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On the set of numbers $\{14, 22, 30, 42, 90\}$

by

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For a fixed integer t, a *size* n P_t*-set* is a set $\{q_1, \ldots, q_n\}$ of distinct positive integers such that $q_i q_j + t$ is the square of an integer whenever $i \neq j$. For example, $\{1, 2, 5\}$ is a P_{-1}-set, while $\{1, 3, 8, 120\}$ is a size 4 P_1-set. A P_t-set S is *extendible* if there exists a positive integer $d \notin S$ such that $S \cup \{d\}$ is still a P_t-set.

Problems related to P_t-sets date back to the time of Diophantus (see Dickson [4, Vol. II, p. 513]). The most famous recent result is in the area of extending P_t-sets and is due to Baker and Davenport [1], who used Diophantine approximation to show that the P_1-set $\{1, 3, 8, 120\}$ is nonextendible. Other methods for arriving at the same result were subsequently described (Kanagasabapathy and Ponnudurai [6], Sansone [9], and Grinstead [5]). Several more recent papers have made efforts to characterize the extendibility of classes of P_t-sets (Brown [3], Mootha and Berzsenyi [7]).

In this paper we introduce a very simple method for assessing the extendibility of P_t-sets of the form $\{a, b, ak, bk\}$, where a, b, and k are integers. The technique is illustrated by demonstrating the nonextendibility of the first identified size 5 P_t-set (see Berzsenyi [2]):

Theorem. *The* P_{-299}*-set* $\{14, 22, 30, 42, 90\}$ *is nonextendible.*

Proof. First, note that if we set $a = 14$, $b = 30$, and $k = 3$, then this set is of the form $\{a, b, ak, bk, 22\}$. Showing that this P_t-set is nonextendible is equivalent to showing that the system of equations

$$(*) \qquad \begin{cases} 14d - 299 = w^2, \\ 30d - 299 = x^2, \\ 42d - 299 = y^2, \\ 90d - 299 = z^2 \end{cases}$$

has exactly one integer solution, $d = 22$, which corresponds to the fifth member of the P_{-299}-set. Eliminating d between $(*)$, we obtain the following

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Pellian equations:

$$
(1) \qquad \begin{cases} y^2 - 3w^2 = 598, \\ z^2 - 3x^2 = 598. \end{cases}
$$

This is a system of two Pellian equations, each having exactly four classes of solutions (see Nagell [8, p. 205]) given by

$$
\begin{aligned}
\mathbf{K}_1 &: \ y_n + \sqrt{3}w_n = z_n + \sqrt{3}x_n = (25 + 3\sqrt{3})(2 + \sqrt{3})^n, \\
\overline{\mathbf{K}}_1 &: \ y_n + \sqrt{3}w_n = z_n + \sqrt{3}x_n = (25 - 3\sqrt{3})(2 + \sqrt{3})^n, \\
\mathbf{K}_2 &: \ y_n + \sqrt{3}w_n = z_n + \sqrt{3}x_n = (29 + 9\sqrt{3})(2 + \sqrt{3})^n, \\
\overline{\mathbf{K}}_2 &: \ y_n + \sqrt{3}w_n = z_n + \sqrt{3}x_n = (29 - 9\sqrt{3})(2 + \sqrt{3})^n,
\end{aligned}
$$

where n is a whole number. These solutions correspond to the linear recurrent sequence $w_n = 4w_{n-1} - w_{n-2}$, $n \geq 2$, where w_0 and w_1 depend on the solution class (and similarly for x_n). Using recurrence relations, we produce explicit expressions for each of the four solution classes:

$$
(2) \begin{cases}
\mathbf{K}_1 : \ w_n = x_n = \left(\dfrac{9 + 25\sqrt{3}}{6}\right)(2 + \sqrt{3})^n + \left(\dfrac{9 - 25\sqrt{3}}{6}\right)(2 - \sqrt{3})^n, \\[2ex]
\overline{\mathbf{K}}_1 : \ w_n = x_n = \left(\dfrac{9 - 25\sqrt{3}}{-6}\right)(2 + \sqrt{3})^n + \left(\dfrac{9 + 25\sqrt{3}}{-6}\right)(2 - \sqrt{3})^n, \\[2ex]
\mathbf{K}_2 : \ w_n = x_n = \left(\dfrac{27 + 29\sqrt{3}}{6}\right)(2 + \sqrt{3})^n + \left(\dfrac{27 - 29\sqrt{3}}{6}\right)(2 - \sqrt{3})^n, \\[2ex]
\overline{\mathbf{K}}_2 : \ w_n = x_n = \left(\dfrac{27 - 29\sqrt{3}}{-6}\right)(2 + \sqrt{3})^n + \left(\dfrac{27 + 29\sqrt{3}}{-6}\right)(2 - \sqrt{3})^n.
\end{cases}
$$

Table 1 is a list of the first 9 solutions $w_n = x_n$ in each of the four classes.

