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EXTREME ORDER STATISTICS IN AN EQUALLY CORRELATED GAUSSIAN ARRAY

Abstract. This paper contains the results concerning the weak convergence of d-dimensional extreme order statistics in a Gaussian, equally correlated array. Three types of limit distributions are found and sufficient conditions for the existence of these distributions are given.

1. Notation and definitions. Let $\{\mathbf{X}_k^{(n)} : k \in \{1,\ldots,n\},\ n \in \mathbb{N}\}$ be a triangular array of d-dimensional random vectors whose mean values and variances satisfy

(i)
$$E\mathbf{X}_k^{(n)} = (EX_{ki}^{(n)} = 0 : i \in \{1,\ldots,d\}),$$
$$V\mathbf{X}_k^{(n)} = (VX_{ki}^{(n)} = 1 : i \in \{1,\ldots,d\}).$$

We assume that

(ii) the rows of the considered array are Gaussian equally correlated sequences.

This means that

$$\operatorname{cov}(X_{ki}^{(n)}, X_{kj}^{(n)}) = \varrho_{ij}^{(0)}, \qquad \operatorname{cov}(X_{ki}^{(n)}, X_{lj}^{(n)}) = \varrho_{ij}^{(n)}$$

for all $i,j \in \{1,\ldots,d\}$, $k,l \in \{1,\ldots,n\}$, $k \neq l$, $n \in \mathbb{N}$. We denote the matrices of covariance coefficients by

$$\mathbf{\Delta}^{(0)} = (\varrho_{ij}^{(0)})_{1\le i,j\le d}, \qquad \mathbf{\Delta}^{(n)} = (\varrho_{ij}^{(n)})_{1\le i,j\le d}.$$

We additionally assume that

(iii)
$$\varrho_{ii}^{(n)} \in (0,1) \qquad \text{for } i \in \{1,\ldots,d\},\ n \in \mathbb{N}.$$

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We also define, for each $\mathbf{t} \in (0,\infty)^d$ and $\mathbf{v} \in (0,1)^d$,

$$\mathbb{A}(\mathbf{t}) = \begin{bmatrix} t_1^{1/2} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & t_d^{1/2} \end{bmatrix}, \qquad \mathbb{B}(\mathbf{v}) = \begin{bmatrix} (1-v_1)^{1/2} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & (1-v_d)^{1/2} \end{bmatrix}.$$

We denote by $\mathbf{M}_n^{(k)}$ (for $k \in \{1,\ldots,n\}$) the d-dimensional vector of the kth extreme order statistics in the sequence

$$\{\mathbf{X}_l^{(n)} : l \in \{1,\ldots,n\}\}.$$

Thus we have

$$M_{ni}^{(n)} \le M_{ni}^{(n-1)} \le \ldots \le M_{ni}^{(1)} \quad \text{for } i \in \{1,\ldots,d\},\ n \in \mathbb{N}.$$

We want to find the limit distributions of the vectors of extreme order statistics normalized by means of sequences of vectors $\mathbf{a}_n = (a_n,\ldots,a_n)$ and $\mathbf{b}_n = (b_n,\ldots,b_n)$, where $b_n = (2\ln n)^{-1/2}$ and $a_n = b_n^{-1} - \frac{1}{2}b_n(\ln\ln n + \ln 4\pi)$. (Notice that all algebraic operations are meant componentwise.)

In 1962 S. M. Berman found the limit distribution of the first extreme order statistics built on the base of a one-dimensional equally correlated Gaussian sequence (see Berman [1]). Mittal's, Ylvisaker's and Pickands's papers (see [4], [5]) give a generalization of this result in the stationary case. In the following section the limit distributions of the kth extreme order statistics built on the base of a multidimensional equally correlated Gaussian array are found.

