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Note on a problem of Ruzsa

by

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1. Introduction. Let $B = \{1 \le b_1 < b_2 < \ldots\}$ be an infinite sequence of integers. For any integer n we define the *counting function* of B up to n to be the number of elements of B not exceeding n; we denote it by $B(n)$. The *lower asymptotic density* $\underline{d}B$ and the *upper asymptotic density* $\overline{d}B$ are defined by

$$\underline{d}B = \liminf_{n\to\infty} B(n)/n, \quad \overline{d}B = \limsup_{n\to\infty} B(n)/n.$$

If $\underline{d}B = \overline{d}B$, we say that B has *asymptotic density* dB, given by the common value.

In [3] I. Z. Ruzsa proved that if $A = \{1 \le a_1 < a_2 < \ldots\}$ is an infinite sequence of integers and if $a_{n+1} \le 2a_n$ for all but at most finitely many values of n, then $P(A)$ has an asymptotic density, where $P(A)$ is the set of all sums of the form $\sum \varepsilon_i a_i$, $\varepsilon_i = 0$ or 1. Ruzsa conjectured that for every pair of numbers $0 \le \alpha \le \beta \le 1$ there exists $A = \{1 \le a_1 < a_2 < \ldots\}$ for which $\underline{d}(P(A)) = \alpha$ and $\overline{d}(P(A)) = \beta$. He also mentioned that an easy argument shows the case $\beta = 1$.

In this paper we prove Ruzsa's conjecture:

THEOREM. *Let $0 \le \alpha \le \beta \le 1$. Then there exists an $A = \{a_1 < a_2 < \ldots\}$ such that*

$$\underline{d}(P(A)) = \alpha \quad \textit{and} \quad \overline{d}(P(A)) = \beta. \tag{1}$$

The finite version of this question may be the following: for which t is it possible to find a sequence $a_1 < \ldots < a_n$ so that there are exactly t distinct integers of the form $\sum_{i=1}^{n} \varepsilon_i a_i$, $\varepsilon_i = 0$ or 1. It was raised in [1] and solved in [2].

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2. The construction. If $\alpha = \beta \neq 0$ then it is easy to see that $d(P(A)) = \alpha$ for $A = \{[2^n/\alpha] \mid n \in \mathbb{N}\}$; if $\alpha = \beta = 0$ then clearly $d(P(A)) = 0$ for $A = \{2^{2^n} \mid n \in \mathbb{N}\}$. So assume that $0 \le \alpha < \beta \le 1$.

We use the following notation: $A = \{a_1 < a_2 < \ldots\}$, $A_n = \{a_1 < \ldots < a_n\}$; $s_n = \sum_{i=1}^n a_i$; $\varrho_n = |P(A_n)|/s_n$; $p_n(x) = |P(A_n) \cap [1,x]|$; $\tau_n = p_{n-1}(a_n)/a_n$.

Let $A = \bigcup_{i=0}^{\infty} \mathcal{B}_i$, where the blocks $\mathcal{B}_i$ will be determined by an iterative process.

First let us define the block $\mathcal{B}_0$. Let $k_0 = \max\{[18\beta/\alpha] + 8,\ [18\beta] + 2,\ 2/(\beta - \alpha)\}$ and let $\mathcal{B}_0 = \{a_1 < \ldots < a_{k_0}\}$, where

$$a_i = \begin{cases} 2^i & \text{if } 0 \le i \le k_0 - 1, \\ \min\{x \mid (2^{k_0+1} - 2)/(x + 2^{k_0}) \le \beta\} & \text{for } i = k_0. \end{cases}$$

Thus $P(A_{k_0}) = [1,\ 2^{k_0} - 1] \cup [x + 1,\ x + 2^{k_0} - 1]$ and so

$$\varrho_{k_0} = (2^{k_0+1} - 1)/(x + 2^{k_0}) \le \beta$$

and an easy calculation shows that $\varrho_{k_0} \ge \beta - 1/k_0$.

