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On an additive function on the set of ideals of an arbitrary number field

by

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1. Introduction. In [1] K. Alladi and P. Erdős showed that if $\beta_\alpha(n) = \sum_{p\mid n} p^\alpha$ then

$$\sum_{n\le x}\beta_1(n) \sim \frac{\pi^2}{12}\cdot\frac{x^2}{\log x}.$$

Many papers have been written concerning the additive function $\beta_1(n)$, see for example [4], [5], [6]. The best result is due to the author [2] and A. Mercier [9]; they have given the asymptotic formula

$$\sum_{n\le x}\beta_1(n) = x^2\Big\{\sum_{i=1}^{m} d_i/\log^i x + O(1/\log^{m+1} x)\Big\} \tag{1}$$

with arbitrary fixed $m\ge 1$ and

$$d_i = \bigg(\sum_{v=0}^{i-1}\frac{(-2)^v}{v!}\zeta^{(v)}(2)\bigg)\frac{(i-1)!}{2^i},\qquad 1\le i\le m. \tag{2}$$

J. M. De Koninck and A. Ivić [5] also got (1), but they did not give the expression of (2).

In 1989, P. Zarzycki [10] studied the distribution of values of an additive function $\mathcal{B}_\alpha(\mathfrak{a})$ on the Gaussian integers given by

$$\mathcal{B}_\alpha(\mathfrak{a}) = \sum_{\mathfrak{p}\mid\mathfrak{a}}{}^{*} N(\mathfrak{p})^\alpha$$

with fixed $\alpha>0$; the asterisk means that the summation is over the non-associate prime divisors $\mathfrak{p}$ of a Gaussian integer $\mathfrak{a}$ and $N(\mathfrak{a}) = N(x+iy) = x^2+y^2$ is the norm of $\mathfrak{a}$. This function is a generalization of the function $\beta_\alpha(n)$. Zarzycki obtained the asymptotic formula for the summatory function $\sum_{\mathfrak{a}\in D}\mathcal{B}_\alpha(\mathfrak{a})$ by using the complex integration technique where D is a certain set of Gaussian integers. The main result he obtained is the following

THEOREM 1. *For $x \to \infty$,*

$$\sum_{N(\mathfrak{a})\le x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) = \frac{\zeta(1+\alpha)L(1+\alpha,\chi_4)}{1+\alpha}\cdot\frac{x^{1+\alpha}}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right)$$

where χ_4 denotes the non-principal Dirichlet character modulo 4.

In [3], I announced that by using an elementary technique one could prove the following

THEOREM 2. *For $x \to \infty$,*

$$\sum_{N(\mathfrak{a})\le x}^{*} \mathcal{B}_\alpha(\mathfrak{a}) = \sum_{n=1}^{N}\left(\sum_{v=0}^{n-1}\frac{(-1)^v(1+\alpha)^v}{v!}\xi^{(v)}(1+\alpha)\right) \times \frac{(n-1)!x^{1+\alpha}}{(1+\alpha)^n\log^n x} + O\left(\frac{x^{1+\alpha}}{\log^{N+1} x}\right)$$

with any fixed positive integer N, where $\xi(s) = \zeta(s)L(s,\chi_4)$ and $\xi^{(v)}(s)$ is the v-th derivative of $\xi(s)$.

In the present paper, we consider the function $\mathcal{B}_\alpha(\mathfrak{a})$ defined on the integral ideals of an arbitrary number field K by the formula

$$\mathcal{B}_\alpha(\mathfrak{a}) = \sum_{\mathfrak{p}|\mathfrak{a}}^{*} N(\mathfrak{p})^\alpha,$$

where the asterisk means that $\mathfrak{p}$ runs over prime ideals of K. We obtain the following

THEOREM 3. *For every number field K,*

$$\sum_{N(\mathfrak{a})\le x} \mathcal{B}_\alpha(\mathfrak{a}) = \sum_{n=1}^{N}\sum_{v=0}^{n-1}\frac{(-1)^v(1+\alpha)^v}{v!}\zeta_K^{(v)}(1+\alpha) \times \frac{(n-1)!x^{1+\alpha}}{(1+\alpha)^n\log^n x} + O\left(\frac{x^{1+\alpha}}{\log^{N+1} x}\right) \tag{3}$$

with any fixed positive integer N, where ζ_K is the Dedekind zeta function of K and $\zeta_K^{(v)}$ its v-th derivative.