2. Main results

Proposition 1. *Assume that the array* $\{\mathbf{X}_k^{(n)} : k \in \{1,\ldots,n\},\ n \in \mathbb{N}\}$ *satisfies conditions* (i)–(iii). *Then the rows of the array can be represented by means of sums of independent vectors in the following way*:

$$(\mathbf{X}_1^{(n)},\ldots,\mathbf{X}_n^{(n)}) \overset{a.s.}{=} (\mathbf{Y}_0^{(n)}\mathbb{A}(\mathbf{r}(n)) + \mathbf{Y}_1^{(n)}\mathbb{B}(\mathbf{r}(n)),\ldots,\mathbf{Y}_0^{(n)}\mathbb{A}(\mathbf{r}(n)) + \mathbf{Y}_n^{(n)}\mathbb{B}(\mathbf{r}(n))),$$

where $\mathbf{r}(n) = (\varrho_{11}^{(n)},\ldots,\varrho_{dd}^{(n)})$, *and* $\{\mathbf{Y}_k^{(n)} : k \in \{0\}\cup\mathbb{N}\}$ *is an independent Gaussian sequence with covariance matrices*

$$\operatorname{cov}(\mathbf{Y}_0^{(n)}) = \left(\frac{\varrho_{ij}^{(n)}}{(\varrho_{ii}^{(n)}\varrho_{jj}^{(n)})^{1/2}}\right)_{1\le i,j\le d}, \tag{1}$$

$$\operatorname{cov}(\mathbf{Y}_k^{(n)}) = \left(\frac{\varrho_{ij}^{(0)} - \varrho_{ij}^{(n)}}{[(1-\varrho_{ii}^{(n)})(1-\varrho_{jj}^{(n)})]^{1/2}}\right)_{1\le i,j\le d}, \tag{2}$$

and with vectors of mean values

$$E\mathbf{Y}_0^{(n)} = E\mathbf{Y}_k^{(n)} = \mathbf{0}$$

(see the one-dimensional case in Berman [1], Galambos [2], Section 3.8, Pickands [5]).

Proof. Fix $n \in \mathbb{N}$. We denote by $\{\mathbf{X}_k^{(n)} : k \in \mathbb{N}\}$ a d-dimensional, Gaussian, equally correlated sequence with

$$\operatorname{cov}(\mathbf{X}_k^{(n)}, \mathbf{X}_m^{(n)}) = \begin{bmatrix} \boldsymbol{\Delta}^{(0)} & \boldsymbol{\Delta}^{(n)} \\ \boldsymbol{\Delta}^{(n)} & \boldsymbol{\Delta}^{(0)} \end{bmatrix} \qquad \text{for } k \neq m\,,$$

and with $E\mathbf{X}_k^{(n)} = \mathbf{0}$ for $k \in \mathbb{N}$. (Thus $\{\mathbf{X}_k^{(n)} : k \in \mathbb{N}\}$ contains the nth row of the considered array.) For $i \in \{1, \ldots, d\}$ the Gaussian sequences of random variables $\{X_{ki}^{(n)} : k \in \mathbb{N}\}$ are equally correlated with parameters $\varrho_{ii}^{(n)}$. Hence they have the following representation (see Berman [1], Galambos [2]):

$$X_{ki}^{(n)} = Y_{0i}^{(n)}(\varrho_{ii}^{(n)})^{1/2} + Y_{ki}^{(n)}(1 - \varrho_{ii}^{(n)})^{1/2} \qquad \text{for } i \in \{1, \ldots, d\},\ k \in \mathbb{N}\,,$$

where the sequences $\{Y_{ki}^{(n)}\colon\ k \in \{0\} \cup \mathbb{N}\}$ consist of independent random Gaussian variables with mean 0 and variance 1. The random variables $Y_{0i}^{(n)}$ can be obtained from the ergodic theorem in the following way:

$$Y_{0i}^{(n)} = (\varrho_{ii}^{(n)})^{-1/2} \underset{k\to\infty}{\text{l.i.m.}}\ \frac{1}{k} \sum_{j=1}^{k} X_{ji}^{(n)} \qquad \text{for } i \in \{1, \ldots, d\}\,. \tag{3}$$