Assume now that the blocks $\mathcal{B}_1, \ldots, \mathcal{B}_{j-1}$ have been defined such that for each $1 \le m \le j - 1$,

$$\mathcal{B}_m = \{a_{N_1^{(m)}} < \ldots < a_{N_2^{(m)}} < \ldots < a_{N_3^{(m)}}\}$$

where $s_{N_3^{(m-1)}} < a_{N_1^{(m)}}$ with $N_1^{(m)} = N_3^{(m-1)} + 1$ and $a_{k_0} < a_{N_1^{(1)}}$ with $N_1(1) = k_0 + 1$. Furthermore, if $k := k_0 + m$, then for every m, $1 \le m \le j$, the following properties are true:

$$\alpha \le \tau_{N_1^{(k)}} \le \alpha + 1/k, \tag{2}$$

$$\varrho_{N_2^{(k)}} > \beta/3, \tag{3}$$

$$\beta - 1/k \le \varrho_{N_3^{(k)}} \le \beta. \tag{4}$$

Our task is to define blocks $\mathcal{B}_j, \mathcal{B}_{j+1}, \ldots$ so that the properties (2)–(4) remain valid for $k = k_0 + m$, $m \ge j$ as well. We verify these parallel with the construction.

In the last section we prove that for $x > s_{k_0}$,

$$\alpha \le |P(A) \cap [1,x]|/x \le \beta. \tag{5}$$

Now we note that (2), (4) and (5) imply

$$\underline{d}(P(A)) = \alpha \quad \text{and} \quad \overline{d}(P(A)) = \beta.$$

Indeed, by (2) and (4) we have

$$\lim_{k\to\infty} |P(A_{N_1^{(k)}})|/a_{N_1^{(k)}} = \alpha, \qquad \lim_{k\to\infty} |P(A_{N_3^{(k)}})|/s_{N_3^{(k)}} = \beta,$$

and by (5) we get (1).

3. Proof of the Theorem. Now we prepare the block $\mathcal{B}_k$.

We use the abbreviations $N_i = N_i^{(k)}$ for $i = 1, 2, 3$ and let $N_0 = N_3^{(k-1)}$. In the first step we make the sequence less dense. Let

$$a_{N_1} = \max\{y \mid y > s_{N_0},\ |P(A_{N_0})|/y \geq \alpha\}. \tag{6}$$

Since $1/2^{k_0} < 1/k_0 < \beta - \alpha$, y exists.

Since $N_1 > k_0 + j$, this definition implies

$$\begin{aligned} 0 \leq \tau_{N_1} - \alpha &= |P(A_{N_0})|/a_{N_1} - \alpha < |P(A_{N_0})|/a_{N_1} - |P(A_{N_0})|/(a_{N_1}+1) \\ &= |P(A_{N_0})|/\{a_{N_1}(a_{N_1}+1) < \alpha/a_{N_1} < 1/N_1 < 1/k, \end{aligned}$$

$1 \leq k_0 < k$, showing (2).

In the next step we do two things: we "stabilize" the density of our sequence and then we make it more dense up to $\beta/3$.

Let $M = a_{N_1}$. Let

$$a_{N_1+i} = (i+1)a_{N_1} \tag{7}$$

for $i = 1, \ldots, M$ and if $t := [a_{N_1}/s_{N_0}] \geq 2$ then let

$$a_{N_1+M+i} = (M+i+1)a_{N_1} + s_{N_0} \tag{8}$$

for $i = 1, \ldots, t-1$. The elements defined in (7) stabilize the density and the ones defined in (8) will make the density close to $\beta/3$, which we now show.

Let $N_2 = N_1 + M + t - 1$. Then $\varrho_{N_2} \geq \beta/3$. Indeed, if $t < 2$ then by (4), (7) and since $k > k_0 > 3/\beta + 1$, we have

$$\varrho_{N_2} > \varrho_{N_1-1}/2 = \varrho_{N_0}/2 > (\beta - 1/(k-1))/2 > \beta/3.$$

Let now $t \geq 2$ and let $M + t \leq j \leq \binom{M+1}{2}$. Clearly $P(A_t) = P(A_{t-1}) \cup \{a_t + P(A_{t-1})\}$ for every $t \in \mathbb{N}$. Since $a_{N_1} > s_{N_0}$ and by (7) we see that $w \in P(A_{N_2}) \cap [ja_{N_1}, (j+1)a_{N_1}]$ if and only if there exist $v \in P(A_{N_0})$ and z, $1 \leq z \leq \binom{M+1}{2}$, so that $w = za_{N_1} + v$. So we have

$$|P(A_{N_2}) \cap [ja_{N_1}, (j+1)a_{N_1})| = t|P(A_{N_0})| \tag{9}$$

and by (8),

$$s_{N_2} \leq \binom{M+t+2}{2} a_{N_1}. \tag{10}$$

Furthermore, if $\binom{M+1}{2} \leq j \leq \binom{M+t+1}{2}$ then it is easy to check that

$$P(A_{N_2}) \cap [ja_{N_1}, (j+1)a_{N_1}] = ja_{N_1} + \{us_{N_0} + P(A_{N_0}) \mid 0 \leq u \leq t\}. \tag{11}$$