Clearly Theorem 3 contains Theorem 2 since every ideal of $\mathbb{Q}(\sqrt{-1})$ is principal.

2. Auxiliary lemmas

LEMMA 1. *Let K be a number field. Then*

$$\pi(x,K) := \sum_{N(\mathfrak{p})\le x}^{*} 1 = \operatorname{Li} x + O(xe^{-c\sqrt{\log x}})$$

with some $c>0$. Moreover, for arbitrary fixed $N\ge 1$,

$$\pi(x,K)=\sum_{n=1}^{N}\frac{(n-1)!x}{\log^n x}+O\Big(\frac{x}{\log^{N+1}x}\Big).$$

For the proof, see [8], Theorem 191, and [7], §4, formula (5).

LEMMA 2. *For arbitrary fixed $N\ge 1$ and $\alpha>0$,*

$$\sum_{N(\mathfrak{p})\le x}^{*}N(\mathfrak{p})^\alpha=\sum_{n=1}^{N}\frac{(n-1)!}{(1+\alpha)^n}\cdot\frac{x^{1+\alpha}}{\log^n x}+O\Big(\frac{x^{1+\alpha}}{\log^{N+1}x}\Big).$$

Proof. We have

$$\begin{aligned}\sum_{N(\mathfrak{p})\le x}^{*}N(\mathfrak{p})^\alpha&=\int_{2-\varepsilon}^{x}t^\alpha\,d\pi(t,K)\\&=x^\alpha\pi(x,K)-\int_{2-\varepsilon}^{x}\alpha t^{\alpha-1}\pi(t,K)\,dt\\&=\sum_{n=1}^{N}\frac{(n-1)!x^{1+\alpha}}{\log^n x}-\int_{2-\varepsilon}^{x}\sum_{n=1}^{N}\frac{(n-1)!\alpha t^\alpha}{\log^n t}\,dt+O\Big(\frac{x^{1+\alpha}}{\log^{N+1}x}\Big)\\&=\frac{1}{1+\alpha}\sum_{n=1}^{N}\frac{(n-1)!x^{1+\alpha}}{\log^n x}-\alpha\sum_{n=2}^{N}\Big(\frac{1}{(1+\alpha)^2}+\frac{1}{(1+\alpha)^3}\\&\quad+\ldots+\frac{1}{(1+\alpha)^n}\Big)\frac{(n-1)!x^{1+\alpha}}{\log^n x}+O\Big(\frac{x^{1+\alpha}}{\log^{N+1}x}\Big)\\&=\sum_{n=1}^{N}\frac{(n-1)!}{(1+\alpha)^n}\cdot\frac{x^{1+\alpha}}{\log^n x}+O\Big(\frac{x^{1+\alpha}}{\log^{N+1}x}\Big).\ \blacksquare\end{aligned}$$

LEMMA 3. *Let $2\le Q<\log^{L+1}x$, for arbitrary fixed $L\ge 1$. Then*

$$\begin{aligned}&\sum_{n=1}^{N}\frac{(n-1)!}{(1+\alpha)^n}\cdot\frac{1}{\log^n x/Q}\\&\quad=\sum_{n=1}^{N}\Big(\sum_{v=0}^{n-1}\frac{(1+\alpha)^v\log^v Q}{v!}\Big)\frac{(n-1)!}{(1+\alpha)^n\log^n x}+O\Big(\frac{\log^N Q}{\log^{N+1}x}\Big).\end{aligned}$$

Proof. By Newton's binomial formula,

$$\sum_{n=1}^{N}\frac{(n-1)!}{(1+\alpha)^n\log^n x/Q}=\sum_{n=1}^{N}\frac{(n-1)!}{(1+\alpha)^n\log^n x}\Big(1-\frac{\log Q}{\log x}\Big)^{-n}$$

$$= \sum_{n=1}^{N} \frac{(n-1)!}{(1+\alpha)^n \log^n x} \sum_{v=0}^{N-1} \binom{-n}{v} (-1)^v \left(\frac{\log Q}{\log x}\right)^v + O\left(\frac{\log^N Q}{\log^{N+1} x}\right)$$

$$= \sum_{k=1}^{N} \frac{1}{\log^k x} \sum_{\substack{n+v=k \\ n\ge 1,\, v\ge 0}} \binom{n+v-1}{v} \frac{(n-1)!}{(1+\alpha)^n} \log^v Q + O\left(\frac{\log^N x}{\log^{N+1} x}\right)$$

$$= \sum_{k=1}^{N} \frac{(k-1)!}{(1+\alpha)^k \log^k x} \sum_{v=0}^{k-1} \frac{(1+\alpha)^v \log^v Q}{v!} + O\left(\frac{\log^N Q}{\log^{N+1} x}\right). \ \blacksquare$$