Because the random vector $\mathbf{Z}_k^{(n)} = \frac{1}{k}\sum_{j=1}^{k} \mathbf{X}_j^{(n)} \mathbb{A}^{-1}(\mathbf{r}(n))$ is normal and $E\mathbf{Z}_k^{(n)} = \mathbf{0}$ its characteristic function $\Psi_k^{(n)}$ is

$$\Psi_k^{(n)}(\mathbf{w}) = \exp(-\tfrac{1}{2}\mathbf{w}\mathbb{O}_k^{(n)}\mathbf{w}') \qquad \text{for } \mathbf{w} \in \mathbb{R}^d\,,$$

where $\mathbb{O}_k^{(n)} = (o_k^{(n)}(p,q))_{1\le p,q\le d}$. It is easy to see that

$$o_k^{(n)}(p,q) = \left[\frac{1}{k}\varrho_{pq}^{(0)} + \left(1 - \frac{1}{k}\right)\varrho_{pq}^{(n)}\right](\varrho_{pp}^{(n)}\varrho_{qq}^{(n)})^{-1/2}\,. \tag{4}$$

Notice that if $\mathbf{Y}_0^{(n)} = (Y_{01}^{(n)}, \ldots, Y_{0d}^{(n)})$ then

$$\begin{aligned} P(\|\mathbf{Z}_k^{(n)} - \mathbf{Y}_0^{(n)}\| > \varepsilon) &= P(\max\{|Z_{ki}^{(n)} - Y_{0i}^{(n)}| :\ i \in \{1, \ldots, d\}\} > \varepsilon) \\ &\le \sum_{i=1}^{d} P(|Z_{ki}^{(n)} - Y_{0i}^{(n)}| > \varepsilon) \le \sum_{i=1}^{d} \frac{E|Z_{ki}^{(n)} - Y_{0i}|^2}{\varepsilon^2}\,. \end{aligned}$$

From (3) we obtain $P(\|\mathbf{Z}_k^{(n)} - \mathbf{Y}_0^{(n)}\| > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0$ for all $\varepsilon > 0$. Hence for each $\mathbf{w} \in \mathbb{R}^d$ we have $\Psi_k^{(n)}(\mathbf{w}) \underset{n\to\infty}{\longrightarrow} \Psi_0^{(n)}(\mathbf{w})$, where $\Psi_0^{(n)}$ is the characteristic function of $\mathbf{Y}_0^{(n)}$. From (4) it results that

$$\Psi_0^{(n)}(\mathbf{w}) = \exp(-\tfrac{1}{2}\mathbf{w}\mathbb{O}_0^{(n)}\mathbf{w}'), \quad \text{where} \quad \mathbb{O}_0^{(n)} = (\varrho_{ij}^{n)}(\varrho_{ii}^{(n)}\varrho_{jj}^{(n)})^{-1/2})_{1\le i,j\le d}\,.$$

We have shown that $\mathbf{Y}_0^{(n)}$ is normally distributed with covariance matrix (1).

Define the random Gaussian sequence

$$\mathbf{Y}_k^{(n)} = [\mathbf{X}_k^{(n)} - \mathbf{Y}_0^{(n)}\mathbb{A}(\mathbf{r}(n))]\mathbb{B}(\mathbf{r}(n))^{-1}\,.$$

From (3) it follows (Rudin [6], Theorem 4.6) that

$$(5)\qquad EX_{ki}^{(n)}Y_{0j}^{(n)} = (\varrho_{jj}^{(n)})^{-1/2}\lim_{m\to\infty}\frac{1}{m}\sum_{p=1}^{m}EX_{ki}^{(n)}X_{pj}^{(n)} = \varrho_{ij}^{(n)}(\varrho_{jj}^{(n)})^{-1/2}\,.$$