Hence

$$2 \leq t \leq a_{N_1}/s_{N_0} = \{|P(A_{N_0})|/s_{N_0}\}\{a_{N_1}/|P(A_{N_0})|\} = \varrho_{N_0}/\tau_{N_1} < \beta/\alpha$$

so we get

$$\beta > 2\alpha. \tag{12}$$

Since $M > a_{N_1} > N_1 > k_0$ by (9), (10) and (12) we get

$$
\begin{aligned}
(13)\quad & \varrho_{N_2} \\
& = |P(A_{N_2})|/s_{N_2} \ge \left\{\binom{M+1}{2} - (M+t)\right\} t|P(A_{N_0})| \bigg/ \binom{M+t+2}{2} a_{N_1} \\
& \ge \frac{\binom{M+1}{2} - (M+t)}{\binom{M+t+2}{2}} (a_{N_1}/s_{N_0} - 1)|P(A_{N_0})|/a_{N_1} \\
& \ge ((1-2t/M)^2 - 2/M)(|P(A_{N_0})|/s_{N_0} - |P(A_{N_0})|/a_{N_1}) \\
& \ge \{(1-2\beta/(\alpha k_0))^2 - 2/k_0\}\{\beta - 1/k_0 - \alpha\} \\
& \ge (1 - 4\beta/(\alpha k_0) - 2/k_0)(\beta/2 - 1/k_0) \ge \beta/3.
\end{aligned}
$$

For the last inequality we use $k_0 > 16\beta/\alpha + 8$ and thus $1 - 4\beta/(\alpha k_0) - 2/k_0 > 3/4$; furthermore, $k_0 > 18/\beta$ and thus $\beta/2 - 1/k_0 > 4\beta/9$. This proves (3).

In the next step we achieve that the sequence will be more dense, satisfying (4). Let $v = s_{N_2}$. Let

$$a_{N_2+i} = 2^i s_{N_2} \tag{14}$$

for $i = 1, \ldots, v$. This definition implies that $a_{N_2+i} > s_{N_2+i-1}$ and so

$$\varrho_{N_2+v} = \varrho_{N_2}. \tag{15}$$

Write for short $N = N_2 + v$; $W = s_N$ and $Y = [W \min\{1/2,\ \beta/\varrho_N - 1\}]$ and $L = s_{N_2}^2$. Let now

$$K_W(z) = |P(A_N) \cap (P(A_N) + W - Y + z)| \tag{16}$$

for $0 \le z \le L$.

Lemma. *There exists a $z^* \in [0,\ L]$ such that*

$$K_W(z^*) \le Y(\varrho_N^2 + 3/s_{N_2}).$$

P r o o f. Let $K_W = \sum_{z=0}^{L} K_W(z)/L$. By (14) we have

$$|P(A_N) \cap [t,\ t+L]| < \varrho_N L + s_{N_2} = (\varrho + 1/s_{N_2})L \tag{17}$$

for $t = 0, \ldots, s_N - L$ and

$$|P(A_N) \cap [W-Y,\ W]| < Y(\varrho_N + 1/s_{N_2}). \tag{18}$$

Write

$$\mathcal{H} = P(A_N) \cap [W-Y, W], \qquad \mathcal{L}_z = W - Y + z + P(A_N).$$

Then by (17) and (18) we have

$$K_W \le \sum_{z=0}^{L} \sum_{u \in \mathcal{H} \cap \mathcal{L}_z} 1 \le (\varrho_N + 1/s_{N_2})^2 \cdot L \cdot Y/L < (\varrho_N^2 + 3/s_{N_2})Y.$$

This implies that

$$K_W(z^*) := \min_{0 \le z \le L} K_W(z) \le K_W < (\varrho_N^2 + 3/s_{N_2})Y,$$

which proves the lemma.

