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Numeration systems and fractal sequences

by

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Let $\mathbb{N}$ denote the set of positive integers. Every sequence $\mathbb{B} = (b_0, b_1, \ldots)$ of numbers in $\mathbb{N}$ satisfying

$$1 = b_0 < b_1 < \ldots \tag{1}$$

is a *basis* for $\mathbb{N}$, as each n in $\mathbb{N}$ has a $\mathbb{B}$-*representation*

$$n = c_0 b_0 + c_1 b_1 + \ldots + c_k b_k, \tag{2}$$

where $b_k \le n < b_{k+1}$ and the coefficients c_i are given by the division algorithm:

$$n = c_k b_k + r_k, \quad c_k = [n/b_k], \quad 0 \le r_k < b_k \tag{3}$$

and

$$r_i = c_{i-1} b_{i-1} + r_{i-1}, \quad c_{i-1} = [r_i/b_{i-1}], \quad 0 \le r_{i-1} < b_{i-1} \tag{4}$$

for $1 \le i < k$. In (2) let i be the least index h such that $c_h \ne 0$; then b_i is the $\mathbb{B}$-*residue of* n. A *proper basis* is a basis other than the sequence $(1, 2, \ldots)$ consisting of all the positive integers.

We extend the above notions to finite sequences $\mathbb{B}_j = (b_0, b_1, \ldots, b_j)$ satisfying

$$1 = b_0 < b_1 < \ldots < b_j$$

for $j \ge 0$. Such a finite sequence is a *finite basis*, and a $\mathbb{B}_j$-*representation* is a sum

$$c_0 b_0 + c_1 b_1 + \ldots + c_j b_j \tag{2$'$}$$

such that if $n = c_0 b_0 + c_1 b_1 + \ldots + c_j b_j$, then there exist integers $r_0, r_1, \ldots, r_j$ such that

$$n = c_j b_j + r_j, \quad c_j = [n/b_{j-1}], \quad 0 \le r_j < b_j \tag{3$'$}$$

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and

(4′) $\quad r_i = c_{i-1}b_{i-1} + r_{i-1}, \quad c_{i-1} = [r_i/b_{i-1}], \quad 0 \le r_{i-1} < b_{i-1}$

for $1 \le i \le j$.

From any basis or finite basis $\mathbb{B}$ we construct an array $A(\mathbb{B})$ of numbers $a(i,j)$ here called the $\mathbb{B}$-*numeration system*. Row 1 of $A(\mathbb{B})$ is the basis $\mathbb{B}$; i.e., $a(1,j) = b_{j-1}$, for $j = 1, 2, \ldots$ Column 1 is the ordered residue class containing 1; i.e., $a(i,1)$ is the ith number n whose $\mathbb{B}$-residue is 1. Generally, column j is the ordered residue class whose least element is b_{j-1}, so that $a(i,j)$ is the ith number n whose $\mathbb{B}$-residue is b_{j-1}. Note that every n in $\mathbb{N}$ occurs exactly once in $A(\mathbb{B})$. As an example, the first six rows of the $\mathbb{B}$-numeration system of the finite basis $\mathbb{B} = (1, 2, 3, 5, 8, 13)$ are

1	2	3	5	8	13
4	7	11	18	21	26
6	10	16	31	34	39
9	15	24	44	47	52
12	20	29	57	60	65
14	23	37	70	73	78

A $\mathbb{B}$-numeration system can also be represented as a sequence $S(\mathbb{B}) = (s_1, s_2, \ldots)$, where

s_n is the number of the row of the array $A(\mathbb{B})$ in which n occurs;

i.e., if $n = a(i,j)$, then $s_n = i$. We call $S(\mathbb{B})$ the *paraphrase* of $\mathbb{B}$. For example, the paraphrase of the finite basis $(1, 2, 3, 5, 8, 13)$ begins with

(5) $\quad$ 1 1 1 2 1 3 2 1 4 3 2 5 1 6 4 3 7 2 8 5 2.

As a second example, let $\mathbb{B}$ be the basis for the ordinary binary system:

$$\mathbb{B} = (1, 2, 2^2, 2^3, 2^4, 2^5, \ldots);$$

in this case, $S(\mathbb{B})$ begins with

(6) 1 1 2 1 3 2 4 1 5 3 6 2 7 4 8 1 9 5 10 3 11 6 12 2 13 7 14 4 15 8 16 1.

Now suppose $S = (s_1, s_2, \ldots)$ is any sequence such that for every i in $\mathbb{N}$ there are infinitely many n such that $s_n = i$; and further, that if $i + 1 = s_n$, then $i = s_m$ for some $m < n$. The *upper-trimmed subsequence* of S is the sequence $\Lambda(S)$ obtained from S by deleting the first occurrence of n, for each n. If $\Lambda(S) = S$, then S is a *fractal sequence*, so named, in [3], because the self-similarity property $\Lambda(S) = S$ implies that S contains a copy of itself, and hence contains infinitely many copies of itself. The sequence begun in (6), and also the paraphrases of trinary and the other -ary number systems, are examples of fractal sequences. Another familiar sequence that is a fractal basis is the sequence $(1, 2, 3, 5, 8, 13, 21, \ldots)$ of Fibonacci numbers.

