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L_p-deviations from zero of polynomials with integral coefficients

by

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Dedicated to the memory of my father

1. Introduction. Let $p(x)$ and $u(x)$ be two non-negative summable functions defined on the interval $[a,b]$, which assume the value zero only on a set of measure zero. Let $\phi_1(x),\phi_2(x),\ldots$ be a finite or denumerably infinite system of linearly independent functions defined on $[a,b]$ which belong to $L^2_{p(x)}([a,b])\cap L^p_{u(x)}([a,b])$, $p\ge 1$ ($L^q_{v(x)}([a,b])$ is the class of those functions $f(x)$ for which the product $v(x)|f(x)|^q$ is summable).

Let $\{\omega_k(x)\}$ be the orthonormal system with weight $p(x)$ that is obtained by the orthogonalization of the original system $\{\phi_k(x)\}$ according to the Schmidt procedure. Then

$$\omega_k(x)=\beta_{1k}\phi_1(x)+\ldots+\beta_{kk}\phi_k(x),\qquad \beta_{kk}=(\Delta_{k-1}/\Delta_k)^{1/2}, \tag{1}$$

and

$$\phi_m(x)=\sum_{s=1}^{m} b_{ms}\omega_s(x),\qquad b_{mm}=(\Delta_m/\Delta_{m-1})^{1/2}, \tag{2}$$

where Δ_k is the Gram determinant of the system of functions $\{\phi_i(x)\}_{i=1}^k$, $\Delta_0=1$.

We consider integrals of the type

$$\int_a^b u(x)|Q_n(x)|^p\,dx,\qquad p\ge 1, \tag{3}$$

where $Q_n(x)$ is a *non-trivial generalized polynomial*, i.e. a function of the

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form

$$Q_n(x) = \sum_{k=1}^{n} \alpha_k \phi_k(x)$$

with coefficients $\alpha_1, \ldots, \alpha_n$ not simultaneously zero.

We prove the following general theorem:

THEOREM 1. *There exists a non-trivial generalized polynomial $Q_n(x)$ with rational integral coefficients such that*

$$I_n = \int\limits_a^b u(x)|Q_n(x)|^p\,dx \le n^{p-1} \Delta_n^{p/(2n)} \sum_{s=1}^{n} A_s, \tag{4}$$

where Δ_n is the Gram determinant of the system $\{\phi_k(x)\}_{k=1}^n$ with respect to the weight function $p(x)$, $A_s = \int_a^b u(x)|\omega_s(x)|^p\,dx$ and $\{\omega_k(x)\}$ is the orthonormal system with weight $p(x)$ that is obtained by the orthogonalization of the system $\{\phi_k(x)\}$.

As applications of Theorem 1 we obtain bounds of the values of the integral (3) for integral polynomials $Q_n(x) = \sum_{k=0}^{n} \alpha_k x^k$ on certain intervals and for several weight functions $p(x)$ and $u(x)$.

(i) In [12], Theorem 1 was proved for $\{\phi_k(x)\} \subset C([a,b])$ and $p(x) = u(x) = 1$. The case $p = 2$ was proved by E. Aparicio [2, 3].

(ii) Concerning the existence of polynomials with rational integral coefficients on intervals of length less than 4 and with arbitrarily small norms (see [9, 6, 2, 8, 14, 4, 5]), D. Hilbert [9] proved the following theorem: If $b-a < 4$, then for all $0 < \delta < 1$, there exists a polynomial $P_n(x)$ with rational integral coefficients, not simultaneously zero, such that $\int_a^b P_n^2(x)\,dx < \delta < 1$.

In the case of uniform norm a similar theorem was proved by Fekete [6], see also [4]. The importance of these polynomials may be seen in [7].

