ACTA ARITHMETICA
LXIX.1 (1995)

On the sums $S_\chi(m)$

by

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1. Introduction. Consider a primitive non-principal Dirichlet character χ modulo q. For any positive integer m define

$$S_\chi(m) = \sum_{a=1}^{q} \chi(a)a^m.$$

These character sums have been studied by Williams [11] and Toyoizumi [10]. The first author proves, as a consequence of a more general theorem in the case where χ is assumed to be the Legendre symbol, that $S_\chi(m) = O(q^{m+1/2}\log q)$. The second author shows a more concrete estimate giving a bound for the constant implicit in the O symbol. He proves the following:

THEOREM. (a) *If* $\chi(-1) = 1$ *then*

$$|S_\chi(m)| \le \frac{2\zeta(2)e^{2\pi}m!}{(2\pi)^{m+1}} q^{m+1/2}.$$

(b) *If* $\chi(-1) = -1$ *then*

$$|S_\chi(m)| \le \left(\frac{2\zeta(3)e^{2\pi}m!}{(2\pi)^{m+1}} + \frac{|L(\chi,1)|}{\pi}\right) q^{m+1/2}.$$

See Toyoizumi [10].

Here L stands for the Dirichlet function of the given character. Toyoizumi uses the generalized Bernoulli numbers and the bound for L due to Pintz [7] and Stephens [9] and based on the work of Burgess [2], [3], in order to complete the bound in the second case.

The purpose of this paper is to give an important improvement on the preceding theorem. In fact, our result is in some sense best possible. For any primitive non-principal character χ modulo q and any natural number m we will prove:

Theorem 1. *If* $\chi(-1) = 1$ *then*

$$|S_\chi(m)| \le q^{m+1/2}\left(\frac{m-1}{2(m+1)}\right).$$

Theorem 2. *If* $\chi(-1) = -1$ *then*

$$\left|S_\chi(m) + \frac{q^{m+1/2}}{\pi i \overline{\tau}_q(\chi)} L(\overline{\chi}, 1)\right| \le q^{m+1/2}\left(\frac{m}{\pi}\int_0^1 \ln\frac{1}{2\sin\pi t} t^{m-1}\, dt\right).$$

Here τ_q is a complex number of modulus one related to the gaussians sums.

In particular, we can deduce the following

Corollary 1. *If* $\chi(-1) = -1$ *then*

$$|S_\chi(m)| \le q^{m+1/2}\left(\frac{1}{\pi}\left|\sum_{n=[m/(2\pi)]}^{\infty}\frac{\chi(n)}{n}\right| + \frac{1}{\pi}\right).$$

Our proof is based on a Fourier analysis approach that leads to a representation of $S_\chi(m)$ from which the theorems follow. In order to prove that the constants are sharp we use the particular case of q being a prime number and χ the Legendre symbol. In fact, we prove

Corollary 2. *Let* χ *be the Legendre symbol and* p *a prime number. Then for fixed* m *and any* $\varepsilon > 0$ *we have*:

(a) *There are infinitely many primes* $p \equiv 1\ (4)$ *such that*

$$S_\chi(m) \ge q^{m+1/2}\left(\frac{m-1}{2(m+1)} - \varepsilon\right).$$

(b) *There are infinitely many primes* $p \equiv 3\ (4)$ *such that*

$$S_\chi(m) \ge q^{m+1/2}\left(\frac{m}{\pi}\int_0^1 \ln\frac{1}{2\sin\pi t} t^{m-1}\, dt - \frac{L(\chi,1)}{\pi} - \varepsilon\right).$$

2. Lemmas. We will base the proof of the theorems on three lemmas. The first is just a partial summation, the second gives the expression of the incomplete sums of a character by means of a Fourier development and in the third we estimate the trigonometric integrals needed for the proof.

For any χ as above, let

$$\tau_q(\chi) = \frac{1}{\sqrt{q}}\sum_{n=1}^{k}\chi(n)e^{2\pi i n/q}.$$

This number has absolute value one. We use the finite Fourier expansion for primitive non-principal characters:

$$\chi(n) = \frac{1}{\sqrt{q}}\tau_q(\chi)\sum_{r=1}^{q}\overline{\chi}(r)e^{-2\pi irn/q}.$$

The fact that $\sum_{m=1}^{k}\chi(m) = 0$ will also be used. See Apostol [1] for these expansions.

