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An application of the projections of C^∞ automorphic forms

by

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1. Introduction. Let k be a positive even integer and S_k be the space of cusp forms of weight k on $\mathrm{SL}_2(\mathbb{Z})$. Let $f(z) \in S_k$ be a normalized Hecke eigenform with the Fourier expansion $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$. The symmetric square L-function attached to $f(z)$ is defined by

$$L_2(s,f) = \prod_p (1-\alpha_p^2 p^{-s})^{-1}(1-\alpha_p\beta_p p^{-s})^{-1}(1-\beta_p^2 p^{-s})^{-1},$$

with $\alpha_p + \beta_p = a(p)$ and $\alpha_p\beta_p = p^{k-1}$. Here the product is taken over all rational primes.

The purpose of this paper is to prove the following theorem:

Theorem. *Let $\Delta_k(z) = \sum_{n=1}^{\infty} \tau_k(n)e^{2\pi i n z} \in S_k$ be the unique normalized Hecke eigenform for $k = 12, 16, 18, 20, 22,$ and 26. Let ϱ be a zero of $\zeta(s)$ or of $L_2(s+k-1, \Delta_k)$ in the critical strip $0 < \mathrm{Re}(s) < 1$, with $\zeta(2\varrho) \neq 0$. Then for each positive integer n,*

$$\begin{aligned}
-\tau_k(n)\Big\{&\zeta(2\varrho)\cdot 2^{-2\varrho}\pi^{-2\varrho}n^{1-2\varrho}\cdot\frac{\Gamma(\varrho)\Gamma(k)}{\Gamma(k-\varrho)} \\
&\qquad + \zeta(2\varrho-1)\cdot 2^{2\varrho-2}\pi^{-1/2}\cdot\frac{\Gamma(\varrho-1/2)\Gamma(k)}{\Gamma(k-1+\varrho)}\Big\} \\
= &\sum_{0<m<n}\tau_k(m)\sigma_{1-2\varrho}(n-m)F(1-\varrho, k-\varrho; k; m/n) \\
&+ \sum_{n<m}(-n/m)^{k-\varrho}\tau_k(m)\sigma_{1-2\varrho}(n-m)F(1-\varrho, k-\varrho; k; n/m),
\end{aligned}$$

where $F(a,b;c;z)$ is the hypergeometric function and $\sigma_s(m)$ is the sum of the s-th powers of positive divisors of m.

Remark. Let $T(n,k;\varrho)$ be the right-hand side of the equality in the theorem. For $0 < \mathrm{Re}(\varrho) \le 1/2$, the following conditions are equivalent:

(A.1) $\mathrm{Re}(\varrho) = 1/2$,
(A.2) $T(n, k; \varrho) = O(\tau_k(n))$.

2. C^∞ automorphic forms. Let $H = \{z = x + \sqrt{-1}y \mid y > 0\}$ be the upper half-plane. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in H$, we put $\gamma\langle z\rangle = (az + b)(cz + d)^{-1}$. We denote by $\mathfrak{M}_k$ the set of functions F which satisfy the following conditions:

(2.1) F is a C^∞ function from H to $\mathbb{C}$,
(2.2) $F(\gamma\langle z\rangle) = (cz + d)^k F(z)$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$.

The function F is called a C^∞ *automorphic form on* $\mathrm{SL}_2(\mathbb{Z})$ *of weight* k, and called *of bounded growth* if for every $\varepsilon > 0$,

$$\int_0^1 \int_0^\infty |F(z)| y^{k-2} e^{-\varepsilon y}\, dy\, dx < \infty.$$

For $F \in \mathfrak{M}_k$ and $f \in S_k$, we define the Petersson inner product

$$\langle f, F\rangle = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash H} f(z)\overline{F(z)} y^{k-2}\, dx\, dy.$$

We quote the following theorem:

Theorem A (Sturm [2]). *Let $F \in \mathfrak{M}_k$ be of bounded growth with the Fourier expansion $F(z) = \sum_{n=1}^\infty a(n, y)e^{2\pi inx}$. Assume $k > 2$. Let*

$$c(n) = 2 \cdot (2\pi n)^{k-1}\Gamma(k-1)^{-1} \int_0^\infty a(n, y)e^{-2\pi ny} y^{k-2}\, dy.$$

Then

$$h(z) = \sum_{n=1}^\infty c(n)e^{2\pi inz} \in S_k \quad \textit{and} \quad \langle g, F\rangle = \langle g, h\rangle \quad \textit{for all } g \in S_k.$$

We shall also use the following properties of the function $L_2(s, f)$. Let $f(z) \in S_k$ be a normalized Hecke eigenform. Then the function $L_2(s, f)$ has an integral representation

$$(1)\quad L_2(s, f) = \frac{\zeta(2s - 2k + 2)}{\zeta(s - k + 1)} \cdot \frac{(4\pi)^s}{\Gamma(s)} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash H} |f(z)|^2 E(z, s - k + 1) y^{k-2}\, dx\, dy.$$

Here $E(z, s)$ is the Eisenstein series

$$(2)\quad E(z, s) = \frac{1}{2} \sum_{c,d\in\mathbb{Z},\ (c,d)=1} y^s |cz + d|^{-2s}.$$

Further, $L_2(s,f)$ has a holomorphic continuation to the whole s-plane (Shimura [1], Zagier [4]).