Hence we obtain (for $k\in\mathbb{N}$)

$$\begin{aligned}
&\operatorname{cov}(Y_{ki}^{(n)}Y_{kj}^{(n)})\\
&\quad= [(1-\varrho_{ii}^{(n)})(1-\varrho_{jj}^{(n)})]^{-1/2}E[X_{ki}^{(n)} - (\varrho_{ii}^{(n)})^{1/2}Y_{0i}^{(n)}][X_{kj}^{(n)} - (\varrho_{jj}^{(n)})^{1/2}Y_{0j}^{(n)}]\\
&\quad= [(1-\varrho_{ii}^{(n)})(1-\varrho_{jj}^{(n)})]^{-1/2}[\varrho_{ij}^{(0)} - (\varrho_{jj}^{(n)})^{1/2}\varrho_{ij}^{(n)}(\varrho_{jj}^{(n)})^{-1/2}\\
&\qquad - (\varrho_{ii}^{(n)})^{1/2}\varrho_{ij}^{(n)}(\varrho_{ii}^{(n)})^{-1/2} + (\varrho_{ii}^{(n)})^{1/2}(\varrho_{jj}^{(n)})^{1/2}\varrho_{ij}^{(n)}(\varrho_{ii}^{(n)}\varrho_{jj}^{(n)})^{-1/2}]\\
&\quad= (\varrho_{ij}^{(0)} - \varrho_{ij}^{(n)})[(1-\varrho_{ii}^{(n)})(1-\varrho_{jj}^{(n)})]^{-1/2}\,.
\end{aligned}$$

In other words, $\mathbf{Y}_k^{(n)}$ has the covariance matrix (2).

The independence of the vectors of the sequence $\{\mathbf{Y}_k^{(n)} : k\in\{0\}\cup\mathbb{N}\}$ results from (5) in the following way:

$$\begin{aligned}
&\operatorname{cov}(Y_{0i}^{(n)}Y_{kj}^{(n)})\\
&\qquad= (1-\varrho_{jj}^{(n)})^{-1/2}[\varrho_{ij}^{(n)}(\varrho_{ii}^{(n)})^{-1/2} - (\varrho_{jj}^{(n)})^{1/2}\varrho_{ij}^{(n)}(\varrho_{ii}^{(n)}\varrho_{jj}^{(n)})^{-1/2}] = 0
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{cov}(Y_{ki}^{(n)}Y_{mj}^{(n)})\\
&\quad= [(1-\varrho_{ii}^{(n)})(1-\varrho_{jj}^{(n)})]^{-1/2}[\varrho_{ij}^{(n)} - (\varrho_{jj}^{(n)})^{1/2}\varrho_{ij}^{(n)}(\varrho_{jj}^{(n)})^{-1/2}\\
&\qquad - (\varrho_{ii}^{(n)})^{1/2}\varrho_{ij}^{(n)}(\varrho_{ii}^{(n)})^{-1/2} + (\varrho_{ii}^{(n)})^{1/2}(\varrho_{jj}^{(n)})^{1/2}\varrho_{ij}^{(n)}(\varrho_{ii}^{(n)}\varrho_{jj}^{(n)})^{-1/2}] = 0
\end{aligned}$$

and so the proof is complete.

THEOREM 1. *Suppose the array* $\{\mathbf{X}_k^{(n)} : k\in\{1,\ldots,n\},\ n\in\mathbb{N}\}$ *satisfies conditions* (i)–(iii), *and additionally the following conditions hold*:

$$\text{(iv)}\qquad \varrho_{ii}^{(n)}\ln n \underset{n\to\infty}{\longrightarrow} \tau_{ii}\in(0,\infty)\qquad \textit{for } i\in\{1,\ldots,d\}\,,$$

$$\text{(v)}\qquad \varrho_{ij}^{(n)}(\varrho_{ii}^{(n)}\varrho_{jj}^{(n)})^{-1/2} \underset{n\to\infty}{\longrightarrow} \varrho_{ij}\qquad \textit{for } i,j\in\{1,\ldots,d\}\,.$$