Let

$$a_{N+1} = W - Y + z^*. \tag{19}$$

Now we deduce a lower estimate for ϱ_{N+1}. By the Lemma we get

$$\begin{aligned} \varrho_{N+1} &= \frac{2W\varrho_N - K_W(z^*)}{2W - Y + z^*} \ge \frac{2W\varrho_N - (\varrho_N^2 + 3/s_{N_2})Y}{2W - Y + z^*} \\ &\ge \varrho_N + Y\varrho_N(1-\varrho_N)/(2W - Y + z^*) \\ &\quad - z^*/(2W - Y) - 3Y/(s_{N_2}(2W - Y)). \end{aligned} \tag{20}$$

Let

$$\omega_N = z^*/(2W - Y) - 3Y/(s_{N_2}(2W - Y)).$$

Clearly

$$\lim_{N \to \infty} \omega_N = 0.$$

First case: $Y = [W \min\{1/2, \beta/\varrho_N - 1\}] = [W(\beta/\varrho_N - 1)]$. Then by (20)

$$\begin{aligned} &\varrho_{N+1} \\ &\ge \varrho_N + W(\beta/\varrho_N - 1)\varrho_N(1-\varrho_N)/\{2W - W(\beta/\varrho_N - 1) + z^*\} - (\omega_N + 1/W) \\ &= \varrho_N + (\beta - \varrho_N)\varrho_N(1-\varrho_N)/\{(3\varrho_N - \beta) + \varrho_N z^*/W\} - (\omega_N + 1/W). \end{aligned} \tag{21}$$

Since $\beta/3 < \varrho_N < \beta \le 1$ the relation $(\beta - \varrho_N)\varrho_N(1-\varrho_N)/\{(3\varrho_N - \beta) + \varrho_N z^*/W\} > 0$ holds. This implies that if W (and so N) is large enough we have

$$\varrho_{N+1} > \varrho_N. \tag{22}$$

Repeating the previous process we define by (14) and (19) the sequence $a_{N+2}, a_{N+3}, \ldots$ More precisely, let $N^{(1)} = N + 1$ and define $a_{N^{(1)}}$ by (14) and $a_{N^{(2)}}$ by (19) and if $N^{(1)}, N^{(2)}, \ldots, N^{(2r)}$ have been defined then let $N^{(2r+1)} = N^{(2r)} + 1$ and define $a_{N^{(2r+1)}}$ by (14) and $a_{N^{(2r+2)}}$ by (19). Then (22) yields that $\beta/\varrho_{N+1} - 1 < \beta/\varrho_N$ so at each step of the iterative process described above we always fall in the first case. Since $\varrho_{N^{(i)}} \le \beta$ and also by (22) we conclude that $\lim_{i\to\infty} \varrho_{N^{(i)}} = \lambda$ exists and clearly $\lambda \le \beta$. Thus by (21) we get

$$\lambda \ge \lambda + (\beta - \lambda)\lambda(1-\lambda)/(3\lambda - \beta),$$

which implies $\lambda = \beta$. Hence there is an $i \in \mathbb{N}$ such that $\beta - 1/k \le \varrho_{N^{(i)}} \le \beta$. So choosing $N_3 = N^{(i)}$ we get (4).

Second case: $Y = [W/2]$. Then by (20) we have

$$\varrho_{N+1} \geq \varrho_N + (W/2)\varrho_N(1-\varrho_N)/(2W - W/2) + \omega'_N, \tag{23}$$

where $\lim \omega'_N = 0$. This implies that $\varrho_{N+1} \geq \varrho_N$ if W (and so N) is large enough. Repeating the previous processes which are defined by (14) and (19) we see that $\lim_{i\to\infty} \varrho_{N^{(i)}} = \mu$ exists. By (23) we conclude that $\mu \geq 1$ thus there is an $i \in \mathbb{N}$ for which $\min\{1/2, \beta/\varrho_{N^{(i)}} - 1\} = \beta/\varrho_{N^{(i)}} - 1$ so we can use case 1.

4. Proof of property (5). We divide the interval $[s_{k_0}, \infty)$ into the union

$$[s_{k_0}, \infty) = \bigcup_{k\geq k_0} [s_{N_3^{(k-1)}}, s_{N_3^{(k)}}).$$

We now prove by induction on k that if

$$s_{N_3^{(k-1)}} \leq x < s_{N_3^{(k)}}$$

for some k then (5) is true.