2. Proof of Theorem 1. We consider an integral of type (3). Substituting in (3) the expressions (2) for the functions $\phi_m(x)$, we obtain

$$I_n = \int\limits_a^b u(x) \Big| \sum_{k=1}^{n} \alpha_k \sum_{s=1}^{k} b_{ks} \omega_s(x) \Big|^p\,dx$$

and by changing the order of summation we get

$$I_n = \int\limits_a^b u(x) \Big| \sum_{s=1}^{n} \Big[\sum_{k=s}^{n} b_{ks} \alpha_k \Big] \omega_s(x) \Big|^p\,dx \tag{5}$$

and hence

$$I_n \le \int_a^b u(x)\Big[\sum_{s=1}^{n} |L_s||\omega_s(x)|\Big]^p\,dx, \tag{6}$$

where

$$L_s = \sum_{k=s}^{n} b_{ks}\alpha_k \qquad (s=1,\ldots,n). \tag{7}$$

By Minkowski's Linear Forms Theorem [13], there exists a system of rational integers $\alpha_1,\ldots,\alpha_n$, not simultaneously zero, such that

$$|L_s| \le \Delta^{1/n} \qquad (s=1,\ldots,n), \tag{8}$$

where Δ is the determinant of the system (7).

By (2), $b_{kk} = \int_a^b \phi_k(x)\omega_k(x)p(x)\,dx$ and the determinant $\Delta = b_{11}\ldots b_{nn}$ becomes $\Delta = \Delta_n^{1/2}$, and therefore,

$$|L_s| \le \Delta_n^{1/(2n)} \qquad (s=1,\ldots,n). \tag{9}$$

From (6) and (9) and taking into account the inequality

$$\Big(\sum_{s=1}^{n} |a_s|\Big)^p \le n^{p-1}\sum_{s=1}^{n} |a_s|^p,$$

(4) follows. ∎

Remark 1. If $p=2$ and $p(x)=u(x)$, since the system $\{\omega_k(x)\}$ is orthonormal, from (5) and (9) we can obtain (see [2, 3])

$$I_n = \sum_{s=1}^{n}\Big(\sum_{k=s}^{n} b_{ks}\alpha_k\Big)^2 \le n\Delta_n^{1/n}. \tag{10}$$

Remark 2. If the functions $u(x)$ and $\{\phi_k(x)\}$ belong to $C([a,b])$, then

$$\begin{aligned} J_n &= \max_{a\le x\le b}\Big|u(x)\sum_{k=1}^{n}\alpha_k\phi_k(x)\Big| \\ &\le \Delta_n^{1/(2n)}\max_{a\le x\le b}\Big(\sum_{s=1}^{n}|u(x)\omega_s(x)|\Big) \le nM_n\Delta_n^{1/(2n)}, \end{aligned} \tag{11}$$

where

$$M_n = \max_{\substack{a\le x\le b\\ 1\le s\le n}} |u(x)\omega_s(x)|.$$

On the other hand, for a fixed natural number n, we may consider σ_n defined by

$$\sigma_n^{-np} = \inf_{Q_n}\int_a^b u(x)|Q_n(x)|^p\,dx, \tag{12}$$

where the infimum is over all non-trivial generalized polynomials with rational integral coefficients.

We then have the following result:

COROLLARY 1. *The inequality*

$$\sigma = \lim_{n\to\infty} \sigma_n \ge \lim_{n\to\infty} \Delta_n^{-1/(2n^2)} \lim_{n\to\infty} \Big(\sum_{s=1}^{n} A_s\Big)^{-1/(pn)} \tag{13}$$

holds if the limits exist.

Remark 3. If $p(x) = u(x) = (1-x^2)^{-1/2}$, estimate (13) is optimal. Consider the system $\{\phi_k(x)\} = \{\widehat{T}_{k-1}(x)\}, k = 1,\ldots,n$, of normalized orthogonal Chebyshev polynomials with positive leading coefficient (as usual, we shall denote by $\widetilde{R}_n(x)$ a polynomial of degree n normalized so that its leading coefficient is 1). Then

$$\begin{aligned}
\sigma_n^{-np} &= \inf_{\alpha_k\in\mathbb{Z}} \int_{-1}^{1} (1-x^2)^{-1/2}\Big|\sum_{k=1}^{n}\alpha_k \widehat{T}_{k-1}(x)\Big|^p\,dx\\
&\ge \|\widetilde{T}_{n-1}(x)\|_{2,p(x)}^{-p} \inf_{0\ne\alpha_n\in\mathbb{Z}} |\alpha_n|^p\\
&\quad\times \inf_{c_k\in\mathbb{R}} \int_{-1}^{1} (1-x^2)^{-1/2}|\widetilde{T}_{n-1}(x) + c_{n-2}\widehat{T}_{n-2}(x)+\ldots|^p\,dx\\
&\ge \|\widetilde{T}_{n-1}(x)\|_{2,p(x)}^{-p} \int_{-1}^{1} (1-x^2)^{-1/2}|\widetilde{T}_{n-1}(x)|^p\,dx.
\end{aligned}$$