3. Proof of Theorem 3. Let $P(\mathfrak{a})$ denote a prime ideal factor of $\mathfrak{a}$ in K of the largest possible norm. We should obtain the corresponding summation (3) for $N(P(\mathfrak{a}))^\alpha$ instead of $\mathcal{B}_\alpha(\mathfrak{a})$. In fact, let

$$\sum_{N(\mathfrak{a})\le x} = \sum\nolimits_1 + \sum\nolimits_2 . \tag{4}$$

Here and below $\sum_1$ means $N(\mathfrak{a}) \le x$ and $P(\mathfrak{a})$ is unique, $\sum_2$ means $N(\mathfrak{a}) \le x$ and $P(\mathfrak{a})$ is not unique. By Theorem 202 of [8], one has

$$\sum_{N(\mathfrak{a})\le x} 1 = O(x), \tag{5}$$

hence

$$\sum\nolimits_2 (\mathcal{B}_\alpha(\mathfrak{a}) - N(P(\mathfrak{a}))^\alpha) = O(x^{\alpha/2}\log x) \sum_{N(\mathfrak{a})\le x} 1 = O(x^{1+\alpha/2}\log x). \tag{6}$$

On the other hand,

$$\begin{aligned} &\sum\nolimits_1 (\mathcal{B}_\alpha(\mathfrak{a}) - N(P(\mathfrak{a}))^\alpha) \\ &= \sum\nolimits_1 \Big\{ \sideset{}{^*}\sum_{\substack{\mathfrak{p}^k \| \mathfrak{a},\, \mathfrak{p}=P(\mathfrak{a}) \\ k\ge 2}} N(\mathfrak{p})^\alpha + \sideset{}{^*}\sum_{\mathfrak{p}|\mathfrak{a},\, \mathfrak{p}\ne P(\mathfrak{a})} N(\mathfrak{p})^\alpha \Big\} \\ &\le \sideset{}{^*}\sum_{\substack{N(\mathfrak{p}^k)\le x \\ k\ge 2}} N(\mathfrak{p})^\alpha \sum\nolimits_{1,\ \mathfrak{p}^k\|\mathfrak{a}} 1 + \sideset{}{^*}\sum_{N(\mathfrak{p})\le x^{1/2}} N(\mathfrak{p})^\alpha \sum\nolimits_{1,\ \mathfrak{p}|\mathfrak{a},\, \mathfrak{p}\ne P(\mathfrak{a})} 1 \\ &\le \sideset{}{^*}\sum_{\substack{N(\mathfrak{p}^k)\le x \\ k\ge 2}} N(\mathfrak{p})^\alpha \sum_{N(\mathfrak{a})\le x/N(\mathfrak{p}^k)} 1 + \sideset{}{^*}\sum_{N(\mathfrak{p})\le x^{1/2}} N(\mathfrak{p})^\alpha \sum_{N(\mathfrak{a})\le x/N(\mathfrak{p})} 1 \\ &= O\Big\{x \sideset{}{^*}\sum_{\substack{N(\mathfrak{p}^k)\le x \\ k\ge 2}} N(\mathfrak{p})^{\alpha-1}\Big\} + O\Big\{x \sideset{}{^*}\sum_{N(\mathfrak{p})\le x^{1/2}} N(\mathfrak{p})^{\alpha-1}\Big\} = O\left(\frac{x^{1+\alpha/2}}{\log x}\right), \end{aligned} \tag{7}$$

where in the last equality we use (5) and Lemma 2.