The exact value of $\tau_q(\chi)$ is known for q any prime number and χ the Legendre symbol. This particular case relates to the class number when $q \equiv 3\ (4)$ is a prime number.

LEMMA 1. *For any χ and any natural m,*

$$\frac{S_\chi(m)}{q^m} = -m\int_0^1\Big(\sum_{a=1}^{[qt]}\chi(a)\Big)t^{m-1}\,dt.$$

Proof. The inner sum is constant in the intervals $(r/q,(r+1)/q)$ so we have

$$\begin{aligned}
-m\int_0^1\Big(\sum_{a=1}^{[qt]}\chi(a)\Big)t^{m-1}\,dt &= -\sum_{r=1}^{q-1}\int_{r/q}^{(r+1)/q}\Big(\sum_{a=1}^{r}\chi(a)\Big)mt^{m-1}\,dt\\
&= -\sum_{r=1}^{q-1}\Big(\sum_{a=1}^{r}\chi(a)\Big(\Big(\frac{r+1}{q}\Big)^m - \Big(\frac{r}{q}\Big)^m\Big)\Big)\\
&= -\sum_{a=1}^{q-1}\chi(a)\Big(\sum_{r=a}^{q-1}\Big(\frac{r+1}{q}\Big)^m - \Big(\frac{r}{q}\Big)^m\Big)\\
&= -\sum_{a=1}^{q-1}\chi(a)\Big(1-\Big(\frac{a}{q}\Big)^m\Big) = 0 + \frac{S_\chi(m)}{q^m}.
\end{aligned}$$

This proves the lemma.

LEMMA 2. (a) *If $\chi(-1) = 1$ then for $\lambda \in [0,1)$, $\lambda \neq r/q$,*

$$\frac{\sqrt{q}}{\pi\overline{\tau}_q(\chi)}\sum_{n=1}^{\infty}\overline{\chi}(n)\frac{\sin 2\pi n\lambda}{n} = \sum_{a=1}^{[\lambda q]}\chi(a).$$

(b) *If $\chi(-1) = -1$ then for $\lambda \in [0,1)$, $\lambda \neq r/q$,*

$$\frac{\sqrt{q}}{i\pi\overline{\tau}_q(\chi)}L(\overline{\chi},1) - \frac{\sqrt{q}}{i\pi\overline{\tau}_q(\chi)}\sum_{n=1}^{\infty}\overline{\chi}(n)\frac{\cos 2\pi n\lambda}{n} = \sum_{a=1}^{[\lambda q]}\chi(a).$$

Proof. Expansions of this type were first used by Pólya [8]. For any

$\lambda \in [0,1)$ set

$$f_\lambda(x) = \begin{cases} 1 & \text{if } x \in [0,\lambda), \\ 0 & \text{if } x \in [\lambda,1) \end{cases}$$

and continue f_λ periodically with period one over the real numbers. Its Fourier development is

$$f_\lambda(x) = \lambda + \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n x}}{n} \left(\frac{\sin 2\pi n\lambda}{2} + \frac{1-\cos 2\pi n\lambda}{2i} \right),$$

which is convergent to the function except for $x = 0$ and $x = \lambda$. Hence we get

$$\begin{aligned} \sum_{a=1}^{q-1} \chi(a) f_\lambda\left(\frac{a}{q}\right) &= \sum_{a=1}^{[\lambda q]} \chi(a) \\ &= \lambda \sum_{a=1}^{q-1} \chi(a) + \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{\sin 2\pi n\lambda}{2n} + \frac{1-\cos 2\pi n\lambda}{2in} \right) \overline{\sum_{a=1}^{q-1} \overline{\chi}(a) e^{-2\pi i n a/q}} \\ &= 0 + \frac{\sqrt{q}}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\frac{\sin 2\pi n\lambda}{2n} + \frac{1-\cos 2\pi n\lambda}{2in} \right) \frac{\overline{\chi}(n)}{\overline{\tau}_q(\chi)}, \end{aligned}$$

where we have used the finite Fourier expansion quoted before, and where the bar means complex conjugation.

