|x-y|<t} |\Phi_t * \Omega_j * a(y)|\,dy\,\frac{dt}{t^{n+1}}\,dx \bigg)^p$$

$$= (2^{j+6}\varrho)^{n(1-p)}\Big(\int\limits_0^{2\varrho}\int\limits_{|x|\le 2^{j+6}\varrho}\int\limits_{|x-y|<t}$$

$$\Big|\int\limits_{\mathbb{R}^n}\Phi_t(y-z)(\Omega_j*a(z)-\Omega_j*a(y))\,dz\Big|\,dy\,dx\frac{dt}{t^{n+1}}\Big)^p$$

$$\le (2^{j+6}\varrho)^{n(1-p)}\|\nabla(\Omega_j*a)\|_\infty^p$$

$$\times\Big(\int\limits_{|x|\le 2^{j+6}\varrho}\int\limits_0^{2\varrho}\int\limits_{|x-y|<t}\int\limits_{\mathbb{R}^n}\Big|\Phi_t(y-z)\frac{y-z}{t}\Big|\,dz\,dy\,\frac{dt}{t^n}\,dx\Big)^p$$

$$\le C(2^j\varrho)^{n(1-p)}\|\nabla(\Omega_j*a)\|_\infty^p(2^j\varrho)^{np}\varrho^p \le C\|\nabla(\Omega_j*a)\|_\infty^p\varrho^p(2^j\varrho)^n.$$

Also,

$$III \le (2^{j+6}\varrho)^{n(1-p)}$$

$$\times\Big(\int\limits_{|x|\le 2^{j+6}\varrho}\int\limits_{2^{j+4}\varrho}^{\infty}\int\limits_{|x-y|<t}\Big|\int\limits_{\mathbb{R}^n}\Phi_t(y-z)\Omega_j*a(z)\,dz\Big|\,dy\,\frac{dt}{t^{n+1}}\,dx\Big)^p$$

$$\le \|\Omega_j*a\|_\infty^p(2^j\varrho)^{np}(2^j\varrho)^{n(1-p)} = C(2^j\varrho)^n\|\Omega_j*a\|_\infty^p.$$

By the cancellation of Ω_j*a, IV can be written as

$$IV \le \int\limits_{|x|\ge 2^{j+6}\varrho}\Big|\sum_{|\delta|=s}\int\limits_0^1\frac{(1-\mu)^{|\delta|}}{|\delta|!}\int\limits_{2^{j+4}\varrho}^{\infty}\int\limits_{|x-y|<t}$$

$$\Big|\frac{1}{t^{s+1}}\int\limits_{\mathbb{R}^n}(D_y^{\delta+1}\Phi)_t(y-\mu z)z^{\delta+1}\Omega_j*a(z)\,dz\Big|\,dy\frac{dt}{t^{n+1}}\,d\mu\Big|^p\,dx$$

$$= \sum_{|\delta|=s}\int\limits_{|x|\ge 2^{j+6}\varrho}\Big|\int\limits_0^1\frac{(1-\mu)^{|\delta|}}{|\delta|!}\int\limits_{2^{j+4}\varrho}^{\infty}\int\limits_{|x-y|<t}$$

$$\Big|\int\limits_{|z|\le 2^{j+4}\varrho}(D_y^{\delta+1}\Phi)_t(y-\mu z)z^{\delta+1}\Omega_j*a(z)\,dz\Big|\,dy\,\frac{dt}{t^{n+s+2}}\,d\mu\Big|^p\,dx.$$

It is clear that the last expression is dominated by

$$\|\Omega_j*a\|_\infty^p(2^j\varrho)^{(s+1)p}\sum_{|\delta|=s}\int\limits_{|x|\ge 2^{j+6}\varrho}\Big|\int\limits_0^1\int\limits_{2^{j+4}\varrho}^{\infty}\int\limits_{|x-y|<t}\int\limits_{|z|\le 2^{j+4}\varrho}$$

$$|(D_y^{\delta+1}\Phi)_t(y-\mu z)|\,dz\,dy\,\frac{dt}{t^{n+s+2}}\,d\mu\Big|^p\,dx.$$

By the compactness of $\operatorname{supp}\Phi$ $(\subset\{x:|x|<1\})$, we have $|y-\mu z|\le t$.