This last inequality follows by Rivlin [11, p. 81]. Here $\|\cdot\|_{p,v(x)}$ is the L_p-norm with weight $v(x)$. In view of Achieser [1, p. 251],

$$\sigma_n^{-np} \ge \Big(\frac{2}{\pi}\Big)^{p/2} \frac{\Gamma\big(\frac12\big)\Gamma\big(\frac{p+1}{2}\big)}{\Gamma\big(\frac{p}{2}+1\big)} \qquad (n\ge 2)$$

and hence

$$\sigma = \lim_{n\to\infty}\sigma_n \le 1.$$

But for this case $\Delta_n = 1$, and

$$\begin{aligned}
A_s &= \int_{-1}^{1} (1-x^2)^{-1/2}|\widehat{T}_{s-1}(x)|^p\,dx = \Big(\frac{2}{\pi}\Big)^{p/2} \frac{\Gamma\big(\frac12\big)\Gamma\big(\frac{p+1}{2}\big)}{\Gamma\big(\frac{p}{2}+1\big)},\\
A_1 &= \pi^{1-p/2}.
\end{aligned}$$

Therefore the limit as $n\to\infty$ of the right-hand side of (13) is also 1.

3. Theorems of Hilbert's type. Let $u(x) = p(x) = [(x-a)(b-x)]^{-1/2}$ and consider the system $\{\phi_k(x) = x^k\}$, $k = 0,1,\ldots,$ on the interval $[a,b]$.

Then the polynomials $\{\omega_k(x)\}$ which form an orthonormal system are the Chebyshev polynomials $\{\widehat{T}_k(x)\}$, $k = 0, 1, \ldots$ (see [10, 11]). Since

$$b_{kk} = \int_a^b [(x-a)(b-x)]^{-1/2} x^k \widehat{T}_k(x)\,dx = \left(\frac{b-a}{4}\right)^k (2\pi)^{1/2},$$
$$b_{00} = \pi^{1/2},$$

it is clear that

$$\Delta_{n+1} = \prod_{k=0}^{n} b_{kk}^2 = \pi^{n+1} 2^n \left(\frac{b-a}{4}\right)^{n(n+1)}.$$

Moreover,

$$A_s = \left(\frac{2}{\pi}\right)^{p/2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)}, \qquad A_0 = \pi^{1-p/2}.$$

Applying the inequality (4), we then have the following result:

Theorem 2. *For every natural number* n, *there exists a non-trivial polynomial* $Q_n(x) = \sum_{k=0}^n \alpha_k x^k$ *with rational integral coefficients such that*

$$\int_a^b [(x-a)(b-x)]^{-1/2} |Q_n(x)|^p\,dx \le \left(\pi + 2^{p/2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)} n\right) 2^{pn/(2n+2)} (n+1)^{p-1} \left(\frac{b-a}{4}\right)^{pn/2} \quad (p = 1, 2, \ldots).$$

We note that the limit as $n \to \infty$ of the right-hand side is zero if $b-a < 4$. Thus we can reword Theorem 2 in the following way (see (ii)):

Theorem 3. *If* $b-a < 4$, *then for all* $0 < \delta < 1$, *there exists a polynomial* $Q_n(x) = \sum_{k=0}^n \alpha_k x^k$ *with rational integral coefficients*, *not simultaneously zero*, *such that*

$$\int_a^b [(x-a)(b-x)]^{-1/2} |Q_n(x)|^p\,dx \le \delta < 1 \qquad (p = 1, 2, \ldots).$$