Then

$$P((\mathbf{M}_n^{(k)} - \mathbf{a}_n)/\mathbf{b}_n \le \mathbf{x}) \underset{n\to\infty}{\longrightarrow} (\Lambda_{\mathbf{t}}^k * \Phi_{\mathbf{t}})(\mathbf{x})\qquad \textit{for } k\in\mathbb{N},\ \mathbf{x}\in\mathbb{R}^d\,,$$

where $\mathbf{t} = (\tau_{11}, \ldots, \tau_{dd})$, $*$ *denotes convolution*,

$$\varLambda_{\mathbf{t}}^k(\mathbf{x}) = \varLambda^k(\mathbf{x}+\mathbf{t}), \qquad \varLambda^k(\mathbf{x}) = \prod_{i=1}^{d} e^{-e^{-x_i}} \sum_{s=0}^{k-1} \frac{(e^{-x_i})^s}{s!}\,,$$

$$\varPhi_{\mathbf{t}}(\mathbf{x}) = \varPhi(2^{-1/2}\mathbf{x}\mathbb{A}^{-1}(\mathbf{t}))\,,$$

and $\varPhi$ *is the distribution function of a Gaussian vector* $\mathbf{Y}_0$, *with* $\operatorname{cov}(\mathbf{Y}_0) = (\varrho_{ij})_{1\le i,j\le d}$ *and* $E\mathbf{Y}_0 = \mathbf{0}$.

Proof. We denote the kth extreme order statistics in the sequence $\{\mathbf{Y}_l^{(n)} : l \in \{1,\ldots,n\}\}$ by $\overline{\mathbf{M}}_n^{(k)}$ (see Proposition 1). Observe that

$$(\mathbf{M}_n^{(k)} - \mathbf{a}_n)/\mathbf{b}_n = \mathbf{I}_n + \mathbf{J}_n^{(k)}\,,$$

where

$$\mathbf{I}_n = (2\ln n)^{1/2}\mathbf{Y}_0^{(n)}\mathbb{A}(\mathbf{r}(n)), \qquad \mathbf{J}_n^{(k)} = [\overline{\mathbf{M}}_n^{(k)} - \mathbf{a}_n\mathbb{B}^{-1}(\mathbf{r}(n))]\mathbb{B}(\mathbf{r}(n))/\mathbf{b}_n\,.$$

Since the vectors $\mathbf{I}_n$ and $\mathbf{J}_n^{(k)}$ are independent, to complete the proof it is enough to show that for all $\mathbf{x} \in \mathbb{R}^d$,

$$P(\mathbf{I}_n \le \mathbf{x}) \underset{n\to\infty}{\longrightarrow} \varPhi_{\mathbf{t}}(\mathbf{x})\,, \tag{6}$$

$$P(\mathbf{J}_n^{(k)} \le \mathbf{x}) \underset{n\to\infty}{\longrightarrow} \varLambda_{\mathbf{t}}^k(\mathbf{x})\,. \tag{7}$$

Condition (v) implies that the distribution functions of the vectors $\mathbf{Y}_0^{(n)}$ (see Proposition 1) converge pointwise to the distribution function of $\mathbf{Y}_0$; moreover, from (iv) it follows that

$$(2\ln n)^{1/2}\mathbb{A}(\mathbf{r}(n)) \underset{n\to\infty}{\longrightarrow} 2^{1/2}\mathbb{A}(\mathbf{t})\,.$$

Hence we obtain (6).

Corollary 2 of Wiśniewski [7] shows that the independence of the components of the limit maximum vector $\overline{\mathbf{M}}^{(1)}$ is equivalent to the independence of the components of the limit vectors of the order statistics $\overline{\mathbf{M}}^{(k)}$ for $k \in \mathbb{N}$. From Example 5.3.1 of Galambos [2] it follows that $\overline{\mathbf{M}}^{(1)}$ has independent components $\overline{M}_i^{(1)}$.