Now grouping together the terms with n and $-n$ we finally get

$$\begin{aligned} \sum_{a=1}^{[\lambda q]} \chi(a) = \frac{\sqrt{q}}{\pi \overline{\tau}_q(\chi)} \sum_{n=1}^{\infty} (\overline{\chi}(n) + \overline{\chi}(-n)) \frac{\sin 2\pi n\lambda}{2n} \\ + \frac{\sqrt{q}}{\pi \overline{\tau}_q(\chi)} \sum_{n=1}^{\infty} (\overline{\chi}(n) - \overline{\chi}(-n)) \frac{1-\cos 2\pi n\lambda}{2in}. \end{aligned}$$

In case (a), $\overline{\chi}(n) = \overline{\chi}(-n)$ and the second series vanishes. In case (b), $\overline{\chi}(n) = -\overline{\chi}(-n)$ and hence the first series vanishes, proving the lemma.

The important role played by the condition $\chi(-1) = 1$ or -1 becomes clear in this lemma.

Lemma 3. *Let m, n be positive integers.*

(a) *Set $I(m,n) = \int_0^1 t^{m-1} \sin(2\pi n t)\, dt$. Then*

$$|I(m,n)| \le \min\left(\frac{1}{m}, \frac{1}{2\pi n}, \frac{2\pi n}{m(m+1)} \right).$$

(b) *Set* $J(m,n)=\int_0^1 t^{m-1}\cos(2\pi nt)\,dt$. *Then*

$$|J(m,n)|\le \min\left(\frac{1}{m},\frac{1}{2\pi n},\frac{m-1}{4\pi^2n^2}\right).$$

(c) *Set* $H(m,n)=\int_0^1 t^{m-1}(1-\cos(2\pi nt))\,dt$. *Then*

$$|H(m,n)|\le \min\left(\frac{1}{m},\frac{4\pi^2n^2}{m(m+1)(m+2)}\right).$$

Proof. Let us prove (a). Parts (b) and (c) will be a consequence of (a) and an appropriate integration by parts.

If we bound $|\sin 2\pi nt|$ by 1 we get the first term in the minimum because $\int_0^1 t^{m-1}\,dt=1/m$. By making a change of variables we get

$$-I(m,n)=\int_0^1 (1-t)^{m-1}\sin(2\pi nt)\,dt$$

and estimating $|\sin 2\pi nt|\le 2\pi nt$ gives

$$|I(m,n)|\le 2\pi n\int_0^1 t(1-t)^{m-1}\,dt=\frac{2\pi n}{m(m+1)},$$

which is the third term of the minimum. Finally,

$$\begin{aligned}
-I(m,n)&=-\int_0^1 t^{m-1}\sin(2\pi nt)\,dt\\
&=-\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} t^{m-1}\sin(2\pi nt)\,dt\\
&=-\sum_{k=0}^{n-1}\int_0^{1/n}\left(t+\frac{k}{n}\right)^{m-1}\sin(2\pi nt)\,dt\\
&=\sum_{k=0}^{n-1}\int_0^{1/(2n)}\left(\left(\frac{1}{n}-t+\frac{k}{n}\right)^{m-1}-\left(t+\frac{k}{n}\right)^{m-1}\right)\sin(2\pi nt)\,dt.
\end{aligned}$$

Here we made use of the fact that $\sin 2\pi n\left(\frac{1}{2n}+t\right)=-\sin 2\pi n\left(\frac{1}{2n}-t\right)$.

Now observe that the integrand is positive so $I(m,n)$ is negative, and this implies that $J(m,n)$ is positive.

If we make a couple of integrations by parts we obtain

$$I(m,n)=\frac{-2\pi n}{m(m+1)}(1+2\pi nI(m+2,n)).$$

This and the fact already proved that $I(m,n)\le 0$ implies that $1+2\pi nI(m,n)\ge 0$, which gives the second term.