First, note that if we choose $a_{N_1^{(k)}}$ at each step ($a_{N_1^{(k)}}$ is the initial element of the block $\mathcal{B}_k$) then since $a_{N_1^{(k)}} > s_{N_1^{(k)}-1}$ we infer that the "density" of A will not be affected in the interval $[s_{N_3^{(k-1)}}, s_{N_3^{(k)}})$ if we select further elements $a_{N_3^{(k)}+1}, a_{N_3^{(k)}+2}, \ldots$

For $x = a_{k_0}$ by the definition of a_{k_0} we get

$$\alpha \leq p(a_{k_0})/a_{k_0} \leq \beta.$$

Now let $k > k_0$ and assume that $s_{N_3^{(k-1)}} \leq x \leq s_{N_3^{(k)}}$. We use the abbreviations $N_i = N_i^{(k)}$, $i = 1, 2, 3$, and $N_0 = N_3^{(k-1)}$ again.

1. Let $s_{N_0} \leq x < a_{N_1}$. Since $p_{N_0}(x)/x$ is a decreasing function of x in this interval we have, by (6),

$$\alpha \leq p(a_{N_1})/a_{N_1} \leq p(x)/x \leq p_{N_0}(s_{N_0})/s_{N_0} \leq \beta.$$

2. Let $a_{N_1} \leq x \leq s_{N_2}$ and let $ja_{N_1} \leq x < (j+1)a_{N_1}$ for some $1 \leq j \leq \binom{M+t+1}{2}$. Let $x' = x - ja_{N_1}$. By the definition of $a_{N_1+1}, \ldots, a_{N_2}$ we conclude by (9) and (11) that

$$p_{N_2}(x)/x = \{j|P(A_{N_1})| + \varepsilon|P(A_{N_0})| + p_{N_2}(x')\}/x,$$

where $\varepsilon = 0$ if $1 \leq j \leq \binom{M+1}{2}$ and $\varepsilon = t-1$ if $\binom{M+1}{2} \leq j \leq \binom{M+t+1}{2}$ (i.e. if $t = [a_{N_1}/s_{N_0}] \geq 2$). The inductive hypothesis

$$a_{N_1}\alpha \leq |P(A_{N_1})| \leq a_{N_1}\beta, \quad \alpha s_{N_0} \leq p(s_{N_0}) \leq \beta s_{N_0},$$

and $\alpha x' \leq p_{N_2}(x') \leq \beta x'$ yield

$$p_{N_2}(x)/x \leq \{jts_{N_0}\beta + \beta x'\}/x \leq \beta\{j(a_{N_1}/s_{N_0})s_{N_0} + x'\}/x = \beta$$

and

$$p_{N_2}(x)/x \geq \{ja_{N_1}\alpha + \alpha x'\}/x = \alpha\{ja_{N_1} + x'\}/x = \alpha.$$

3. $s_{N_2} < x \leq s_N$ where $N = N_2 + v$ was defined in (14). By (14),

$$[a_{N_2+i}, a_{N_2+i+1}] \cap P(A_N) = a_{N_2+i} + P(A_{N_2}). \tag{24}$$

Thus if $a_{N_2+i} \leq x < 2a_{N_2+i}$ and $x' = x - a_{N_2+i}$ then by the inductive hypothesis again and by (24),

$$p_N(x)/x \leq \{\beta a_{N_2+i} + x'\beta\}/x = \beta \tag{25}$$

and

$$p_N(x)/x \geq \{\alpha a_{N_2+i} + x'\alpha\}/x = \alpha. \tag{26}$$

4. Finally, let $x \in [a_{N+1}, s_{N+1}]$. Since $a_{N+1} < s_N$ it follows that if $x \leq s_N$ then $p_N(x)/x \geq \alpha$. This implies that

$$p_{N+1}(x)/x \geq \alpha \tag{27}$$

for every $x \in [a_{N+1}, s_{N+1}]$.

Now we only have to prove that $p_{N+1}(x)/x \leq \beta$. If $x \geq W$ then by $Y \leq W(\beta/\varrho_N - 1)$ we have

$$\begin{aligned} p_{N+1}(x)/x &\leq \{\varrho_N x + Y\varrho_N\}/x = \varrho_N + Y\varrho_N/x \\ &\leq \varrho_N + Y\varrho_N/W \leq \varrho_N + (\beta - \varrho_N) = \beta. \end{aligned}$$

If $a_{N+1} \leq x < W$ then

$$p_{N+1}(x)/x = \varrho_N x + (x - a_{N+1})\varrho_N/x < \varrho_N + Y\varrho_N/W = \beta. \tag{28}$$

Now, to define $a_{N+2}, a_{N+3}, \ldots$ by (14) and (19) we can apply the same ideas as in items 3 and 4; in this way we conclude that (25)–(28) hold for every x with $s_N \leq x \leq s_{N_3}$, so that (5) holds and this completes the proof of the Theorem.

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