Table 1. Some solutions w_n and x_n

n	$w_n = x_n \in \mathbf{K}_1$	$w_n = x_n \in \overline{\mathbf{K}}_1$	$w_n = x_n \in \mathbf{K}_2$	$w_n = x_n \in \overline{\mathbf{K}}_2$
0	3	-3	9	-9
1	31	19	47	11
2	121	79	179	53
3	453	297	669	201
4	1691	1109	2499	751
5	6311	4139	9319	2803
6	23553	15447	34779	10461
7	87901	57649	129898	39041
8	328051	215149	484409	145703

Because we have derived closed expressions for w_n and x_n, we can set $w = w_j$ and $x = x_i$, for some whole numbers i and j. From $(*)$, it becomes clear that since $x^2/w^2 = x_i^2/w_j^2 = (30d - 299)/(14d - 299)$,

$$\frac{x_i}{w_j} \approx \sqrt{\frac{15}{7}} = 1.4638501\ldots \text{ for large } d.$$

This provides us with an additional constraint which must be satisfied simultaneously with (1) for sufficiently large values of d. Hence, if there is an integer $d \neq 22$ that solves $(*)$, and d is large, then we expect x_i/w_j to be asymptotically equal to $1.4638501\ldots$ For computational purposes, it is necessary to formalize what we mean by "sufficiently large" values of d. We define

$$\varepsilon(d) \equiv \left| \sqrt{\frac{30d - 299}{14d - 299}} - \sqrt{\frac{15}{7}} \right| = \left| \frac{x_i}{w_j} - \sqrt{\frac{15}{7}} \right|$$

and note that $\varepsilon(d) \to 0$ as $d \to \infty$. In particular, observe that for $d \geq 8.34 \times 10^8$ (i.e., $w_j \geq 1.08 \times 10^5$ and $x_i \geq 1.58 \times 10^5$) we must have $\varepsilon(d) \leq 10^{-8}$. Table 1 lists all values of $x_i \leq 1.58 \times 10^5$, and simple trial and error of these values indicates that the only solution in this range corresponds to $d = 22$. Hence, *x_i and w_j must be so large that $d \geq 8.34 \times 10^8$ and $\varepsilon(d) \leq 10^{-8}$.*

We now demonstrate that no selection of large x_i and w_j (i.e., $x_i \geq 1.58 \times 10^5$ and $w_j \geq 1.08 \times 10^5$) meets this requirement. By *selection*, we mean a choice of two classes from which to assign values to x and w, e.g., $x = x_i \in \mathbf{K}_1$ and $w = w_j \in \overline{\mathbf{K}}_2$, or $x = x_i \in \mathbf{K}_2$ and $w = w_j \in \overline{\mathbf{K}}_2$, etc. Clearly, there are a total of 16 possible selections that we must consider, and we treat each case separately:

Case 1: $x = x_i \in \mathbf{K}_1$ and $w = w_j \in \mathbf{K}_1$. From $(*)$, we see that $x > w$, which implies that $i > j$. We must attempt to minimize $\varepsilon(d)$, and the best we can do is to choose $i = j + 1$, implying that $x/w = w_{j+1}/w_j$. From (2), we find that $\varepsilon(d)$ decreases monotonically for increasing d. But

$$\lim_{d\to\infty} \varepsilon(d) = \lim_{j\to\infty} \left| \frac{w_{j+1}}{w_j} - \sqrt{\frac{15}{7}} \right| = 2.2682006\ldots \gg 10^{-8}.$$

Hence, selecting both x and w from $\mathbf{K}_1$ cannot satisfy $(*)$ for large values of d.

Case 2: $x = x_i \in \overline{\mathbf{K}}_1$ and $w = w_j \in \mathbf{K}_1$. Again, because $x > w$, we are forced to choose $i = j + 1$ to minimize $\varepsilon(d)$. $\varepsilon(d)$ decreases monotonically with increasing d, and we find from (2) that

$$\lim_{d\to\infty} \varepsilon(d) = \lim_{j\to\infty} \left| \frac{x_{j+1}}{w_j} - \sqrt{\frac{15}{7}} \right| = 0.9837784\ldots \gg 10^{-8}.$$

Conclude that this particular selection of x and w does not yield a large solution to $(*)$.

The remaining fourteen cases are treated similarly, and the results are summarized in Table 2. For each selection, the "best" index choice (which minimizes $\varepsilon(d)$) and $M = \lim_{d\to\infty} \varepsilon(d)$ are shown.

Table 2. Summary of 16 cases

	$x = x_i \in \mathbf{K}_1$	$x = x_i \in \overline{\mathbf{K}}_1$	$x = x_i \in \mathbf{K}_2$	$x = x_i \in \overline{\mathbf{K}}_2$
$w = w_j \in \mathbf{K}_1$	$i = j+1$ $M = 2.2682007$	$i = j+1$ $M = 0.9837784$	$i = j$ $M = 0.0127771$	$i = j+1$ $M = 0.1937306$
$w = w_j \in \overline{\mathbf{K}}_1$	$i = j$ $M = 0.0609116$	$i = j+1$ $M = 2.2682007$	$i = j$ $M = 0.7876547$	$i = j+1$ $M = 1.0635658$
$w = w_j \in \mathbf{K}_2$	$i = j+1$ $M = 1.0635658$	$i = j+1$ $M = 0.1937306$	$i = j+1$ $M = 2.2682007$	$i = j+1$ $M = 0.3413048$
$w = w_j \in \overline{\mathbf{K}}_2$	$i = j$ $M = 0.7876547$	$i = j$ $M = 0.0127771$	$i = j$ $M = 1.8607830$	$i = j+1$ $M = 2.2682007$

Note that in all cases, $\lim_{d\to\infty} \varepsilon(d)$ is much greater than 10^{-8}, which means that we have safely precluded the possibility of a "large" solution to $(*)$.

As we have already exhausted all possibilities in Table 1, we conclude that the P_{-299}-set $\{14, 22, 30, 42, 90\}$ is nonextendible. ∎

This same approach can be taken in quickly assessing the extendibility of any P_t-set of the form $\{a, b, ak, bk\}$.

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