3. Proof of the theorems

Proof of Theorem 1. If we use the Lemmas 1 and 2 together we get in this case

$$\begin{aligned}\frac{S_\chi(m)}{q^m} &= -m\int_0^1\Big(\sum_{a=1}^{[qt]}\chi(a)\Big)t^{m-1}\,dt\\ &= \frac{-m\sqrt{q}}{\pi\overline{\tau}_q(\chi)}\int_0^1\left(\sum_{n=1}^{\infty}\overline{\chi}(n)\frac{\sin 2\pi nt}{n}\right)t^{m-1}\,dt\\ &= \frac{m(m-1)\sqrt{q}}{2\pi^2\overline{\tau}_q(\chi)}\int_0^1\left(\sum_{n=1}^{\infty}\overline{\chi}(n)\frac{1-\cos 2\pi nt}{n^2}\right)t^{m-2}\,dt.\end{aligned}$$

After an integration by parts, observe that the last expression together with the trivial estimate $2\zeta(2)$ for the series already improves the theorem of Toyoizumi reducing the dependence of the constant to the order m. Now we will use Lemma 3 to get the theorem.

From the first expression we have

$$\left|\frac{S_\chi(m)}{q^m}\right| \le \frac{\sqrt{q}}{\pi}m\sum_{n=1}^{\infty}\frac{1}{n}|I(m,n)|.$$

By Lemma 3, $I(m,n)$ is negative so we have

$$\left|\frac{S_\chi(m)}{q^{m+1/2}}\right| \le \frac{1}{\pi}m\sum_{n=1}^{\infty}\frac{1}{n}|I(m,n)| = -\frac{1}{\pi}m\int_0^1\sum_{n=1}^{\infty}\frac{\sin 2\pi nt}{n}t^{m-1}\,dt$$

and taking into account that

$$\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2\pi nt}{n} = \frac{1}{2}-t$$

and that

$$m\int_0^1\left(t-\frac{1}{2}\right)t^{m-1}\,dt = \frac{m-1}{2(m+1)}$$

we get the theorem.

Proof of Theorem 2. If $\chi(-1)=-1$, then Lemmas 1 and 2 give

$$\begin{aligned}\frac{S_\chi(m)}{q^m} &= -m\int_0^1\Big(\sum_{a=1}^{[qt]}\chi(a)\Big)t^{m-1}\,dt\\ &= -\frac{\sqrt{q}}{\pi i\overline{\tau}_q(\chi)}L(\overline{\chi},1) + \frac{m\sqrt{q}}{\pi i\overline{\tau}_q(\chi)}\int_0^1\left(\sum_{n=1}^{\infty}\overline{\chi}(n)\frac{\cos 2\pi nt}{n}\right)t^{m-1}\,dt\end{aligned}$$

$$= -\frac{m\sqrt{q}}{\pi i\overline{\tau}_q(\chi)} \int_0^1 \left(\sum_{n=1}^{\infty} \overline{\chi}(n)\frac{1-\cos 2\pi nt}{n}\right) t^{m-1}\,dt$$

$$= -\frac{\sqrt{q}}{\pi i\overline{\tau}_q(\chi)} L(\overline{\chi},1) - \frac{m(m-1)\sqrt{q}}{2\pi^2 i\overline{\tau}_q(\chi)} \int_0^1 \left(\sum_{n=1}^{\infty} \overline{\chi}(n)\frac{\sin 2\pi nt}{n^2}\right) t^{m-2} dt$$

after an integration by parts. Again taking the bound $\zeta(2)$ in the last series we get an improvement over Toyoizumi's result.

Let us prove Corollary 1 first.

In this case, by breaking the sum from $n = 1$ to $n = N$ in the third expression, we have

$$\left|\frac{S_\chi(m)}{q^m}\right| \le \frac{\sqrt{q}}{\pi}|L_N(\chi,1)| + \frac{\sqrt{q}}{\pi}\sum_{n=1}^{N}\frac{m}{n}|H(m,n)| + \frac{\sqrt{q}}{\pi}\sum_{n=N+1}^{\infty}\frac{m}{n}|J(m,n)|,$$

where L_N means the sum from $N+1$ to infinity, and where $H(m,n)$ and $J(m,n)$ are the integrals defined in Lemma 3.