It is clear that in the case $p = 2$ by (10) we can get

$$\int_a^b [(x-a)(b-x)]^{-1/2} Q_n^2(x)\,dx \le \pi(n+1) 2^{n/(n+1)} \left(\frac{b-a}{4}\right)^n.$$

Theorem 4. *For every natural number* n, *there exists a non-trivial polynomial* $Q_n(x)$ *with rational integral coefficients*, *of degree* $\le n$, *such that*

$$I_{n+1} = \int_a^b |Q_n(x)|\,dx \le 2\left(\frac{b-a}{2}\right)(n+1)\left(\frac{b-a}{4}\right)^{n/2}.$$

P r o o f. Consider the Chebyshev polynomials $\{\widehat{U}_k(x)\}$ of the second kind which form an orthonormal system with weight $p(x) = [(x-a)(b-x)]^{1/2}$ on the interval $[a,b]$ (see [10, 11]).

From Theorem 1 it follows that

$$I_{n+1} \le \Delta_{n+1}^{1/(2n+2)} \sum_{s=0}^{n} A_s. \tag{14}$$

Since

$$\begin{aligned} b_{kk} &= \int_a^b [(x-a)(b-x)]^{1/2} x^k \widehat{U}_k(x)\,dx \\ &= (\pi/2)^{1/2}\left(\frac{b-a}{2}\right)\left(\frac{b-a}{4}\right)^k, \qquad k=0,1,\ldots, \end{aligned}$$

it follows that

$$\Delta_{n+1} = (\pi/2)^{n+1}\left(\frac{b-a}{2}\right)^{2(n+1)}\left(\frac{b-a}{4}\right)^{n(n+1)}. \tag{15}$$

Moreover,

$$A_s = \int_a^b |\widehat{U}_s(x)|\,dx = 2(2/\pi)^{1/2}, \qquad s=0,1,\ldots,$$

and therefore

$$\sum_{s=0}^{n} A_s = 2(2/\pi)^{1/2}(n+1). \tag{16}$$

From (14)–(16) the theorem follows. ■

We next turn to the least-squares approximation problem on an interval:

THEOREM 5. *For every natural number n, there exists a non-trivial polynomial $Q_n(x) = \sum_{k=0}^{n} \alpha_k x^k$ with rational integral coefficients such that*

$$I_{n+1} = \int_a^b [(x-a)(b-x)]^{1/2} Q_n^2(x)\,dx \le \frac{\pi}{2}\left(\frac{b-a}{2}\right)^2 (n+1)\left(\frac{b-a}{4}\right)^n.$$

P r o o f. By Remark 1 we have

$$I_{n+1} \le (n+1)\Delta_{n+1}^{1/(n+1)}. \tag{17}$$

Let $\{\widehat{U}_k(x)\}$ be the orthonormal system obtained by the orthogonalization of $\{x^k\}$ with weight $[(x-a)(b-x)]^{1/2}$. Since

$$b_{kk} = (\pi/2)^{1/2}\left(\frac{b-a}{2}\right)\left(\frac{b-a}{4}\right)^k, \qquad k=0,1,\ldots,$$

it follows that

$$\Delta_{n+1} = (\pi/2)^{n+1}\left(\frac{b-a}{2}\right)^{2(n+1)}\left(\frac{b-a}{4}\right)^{n(n+1)}. \tag{18}$$

From (17) and (18) the theorem follows. ∎

THEOREM 6. *For every natural number n, there exists a non-trivial polynomial $Q_n(x) = \sum_{k=0}^{n} \alpha_k x^k$ with rational integral coefficients such that*

$$I_{n+1} = \int_a^b \left(\frac{b-x}{x-a}\right)^{1/2} Q_n^2(x)\,dx \le \pi\left(\frac{b-a}{2}\right)(n+1)\left(\frac{b-a}{4}\right)^{n}.$$