Now we make two steps:

(i) We first sum over all $N(\mathfrak{a}) \le x$ with $N(P(\mathfrak{a})) \le x/\log^{M+1} x$, $M = [(N+1)/\alpha]$, including those with $P(\mathfrak{a})Q(\mathfrak{a}) \mid \mathfrak{a}$, $N(Q(\mathfrak{a})) = N(P(\mathfrak{a}))$. Then

$$\sum_{N(\mathfrak{a})\le x} N(P(\mathfrak{a}))^\alpha \le \sum_{N(\mathfrak{a})\le x} \frac{x^\alpha}{\log^{\alpha(M+1)} x} = O\biggl(\frac{x^{1+\alpha}}{\log^{N+1} x}\biggr).$$

(ii) Next we sum over all $N(\mathfrak{a}) \le x$ such that $N(P(\mathfrak{a})) > x/\log^{M+1} x$. Assume $\mathfrak{a} = \mathfrak{p}_k \mathfrak{p}_{k-1}^{\alpha_{k-1}} \ldots \mathfrak{p}_1^{\alpha_1}$, $N(\mathfrak{p}_k) > N(\mathfrak{p}_{k-1}) \ge \ldots \ge N(\mathfrak{p}_1)$. Let $\mathfrak{q} = \mathfrak{p}_{k-1}^{\alpha_{k-1}} \ldots \mathfrak{p}_1^{\alpha_1}$. Noting that $N(\mathfrak{p}_{k-1}) \le N(\mathfrak{q}) < N(\mathfrak{p}_k)$, if x is large enough, by Lemmas 2 and 3, one has

$$\begin{aligned}
\sum_{N(\mathfrak{p}_k\mathfrak{q})\le x} N(\mathfrak{p}_k)^\alpha &= \sum_{N(\mathfrak{q})<\log^{M+1} x}\ \sum^{*}_{x/\log^{M+1} x<N(\mathfrak{p})\le x/N(\mathfrak{q})} N(\mathfrak{p})^\alpha \\
&= \sum_{N(\mathfrak{q})<\log^{M+1} x} \biggl\{\sum_{n=1}^{N} \frac{(n-1)!}{(1+\alpha)^n}\cdot\frac{y^{1+\alpha}}{\log^n y} + O\biggl(\frac{y^{1+\alpha}}{\log^{N+1} y}\biggr)\biggr\}\bigg|_{x/\log^{M+1} x}^{x/N(\mathfrak{q})} \\
&= \sum_{N(\mathfrak{q})<\log^{M+1} x} \frac{1}{N(\mathfrak{q})^{1+\alpha}} \sum_{n=1}^{N} \frac{(n-1)!}{(1+\alpha)^n}\cdot\frac{x^{1+\alpha}}{\log^n x/N(\mathfrak{q})} + O\biggl(\frac{x^{1+\alpha}}{\log^{N+1} x}\biggr) \\
&= \sum_{N(\mathfrak{q})<\log^{M+1} x} \frac{1}{N(\mathfrak{q})^{1+\alpha}} \sum_{n=1}^{N} \biggl(\sum_{v=0}^{n-1} \frac{(1+\alpha)^v \log^v N(\mathfrak{q})}{v!}\biggr) \frac{(n-1)!}{(1+\alpha)^n}\cdot\frac{x^{1+\alpha}}{\log^n x} \\
&\quad + O\biggl(\frac{x^{1+\alpha}(\log\log x)^N}{\log^{N+1} x}\biggr) \\
&= \sum_{n=1}^{N} \biggl(\sum_{v=0}^{n-1} \frac{(1+\alpha)^v}{v!} \sum_{N(\mathfrak{q})<\log^{M+1} x} \frac{\log N(\mathfrak{q})}{N(\mathfrak{q})^{1+\alpha}}\biggr) \frac{(n-1)!}{(1+\alpha)^n}\cdot\frac{x^{1+\alpha}}{\log^n x} \\
&\quad + O\biggl(\frac{x^{1+\alpha}(\log\log x)^N}{\log^{N+1} x}\biggr) \\
&= \sum_{n=1}^{N}\sum_{v=0}^{n-1} \frac{(-1)^v(1+\alpha)^v}{v!} \zeta_K^{(v)}(1+\alpha) \frac{(n-1)!}{(1+\alpha)^n}\cdot\frac{x^{1+\alpha}}{\log^n x} \\
&\quad + O\biggl(\frac{x^{1+\alpha}(\log\log x)^N}{\log^{N+1} x}\biggr).
\end{aligned}$$

In the last equality we use the formulas (cf. [8], Theorem 141)

$$\sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \zeta_K(s), \qquad \operatorname{Re} s > 1,$$

and

$$\sum_{\mathfrak{a}} \frac{\log^v N(\mathfrak{a})}{N(\mathfrak{a})^s} = (-1)^v \zeta_K^{(v)}(s), \qquad \operatorname{Re} s > 1.$$

Noting that N can be any positive integer, by (4), (6) and (7), we complete the proof of Theorem 3. ∎

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