3. Proof of Theorem. Let $e(x)=e^{2\pi ix}$. The Eisenstein series (2) has the Fourier expansion $E(z,s)=\sum_{m=-\infty}^{\infty}a_m(y,s)e(mx)$ with

$$a_0(y,s)=y^s+\pi^{1/2}\Gamma(s-1/2)\Gamma(s)^{-1}\zeta(2s-1)\zeta(s)^{-1}y^{1-s} \tag{3}$$

and

$$\begin{aligned} a_m(y,s)&=\zeta(2s)^{-1}\sigma_{1-2s}(m)\cdot 2\pi^s|m|^{s-1/2}\Gamma(s)^{-1}y^{1/2}K_{s-1/2}(2\pi|m|y) \\ &=\zeta(2s)^{-1}\sigma_{1-2s}(m)\cdot y^{1-s}\int_{-\infty}^{\infty}e(-my\xi)(1+\xi^2)^{-s}\,d\xi \end{aligned} \tag{4}$$

for $m\neq 0$. Here we have used the integral representation in [3, p. 172] for the modified Bessel function $K_\nu(t)$. Then there exist positive constants c_1 and c_2 depending only on s such that

$$|a_0(y,s)|\le c_1(y^{\mathrm{Re}(s)}+y^{1-\mathrm{Re}(s)}) \tag{5}$$

and

$$|a_m(y,s)|\le c_2y^{\mathrm{Re}(s)}|\sigma_{1-2s}(m)|e^{-\pi|m|y/2} \tag{6}$$

for $m\neq 0$.

Lemma 1. *For $f(z)\in S_k$ and $s\in\mathbb{C}$ in $0<\mathrm{Re}(s)<1$, $f(z)E(z,s)$ is a C^∞ automorphic form of bounded growth.*

Proof. It is easy to see that $f(z)E(z,s)$ is a C^∞ automorphic form. We show $f(z)E(z,s)$ is of bounded growth. Let $f(z)=\sum_{n=1}^{\infty}a(n)e(nz)$. Then

$$f(z)E(z,s)=\sum_{n=-\infty}^{\infty}b_s(n,y)e(nx)$$

with

$$b_s(n,y)=\sum_{m=1}^{\infty}a(m)a_{n-m}(y,s)e^{-2\pi my}.$$

By (5), (6) and $a(m)=O(m^{k/2})$, there exists a positive constant c_3 depending only on s such that

$$\begin{aligned} &\sum_{n=-\infty}^{\infty}\sum_{m=1}^{\infty}\int_0^{\infty}|a(m)a_{n-m}(y,s)|y^{k-2}e^{-(2\pi m+\varepsilon)y}\,dy \\ &\quad\le c_3\sum_{n=-\infty}^{\infty}\Big\{\sum_{m=1}^{\infty}m^{k/2}(n+m)^{-k+2-3\,\mathrm{Re}(s)}+n^{-k/2+1-\max(\mathrm{Re}(s),1-\mathrm{Re}(s))}\Big\}. \end{aligned}$$

The last series is convergent for $k\ge 12$ and $\mathrm{Re}(s)>0$, hence $f(z)E(z,s)$ is of bounded growth. ■

LEMMA 2. *Let* $f(z) \in S_k$ *be a normalized Hecke eigenform. Let* ϱ *be a zero of* $\zeta(s)$ *or of* $L_2(s+k-1,f)$ *in the critical strip* $0 < \mathrm{Re}(s) < 1$ *with* $\zeta(2\varrho) \neq 0$. *Then*

$$\langle f(z)E(z,\varrho), f(z)\rangle = 0.$$

Proof. By (1),

$$L_2(s,f) = \frac{\zeta(2s-2k+2)}{\zeta(s-k+1)} \cdot \frac{(4\pi)^s}{\Gamma(s)} \langle f(z)E(z,s-k+1), f(z)\rangle.$$

Since $L_2(s,f)$ is entire, $\langle f(z)E(z,\varrho), f(z)\rangle = 0$ for $\varrho \in \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\}$ such that $\zeta(\varrho) = 0$ with $\zeta(2\varrho) \neq 0$. We also see that $\langle f(z)E(z,\varrho), f(z)\rangle = 0$ for $\varrho \in \{s \in \mathbb{C} \mid 0 < \mathrm{Re}(s) < 1\}$ such that $L_2(\varrho+k-1,f) = 0$ with $\zeta(2\varrho) \neq 0$. ■