Proof. In this case we consider the polynomials $\{\widehat{W}_k(x)\}$ which form an orthonormal system on $[a,b]$ with weight $[(b-x)/(x-a)]^{1/2}$ (see [10, 11]). But now

$$b_{kk} = \int_a^b \left(\frac{b-x}{x-a}\right)^{1/2} x^k \widehat{W}_k(x)\,dx = \pi^{1/2}\left(\frac{b-a}{2}\right)^{1/2}\left(\frac{b-a}{4}\right)^{k},$$

and therefore

$$\Delta_{n+1} = \pi^{n+1}\left(\frac{b-a}{2}\right)^{n+1}\left(\frac{b-a}{4}\right)^{n(n+1)},$$

so that (10) becomes

$$I_{n+1} \le \pi\left(\frac{b-a}{2}\right)(n+1)\left(\frac{b-a}{4}\right)^{n}. \ ∎$$

Following the notation used by Achieser [1, pp. 249–254], let

$$\omega(x) = \left(1-\frac{x}{a_1}\right)\left(1-\frac{x}{a_2}\right)\cdots\left(1-\frac{x}{a_{2q}}\right)$$

be a polynomial which is positive in $(-1,1)$, and can have simple roots at one or both ends of $(-1,1)$. The polynomial $\omega(x)$ is of degree $2q-1$ if $a_{2q} = \infty$ and $|a_k| < \infty$, $k = 1,\ldots,2q-1$. Set

$$\begin{aligned} x &= \frac{1}{2}\left(v+\frac{1}{v}\right) \qquad (|v| \le 1),\\ a_k &= \frac{1}{2}\left(c_k+\frac{1}{c_k}\right) \qquad (|c_k| \le 1,\ k = 1,\ldots,2q),\\ \Omega(v) &= \prod_{k=1}^{2q}\sqrt{v-c_k}, \end{aligned} \tag{19}$$

$$L_m = \begin{cases} 2^{-m+1} \prod_{k=1}^{2q} (1+c_k^2)^{1/2} & (m > q), \\ 2^{-q+1} \prod_{k=1}^{2q} (1+c_k^2)^{1/2} (1+c_1 c_2 \ldots c_{2q})^{-1} & (m = q). \end{cases} \tag{19 [cont.]}$$

We consider the weight functions

$$u(x) = \frac{(1-x^2)^{(p-1)/2}}{[\omega(x)]^{p/2}} \quad \text{and} \quad p(x) = \frac{(1-x^2)^{1/2}}{\omega(x)}.$$

Let $\{\omega_k(x)\}_{k=0}^n$ be the orthonormal system with weight $p(x)$ that is obtained by the orthogonalization of $\{x^k\}_{k=0}^n$. By Achieser [1, p. 251], the system of monic polynomials $\{\widetilde{U}_m(x;\omega)\}_{m\ge q}$ of degree m in x,

$$\widetilde{U}_m(x;\omega) = L_{m+1}\left\{ v^{2q-m-1} \frac{\Omega(1/v)}{\Omega(v)} - v^{m+1-2q} \frac{\Omega(v)}{\Omega(1/v)} \right\} \frac{\sqrt{\omega(x)}}{1/v - v},$$

is orthogonal on $[-1,1]$ with weight function $p(x)$. Hence

$$\{\omega_0(x), \omega_1(x), \ldots, \omega_{q-1}(x), \widehat{U}_q(x;\omega), \ldots, \widehat{U}_n(x;\omega)\}$$

is an orthonormal system with weight $p(x)$ on $[-1,1]$.

Since

$$b_{kk} = \int_{-1}^{1} \frac{(1-x^2)^{1/2}}{\omega(x)} x^k \omega_k(x)\, dx = \|\widetilde{\omega}_k\|_{2,p(x)}, \qquad k = 0,1,\ldots,q-1,$$

and

$$b_{kk} = \int_{-1}^{1} \frac{(1-x^2)^{1/2}}{\omega(x)} x^k \widehat{U}_k(x;\omega)\, dx = \left(\frac{\pi}{2}\right)^{1/2} L_{k+1}, \qquad k = q,\ldots,n,$$

it follows that

$$\Delta_{n+1} = \Big(\prod_{k=0}^{q-1} \|\widetilde{\omega}_k\|_{2,p(x)}^2\Big) \left(\frac{\pi}{2}\right)^{n-q+1} 2^{-(n+q)(n-q+1)} \prod_{k=1}^{2q} (1+c_k^2)^{n-q+1}. \tag{20}$$