This implies $|y|\le t+|z|\le t+2^{j+4}\varrho$. On the other hand, $|x-y|<t$. Hence $|x|<t+|y|\le 2t+2^{j+4}\varrho$. Since $|x|\ge 2^{j+6}\varrho$, we have $|x|/4\le t$. Applying Fubini's Theorem to interchange the order of the integrals $\int_{|x-y|<t}$ and $\int_{|z|\le 2^{j+4}\varrho}$, we bound the last expression by

$$C\|\Omega_j*a\|_\infty^p(2^j\varrho)^{(n+s+1)p}\int\limits_{|x|\ge 2^{j+6}\varrho}\Big|\int\limits_{|x|/4}^{\infty}\frac{dt}{t^{n+s+2}}\Big|^p\,dx \le C\|\Omega_j*a\|_\infty^p(2^j\varrho)^n.$$

Finally, by mimicking the estimate for $P_2(j)$ in [3], we have

$$II\le C\|\Omega_j*a\|_\infty^p(2^j\varrho)^n.$$

One combines all the estimates of I–V and obtains

$$\|\Omega_j*a\|_{H_0^p}^p \le C\|\Omega_j*a\|_\infty^p(2^j\varrho)^n + C\varrho^p\|\nabla(\Omega_j*a)\|_\infty^p(2^j\varrho)^n.$$

Since

$$\varrho^p\|\nabla(\Omega_j*a)\|_\infty^p = \|\Omega_j*\varrho\nabla a\|_\infty^p$$

and $\varrho\nabla a$ is a $(p,1,\infty,s)$-atom supported in $B(0,\varrho)$ it is enough to estimate

$$\|\Omega_j*a\|_{H_0^p}^p \le C\|\Omega_j*a\|_\infty^p(2^j\varrho)^n$$

for any $(p,1,\infty,s)$-atom a. By Lemma 2.7, noting that $p=n/(n+k)$, $s=k$, we have

$$\|\Omega_j*a\|_\infty^p(2^j\varrho)^n \le C(2^j\varrho)^n(2^j\varrho)^{-\alpha p}(2^j\varrho)^{(b-1)kp}2^{-jp}$$
$$+ C(2^j\varrho)^n(2^j\varrho)^{-\alpha p}(2^j\varrho)^{(b-1)kp}$$
$$\times\Big(\int\limits_0^1\min\{2^{-j}(2^j\varrho)^b t, 2^j(2^j\varrho)^{-b}t^{-1}\}\,dt\Big)^p.$$

Note that from the hypothesis that $\alpha\ge kb+n$, if $2^j\varrho\ge 1$ then

$$(2^j\varrho)^n(2^j\varrho)^{-\alpha p}(2^j\varrho)^{(b-1)kp}\le(2^j\varrho)^{n-np-kp}.$$

Since $p=n/(n+k)$, we have $n-np-kp=0$ and thus

$$\|\Omega_j*a\|_\infty^p(2^j\varrho)^n \le C2^{-jp}+C\Big(\int\limits_0^1\min\{2^{-j}(2^j\varrho)^b t, 2^j(2^j\varrho)^{-b}t^{-1}\}\,dt\Big)^p.$$

A simple computation on the above integral yields

$$\sum_{2^j\varrho\ge 1}\|\Omega_j*a\|_{H_0^p}^p$$
$$\le \sum_{2^j\varrho\ge 1}\|\Omega_j*a\|_\infty^p(2^j\varrho)^n$$
$$\le C+C\sum_j\min\{(2^{-j}(2^j\varrho)^b)^p, (2^j(2^j\varrho)^{-b}(1-\ln 2^j(2^j\varrho)^{-b}))^p\}.$$

Since $b \neq 1$, the last sum is uniformly bounded. The theorem is proved.

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DEPARTMENT OF MATHEMATICS
OREGON STATE UNIVERSITY
CORVALLIS, OREGON 97331
U.S.A.

DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF WISCONSIN-MILWAUKEE
MILWAUKEE, WISCONSIN 53201
U.S.A.

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