According to Lemma 3 we have

$$|H(m,n)| \le \frac{4\pi^2 n^2}{m(m+1)(m+2)} \quad \text{and} \quad |J(m,n)| \le \frac{m-1}{4\pi^2 n^2};$$

so we get

$$\sum_{n=1}^{N}\frac{m}{n}|H(m,n)| \le \sum_{n=1}^{N}\frac{m}{n}\cdot\frac{4\pi^2 n^2}{m(m+1)(m+2)} \le \frac{2\pi^2 N(N+1)}{(m+1)(m+2)}$$

and

$$\sum_{n=N+1}^{\infty}\frac{m}{n}|J(m,n)| \le \sum_{n=N+1}^{\infty}\frac{m}{n}\cdot\frac{m-1}{4\pi^2 n^2} \le \frac{m(m-1)}{8\pi^2(N+1/2)^2}$$

and finally, taking $N = \left[\frac{m+1}{2\pi}\right]$ and collecting all the terms, we have

$$\left|\frac{S_\chi(m)}{q^m}\right| \le \frac{\sqrt{q}}{\pi}(|L_N(\chi,1)|+1),$$

which proves the corollary.

Observe that $|L_N(\chi,1)|$ can be trivially bound by $|L(\chi,1)| + \log m$.

In order to prove Theorem 2 it is enough to make the same reasoning as in the case of Theorem 1, taking into account that $J(m,n)$ is positive, as was proved in Lemma 3, and finally using the fact that

$$\sum_{n=1}^{\infty}\frac{\cos 2\pi nt}{n} = -\ln(2\sin \pi t) \quad \text{for } t \in (0,1).$$

Proof of Corollary 2. We use an argument similar to the one used by Montgomery [5]. We prove (a), the proof of (b) being identical.

Set $P = 4\prod_{q\le N} q$, where q is prime. Using quadratic reciprocity, we can find an a such that if $p \equiv a\ (P)$ then $(\frac{q}{p}) = +1$ for all primes $q \le N$. Now for those $p \equiv a\ (P)$ we have

$$\begin{aligned}
&\frac{1}{p^{m+1/2}}\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)a^m \\
&\quad = \frac{m}{\pi}\int_0^1 \sum_{n=1}^{\infty}\left(\frac{n}{p}\right)\frac{\sin 2\pi nt}{n}(1-t)^{m-1}\,dt \\
&\quad = \frac{m}{\pi}\int_0^1 \left(\sum_{n=1}^{\infty}\frac{\sin 2\pi nt}{n} - \sum_{n=N+1}^{\infty}\frac{\sin 2\pi nt}{n} + \sum_{n=N+1}^{\infty}\left(\frac{n}{p}\right)\frac{\sin 2\pi nt}{n}\right) \\
&\qquad\qquad \times (1-t)^{m-1}\,dt
\end{aligned}$$

by the fact that for any $n \le N$, $\left(\frac{n}{p}\right) = +1$.

Hence

$$\begin{aligned}
&\frac{1}{p^{m+1/2}}\sum_{a=1}^{p-1}\left(\frac{a}{p}\right)a^m \\
&\qquad \ge \frac{m}{\pi}\int_0^1 \pi(1/2-t)(1-t)^{m-1}\,dt - \frac{2m}{\pi}\sum_{n=N+1}^{\infty}\frac{|I(m,n)|}{n} \\
&\qquad = \frac{m-1}{2(m+1)} - \frac{2m}{\pi}\sum_{n=N+1}^{\infty}\frac{|I(m,n)|}{n} \\
&\qquad \ge \frac{m-1}{2(m+1)} - \frac{m}{\pi^2}\sum_{n=N+1}^{\infty}\frac{1}{n^2} \ge \frac{m-1}{2(m+1)} - \frac{m}{\pi^2}\cdot\frac{1}{N}
\end{aligned}$$

by Lemma 3. So it is enough to take N such that $N > [m/(\varepsilon\pi^2)]$.

Remarks. 1. For a fixed m, the number of p's for which we are near to reaching the best constant has order of magnitude at least e^m, so even though Theorems 1 and 2 cannot be improved in general, it seems possible to get some improvement if we fix the relative sizes of m and p. This should be achieved if we use, in the integral expression for $S_\chi(m)$, the results about the behaviour of small segments of the incomplete sums for χ and the P/olya–Vinogradov inequality. The results of Burgess already quoted and the work of Montgomery and Vaughan [6] deal with this type of sums.

2. The estimate for the constant in the P/olya–Vinogradov inequality given by Hildebrand [4] and the bounds for $L(\chi,1)$ due to Stephens [9] and Pintz [7] can be used in order to complete the theorems as far as the bound for $L(\chi,1)$ is concerned.

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Received on 23.3.1993
and in revised form on 20.6.1994 (2399)