On the other hand,

$$\begin{aligned} A_s &= \int_{-1}^{1} \frac{(1-x^2)^{(p-1)/2}}{[\omega(x)]^{p/2}} |\widehat{U}_s(x;\omega)|^p\, dx \\ &= \left(\frac{\pi}{2}\right)^{-p/2} L_{s+1}^{-p} \int_{-1}^{1} \left| \frac{(1-x^2)^{1/2}}{[\omega(x)]^{1/2}} \widetilde{U}_s(x;\omega) \right|^p \frac{dx}{(1-x^2)^{1/2}} \\ &= \left(\frac{\pi}{2}\right)^{-p/2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}+1\right)} \qquad (s \ge q). \end{aligned}$$

Therefore

$$\sum_{s=0}^{n} A_s = \Big[\sum_{s=0}^{q-1} A_s + \Big(\frac{\pi}{2}\Big)^{-p/2} \frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{p+1}{2}\big)}{\Gamma\big(\frac{p}{2}+1\big)}(n-q+1)\Big]. \tag{21}$$

From (4), (20) and (21), we deduce the following result:

THEOREM 7. *Suppose that*

$$\omega(x) = \Big(1-\frac{x}{a_1}\Big)\Big(1-\frac{x}{a_2}\Big)\ldots\Big(1-\frac{x}{a_{2q}}\Big) > 0$$

in $(-1,1)$, *and* $\omega(x)$ *can have simple roots at one or both ends of the interval* $(-1,1)$.*For every natural number* $n \ge q$, *there exists a non-trivial polynomial* $Q_n(x) = \sum_{k=0}^{n} \alpha_k x^k$ *with rational integral coefficients such that*

$$\begin{aligned}
&\int_{-1}^{1} \Big|\frac{(1-x^2)^{1/2}}{[\omega(x)]^{1/2}} Q_n(x)\Big|^p \frac{dx}{(1-x^2)^{1/2}} \\
&\le (n+1)^{p-1}\Big\{\Big(\prod_{k=0}^{q-1} \|\widetilde{\omega}_k\|^2_{2,p(x)}\Big)\Big(\frac{\pi}{2}\Big)^{n-q+1} 2^{-(n+q)(n-q+1)} \\
&\qquad\qquad \times \prod_{k=1}^{2q}(1+c_k^2)^{n-q+1}\Big\}^{p/(2n+2)} \\
&\quad \times \Big[\sum_{s=0}^{q-1} \|\omega_s\|^p_{p,u(x)} + \Big(\frac{\pi}{2}\Big)^{-p/2} \frac{\Gamma\big(\frac{1}{2}\big)\Gamma\big(\frac{p+1}{2}\big)}{\Gamma\big(\frac{p}{2}+1\big)}(n-q+1)\Big].
\end{aligned}$$

4. Theorem of Fekete's type. As before we follow the notation used by Achieser [1, p. 249]. Given a polynomial

$$\omega(x) = \Big(1-\frac{x}{a_1}\Big)\Big(1-\frac{x}{a_2}\Big)\ldots\Big(1-\frac{x}{a_{2q}}\Big)$$

which is positive in $[-1,1]$. The degree of $\omega(x)$ is $2q-1$ if $a_{2q}=\infty$ and $|a_k|<\infty$ $(k=1,\ldots,2q-1)$. We use the notation (19).

Let $\{\omega_k(x)\}$ be the orthonormal system with weight

$$p(x) = \frac{1}{\omega(x)(1-x^2)^{1/2}}$$

obtained by the orthogonalization of the system $\{x^k\}$. By Achieser [1, p. 250] it is known that the system of monic polynomials $\{\widetilde{T}_m(x;\omega)\}$, $m \ge q$,

$$\widetilde{T}_m(x;\omega) = \frac{L_m}{2}\Big\{v^{2q-m}\frac{\Omega(1/v)}{\Omega(v)} + v^{m-2q}\frac{\Omega(v)}{\Omega(1/v)}\Big\}\sqrt{\omega(x)}$$

is orthogonal on $[-1,1]$ with weight $p(x)$.