Proof of Theorem. By Theorem A and Lemma 1, there exists

$$h(z,s) = \sum_{n=1}^{\infty} c(n,s)e(nz) \in S_k$$

such that $\langle g,\ \Delta_k \cdot E(z,s)\rangle = \langle g,h\rangle$ for all $g \in S_k$. Here

$$c(n,s) = \gamma_k(n) \int_0^{\infty} b_s(n,y)e^{-2\pi ny}y^{k-2}\,dy$$

with

$$\gamma_k(n) = 2\cdot(2\pi n)^{k-1}\Gamma(k-1)^{-1}$$

and

$$b_s(n,y) = \sum_{m=1}^{\infty} \tau_k(m)a_{n-m}(y,s)e^{-2\pi my}.$$

Using (3) and (4), for $\mathrm{Re}(s) > 1/2$ we have

$$\begin{aligned}(7)\quad & c(n,s) \\ &= \frac{\gamma_k(n)}{\zeta(2s)} \sum_{\substack{m=1\\ m\neq n}}^{\infty} \tau_k(m)\sigma_{1-2s}(n-m) \\ &\quad \times \int_0^{\infty} y^{k-1-s}e^{-2\pi(m+n)y} \int_{-\infty}^{\infty} e(-(n-m)y\xi)(1+\xi^2)^{-s}\,d\xi\,dy \\ &\quad + \gamma_k(n)\tau_k(n)\biggl\{\frac{\Gamma(k-1+s)}{(4\pi n)^{k-1+s}} + \frac{\pi^{1/2}\Gamma\bigl(s-\frac12\bigr)}{\Gamma(s)}\cdot\frac{\zeta(2s-1)}{\zeta(2s)}\cdot\frac{\Gamma(k-s)}{(4\pi n)^{k-s}}\biggr\} \\ &= \frac{\gamma_k(n)\Gamma(k-s)}{\zeta(2s)} \sum_{\substack{m=1\\ m\neq n}}^{\infty} \tau_k(m)\sigma_{1-2s}(n-m)\end{aligned}$$

$$\times \int_{-\infty}^{\infty} \{2\pi(m+n)+2\pi i(n-m)\xi\}^{-k+s}(1+\xi^2)^{-s}\,d\xi$$
$$+\gamma_k(n)\tau_k(n)\left\{\frac{\Gamma(k-1+s)}{(4\pi n)^{k-1+s}}+\frac{\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\cdot\frac{\zeta(2s-1)}{\zeta(2s)}\cdot\frac{\Gamma(k-s)}{(4\pi n)^{k-s}}\right\}.$$

Here the interchange of summation and integration is justified by using (5) and (6), and by Fubini's theorem, the last equality also holds in the region $\mathrm{Re}(s) > -k+1$.

For $p, q \in \mathbb{C}$ and $0 < c < 1$,

$$\begin{aligned} f_c(p,q) &:= \int_{-\infty}^{\infty} (1-it)^{-p}(1+it)^{-p}(1+ict)^{-q}\,dt \\ &= 2^{2-2p}\pi\cdot(1+c)^{-q}\Gamma(p)^{-1}\Gamma(1-p)^{-1} \\ &\quad\times \int_0^1 t^{-p}(1-t)^{q+2p-2}\left(1-\left(\frac{1-c}{1+c}\right)t\right)^{-q} dt \\ &= 2^{2-2p}\pi\cdot(1+c)^{-q}\Gamma(2p+q-1)\Gamma(p)^{-1}\Gamma(p+q)^{-1} \\ &\quad\times F\left(1-p,q;p+q;\frac{1-c}{1+c}\right). \end{aligned}$$

Therefore, for $m < n$,

$$\begin{aligned} (8)\qquad &\int_{-\infty}^{\infty} \{2\pi(m+n)+2\pi i(n-m)\xi\}^{-k+s}(1+\xi^2)^{-s}\,d\xi \\ &= (2n)^{-k+s}2^{2-2s}\pi\cdot\Gamma(k-1+s)\Gamma(s)^{-1}\Gamma(k)^{-1}F(1-s,k-s;k;m/n) \end{aligned}$$

and for $m > n$,

$$\begin{aligned} (9)\qquad &\int_{-\infty}^{\infty} \{2\pi(m+n)+2\pi i(n-m)\xi\}^{-k+s}(1+\xi^2)^{-s}\,d\xi \\ &= (-2m)^{-k+s}2^{2-2s}\pi\cdot\Gamma(k-1+s)\Gamma(s)^{-1}\Gamma(k)^{-1}F(1-s,k-s;k;n/m). \end{aligned}$$

From Lemma 2 and $\dim S_k = 1$, we have $h(z,\varrho)=0$, hence $c(n,\varrho)=0$ for every positive integer n. Combining (7), (8) and (9), we conclude the proof of Theorem. ∎

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