Additionally, Theorems 2.2.2 and 1.5.3 of Leadbetter, Lindgren and Rootzén [3] imply that

$$P(\overline{M}_i^{(k)} \le x_i) = e^{-e^{-x_i}} \sum_{s=0}^{k-1} \frac{(e^{-x_i})^s}{s!}\,.$$

Hence, we get

$$P((\overline{\mathbf{M}}_n^{(k)} - \mathbf{a}_n)/\mathbf{b}_n \le \mathbf{x}) \underset{n\to\infty}{\longrightarrow} \varLambda^k(\mathbf{x})\,.$$

We note that

$$P(\mathbf{J}_n^{(k)} \le \mathbf{x}) = P((\mathbf{M}_n^{(k)} - \mathbf{A}_n)/\mathbf{B}_n \le \mathbf{x})$$

where

$$\mathbf{A}_n = \mathbf{a}_n \mathbb{B}^{-1}(\mathbf{r}(n)), \qquad \mathbf{B}_n = \mathbf{b}_n \mathbb{B}^{-1}(\mathbf{r}(n))\,.$$

From a multidimensional version of Khinchin's theorem it follows that to complete the proof of (7) we must show that

$$\frac{A_{ni}-a_n}{b_n} \underset{n\to\infty}{\longrightarrow} \tau_{ii} \tag{8}$$

and

$$\frac{B_{ni}}{b_n} \underset{n\to\infty}{\longrightarrow} 1\,. \tag{9}$$

Now, (9) follows from $\varrho_{ii}^{(n)} \underset{n\to\infty}{\longrightarrow} 0$ (see (iv)). Since

$$(1-\varrho_{ii}^{(n)})^{-1/2} = 1 + \tfrac{1}{2}\varrho_{ii}^{(n)} + O((\varrho_{ii}^{(n)})^2) \qquad \text{as } \varrho_{ii}^{(n)} \to 0\,,$$

we have

$$\frac{A_{ni}-a_n}{b_n} = [\tfrac{1}{2}\varrho_{ii}^{(n)} + O((\varrho_{ii}^{(n)})^2)](2\ln n + o(\ln n)) \underset{n\to\infty}{\longrightarrow} \tau_{ii}\,,$$

and this completes the proof.

THEOREM 2. *If the array* $\{\mathbf{X}_k^{(n)} : k\in\{1,\ldots,n\},\, n\in\mathbb{N}\}$ *satisfies conditions* (i)–(iii) *and*

$$\varrho_{ii}^{(n)} \ln n \underset{n\to\infty}{\longrightarrow} 0 \qquad \textit{for } i\in\{1,\ldots,d\}\,, \tag{iv$'$}$$

then

$$P((\mathbf{M}_n^{(k)}-\mathbf{a}_n)/\mathbf{b}_n \le \mathbf{x}) \underset{n\to\infty}{\longrightarrow} \Lambda^k(\mathbf{x}) \qquad \textit{for } k\in\mathbb{N},\ \mathbf{x}\in\mathbb{R}^d\,.$$

Proof. Notice that (see the proof of Theorem 1)

$$\begin{aligned} P(\max\{|I_{ni}| : i\in\{1,\ldots,d\}\} > \varepsilon) &\le \sum_{i=1}^d P(|I_{ni}|>\varepsilon) \\ &\le \sum_{i=1}^d \frac{EI_{ni}^2}{\varepsilon^2} = \frac{1}{\varepsilon^2}\sum_{i=1}^d 2\varrho_{ii}^{(n)} E(Y_{0i}^{(n)})^2 \ln n\,. \end{aligned}$$

Hence the condition

$$P(\|I_n\|>\varepsilon) \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{for all } \varepsilon>0$$

follows from (iv)$'$. Now, the proof is similar to that of (2).