Consider the orthonormal system

$$\{\omega_0(x),\omega_1(x),\ldots,\omega_{q-1}(x),\widehat{T}_q(x;\omega),\ldots,\widehat{T}_n(x;\omega)\}$$

with weight $p(x)$.

By Remark 2,

$$\max_{-1\le x\le 1}\left|\frac{1}{\sqrt{\omega(x)}}\sum_{k=0}^{n}\alpha_k x^k\right|\le (n+1)M_{n+1}\Delta_{n+1}^{1/(2n+2)}, \tag{22}$$

where

$$\begin{aligned}(23)\quad & M_{n+1}\\ &= \max\Big\{\max_{\substack{-1\le x\le 1\\ 0\le s\le q-1}}\left|\frac{1}{\sqrt{\omega(x)}}\omega_s(x)\right|,\ \max_{\substack{-1\le x\le 1\\ q\le s\le n}}\left|\frac{1}{\sqrt{\omega(x)}}\widehat{T}_s(x;\omega)\right|\Big\}\\ &= \max\Big\{\max_{\substack{-1\le x\le 1\\ 0\le s\le q-1}}\left|\frac{1}{\sqrt{\omega(x)}}\omega_s(x)\right|,\ \Big(\frac{2}{\pi}\Big)^{1/2}(1+c_1\ldots c_{2q})^{-1/2},\ \Big(\frac{2}{\pi}\Big)^{1/2}\Big\},\end{aligned}$$

and Δ_{n+1} is the Gram determinant of the system $\{x^k\}_{k=0}^n$ with weight $p(x)$. Since

$$b_{kk}=\int_{-1}^{1}\omega_k(x)x^k p(x)\,dx=\|\widetilde{\omega}_k\|_{2,p(x)},\qquad k=0,\ldots,q-1,$$

and

$$b_{kk}=\int_{-1}^{1}\widehat{T}_k(x;\omega)\frac{x^k}{\omega(x)(1-x^2)^{1/2}}\,dx=(\pi L_k L_{k+1})^{1/2},\qquad k\ge q,$$

we have

$$\begin{aligned}(24)\quad & \Delta_{n+1}\\ &=\Big(\prod_{k=0}^{q-1}\|\widetilde{\omega}_k\|_{2,p(x)}^2\Big)\pi^{n-q+1}\,2^{-n^2+(q-1)^2}\frac{1}{1+c_1\ldots c_{2q}}\prod_{k=1}^{2q}(1+c_k^2)^{n-q+1}.\end{aligned}$$

From (22)–(24), we have thus proved the following:

Theorem 8. *Suppose that*

$$\omega(x)=\Big(1-\frac{x}{a_1}\Big)\Big(1-\frac{x}{a_2}\Big)\ldots\Big(1-\frac{x}{a_{2q}}\Big)$$

is positive in $[-1,1]$. *For every natural number* $n\ge q$, *there exists a nontrivial polynomial* $Q_n(x)=\sum_{k=0}^n\alpha_k x^k$ *with rational integral coefficients such that*

$$
\begin{aligned}
&\max_{-1\le x\le 1}\left|\frac{1}{\sqrt{\omega(x)}}\sum_{k=0}^{n}\alpha_k x^k\right| \\
&\qquad \le (n+1)\max\Bigg\{\max_{\substack{-1\le x\le 1\\ 0\le s\le q-1}}\left|\frac{1}{\sqrt{\omega(x)}}\omega_s(x)\right|, \\
&\qquad\qquad\qquad \left(\frac{2}{\pi}\right)^{1/2}(1+c_1\ldots c_{2q})^{-1/2}, \left(\frac{2}{\pi}\right)^{1/2}\Bigg\} \\
&\qquad \times\Bigg\{\Big(\prod_{k=0}^{q-1}\|\widetilde{\omega}_k\|^2_{2,p(x)}\Big)\pi^{n-q+1}\,2^{-n^2+(q-1)^2} \\
&\qquad\qquad \times\frac{1}{1+c_1\ldots c_{2q}}\prod_{k=1}^{2q}(1+c_k^2)^{n-q+1}\Bigg\}^{1/(2n+2)}.
\end{aligned}
$$

References

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