THEOREM 3. *If the array* $\{\mathbf{X}_k^{(n)} : k\in\{1,\ldots,n\},\ n\in\mathbb{N}\}$ *satisfies conditions* (i)–(iii), (v) *and*

$$\varrho_{ii}^{(n)} \ln n \underset{n\to\infty}{\longrightarrow} \infty \qquad \textit{for } i\in\{1,\ldots,d\}\,, \tag{iv$''$}$$

then

$$P([\mathbf{M}_n^{(k)} - \mathbf{a}_n \mathbb{B}(\mathbf{r}(n))]\mathbb{A}^{-1}(\mathbf{r}(n)) \le \mathbf{x}) \underset{n\to\infty}{\longrightarrow} \Phi(\mathbf{x}) \qquad \textit{for } k \in \mathbb{N},\ \mathbf{x} \in \mathbb{R}^d\,.$$

Proof. We notice that

$$[\mathbf{M}_n^{(k)} - \mathbf{a}_n \mathbb{B}(\mathbf{r}(n))]\mathbb{A}^{-1}(\mathbf{r}(n)) = \mathbf{Y}_0^{(n)} + \mathbf{N}_n^{(k)}\,,$$

where (see the proof of Theorem 1)

$$\mathbf{N}_n^{(k)} = (\overline{\mathbf{M}}_n^{(k)} - \mathbf{a}_n)\mathbb{B}(\mathbf{r}(n))\mathbb{A}^{-1}(\mathbf{r}(n))\,.$$

To complete the proof it is enough to show that

$$P(\|\mathbf{N}_n^{(k)}\| > \varepsilon) \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{for all } \varepsilon > 0,\ k \in \mathbb{N}\,. \tag{10}$$

It is easy to see that

$$\begin{aligned} P(\max\{|N_{ni}^{(k)}| : i \in \{1,\ldots,d\}\} > \varepsilon) &\le \sum_{i=1}^d P(|N_{ni}^{(k)}| > \varepsilon) \\ &\le \sum_{i=1}^d P\left(\left|\frac{\overline{M}_{ni}^{(k)} - a_n}{b_n}\right| > \varepsilon(2\varrho_{ii}^{(n)} \ln n)^{1/2}\right). \end{aligned} \tag{11}$$

Since the limit distributions of the sequences $\{(\overline{M}_{ni}^{(k)} - a_n)/b_n : n \in \mathbb{N}\}$ exist for $i \in \{1,\ldots,d\}$, $k \in \mathbb{N}$ (see for example Galambos [2]), the condition (10) follows from (iv)$''$ and (11).

We emphasize that in the situation considered in Theorem 3 all extreme order statistics have identical limit distributions.

Finally, we formulate a result which is easy to obtain by the method of proof of Proposition 1 and Theorem 3.

Theorem 4. *If a d-dimensional, normalized, Gaussian sequence $\{\mathbf{X}_n : n \in \mathbb{N}\}$ is equally correlated with covariance matrix*

$$\operatorname{cov}(\mathbf{X}_m, \mathbf{X}_n) = \begin{pmatrix} \mathbf{\Delta}^{(0)} & \mathbf{\Delta}^{(1)} \\ \mathbf{\Delta}^{(1)} & \mathbf{\Delta}^{(0)} \end{pmatrix} \qquad (\textit{for } n \ne m)$$

and $\varrho_{ii}^{(1)} \in (0,1)$ for $i \in \{1,\ldots,d\}$, then

$$P([\mathbf{M}_n^{(k)} - \mathbf{a}_n \mathbb{B}(\mathbf{r}(n))]\mathbb{A}^{-1}(\mathbf{r}(n)) \le \mathbf{x}) \underset{n\to\infty}{\longrightarrow} \Phi_1(\mathbf{x}) \qquad \textit{for } k \in \mathbb{N},\ \mathbf{x} \in \mathbb{R}^d\,,$$

where Φ_1 is the distribution function of a Gaussian vector $\mathbf{Y}$ with

$$\operatorname{cov}(\mathbf{Y}) = \left(\frac{\varrho_{ij}^{(1)}}{(\varrho_{ii}^{(1)}\varrho_{jj}^{(1)})^{1/2}}\right)_{1\le i,j\le d} \qquad \textit{and} \quad E\mathbf{Y} = \mathbf{0}\,.$$

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