To determine which bases are fractal bases, we shall extend finite bases one term at a time, with attention to certain shift functions. To define them, let $\mathbb{B}_j = (b_0, b_1, \ldots, b_j)$, where $j \geq 1$, and for each n in $\mathbb{N}$, let the $\mathbb{B}_{j-1}$-representation of n be given by

$$n = \sum_{h=0}^{j-1} c_h b_h; \tag{7}$$

then the *shift-function* $f_{\mathbb{B}_j}$ is defined by

$$f_{\mathbb{B}_j}(n) = \sum_{h=0}^{j-1} c_h b_{h+1}. \tag{8}$$

We call $\mathbb{B}_j$ an *affable finite basis* if the sum in (8) is a $\mathbb{B}_j$-representation whenever the sum in (7) is a $\mathbb{B}_{j-1}$-representation. To see what can go wrong, consider the finite basis $\mathbb{B}_j = \mathbb{B}_3 = (1, 3, 6, 10)$: here the $\mathbb{B}_2$-representation of 5 is $2{\cdot}1+1{\cdot}3$, so that $f_{\mathbb{B}_2}(5) = 2{\cdot}3+1{\cdot}6 = 12$; but alas, the $\mathbb{B}_3$-representation of 12 is $2 \cdot 1 + 1 \cdot 10$, not $2 \cdot 3 + 1 \cdot 6$. Theorem 1 gives lower bounds on successive b_i's that ensure that $\mathbb{B}_j$ is affable.

LEMMA 1. *If the sum in* (2) *is a* $\mathbb{B}$*-representation*, *then*

$$\sum_{h=0}^{i} c_h b_h < b_{i+1} \tag{9}$$

for $i = 0, 1, \ldots, k$; *conversely*, *if* (9) *holds for* $i = 0, 1, \ldots, k$, *then each of the* $k+1$ *sums is a* $\mathbb{B}$*-representation. Similarly*, *if* $n < b_j$ *and the sum in* (2′) *is a* $\mathbb{B}_j$*-representation*, *then* (9) *holds for* $i = 0, 1, \ldots, j-1$; *and conversely*, *if* (9) *holds for* $i = 0, 1, \ldots, j-1$, *then each of the* j *sums is a* $\mathbb{B}_j$*-representation.*

Proof. The proof for $\mathbb{B}$-representations is essentially given in [4]. A similar proof for $\mathbb{B}_j$-representations is given here for the sake of completeness. First, suppose $n < b_j$ and that n equals the sum (2′), a $\mathbb{B}_j$-representation. Then by (4′),

$$\begin{aligned}
b_1 &> r_1 = c_0 b_0,\\
b_2 &> r_2 = c_1 b_1 + r_1 = c_1 b_1 + c_0 b_0,\\
b_3 &> r_3 = c_2 b_2 + r_2 = c_2 b_2 + c_1 b_1 + c_0 b_0,\\
&\;\vdots\\
b_{j-1} &> r_{j-1} = c_{j-2} b_{j-2} + r_{j-2} = \sum_{h=0}^{j-2} c_h b_h.
\end{aligned}$$

These $j-1$ inequalities together with $n < b_j$ show that (9) holds for $i = 0, 1, \ldots, j-1$.

For the converse, suppose $c_0, c_1, \ldots, c_{j-1}$ are nonnegative integers such that the sum in (9) is a $\mathbb{B}_j$-representation for $i = 0, 1, \ldots, j-1$. Let $r_0 = 0$ and

$$r_i = c_0 b_0 + c_1 b_1 + \ldots + c_{i-1} b_{i-1}$$

for $1 \le i < j$. Clearly $r_0 < b_0$, and $r_{i-1} < b_{i-1}$ for $i = 2, 3, \ldots, j$, by (9), so that conditions (4′) hold. Write the sum $c_0 b_0 + c_1 b_1 + \ldots + c_{j-1} b_{j-1}$ as n; then condition (3′) holds, since $r_j < b_j$, by (9). ∎

THEOREM 1. *Suppose* $j \ge 2$. *Let* $\mathbb{B}_j = (b_0, b_1, \ldots, b_j)$ *be a finite basis. The following statements are equivalent*:

(i) $\mathbb{B}_j$ *is an affable finite basis.*

(ii) *If* $c_0, c_1, \ldots, c_{j-2}$ *are nonnegative integers satisfying the* $j-1$ *inequalities*

$$\begin{gathered} c_0 b_0 < b_1, \\ c_0 b_0 + c_1 b_1 < b_2, \\ \vdots \\ c_0 b_0 + c_1 b_1 + \ldots + c_{j-2} b_{j-2} < b_{j-1}, \end{gathered}$$

then the following $j-1$ *inequalities also hold*:

$$\begin{gathered} c_0 b_1 < b_2, \\ c_0 b_1 + c_1 b_2 < b_3, \\ \vdots \\ c_0 b_1 + c_1 b_2 + \ldots + c_{j-2} b_{j-1} < b_j. \end{gathered}$$

(iii) $b_i \ge f_{\mathbb{B}_{i-1}}(b_{i-1} - 1) + 1$ *for* $i = 2, \ldots, j$.

(iv) $f_{\mathbb{B}_j}$ *is strictly increasing on the set* $\{m \in \mathbb{N} : 1 \le m \le b_{j-1}\}$.

Proof. A proof is given in four parts: (i)⇔(ii), (iii)⇒(ii) and (iv), (i)⇒(iii), and (iv)⇒(iii).

Part 1: (i)⇔(ii). Suppose $\mathbb{B}_j$ is an affable finite basis and $c_0, c_1, \ldots, c_{j-2}$ are nonnegative integers satisfying $\sum_{h=0}^{i} c_h b_h < b_{i+1}$ for $i = 0, 1, \ldots, j-2$. By Lemma 1, each of these $j-1$ sums is a $\mathbb{B}_{j-1}$-representation. Since $\mathbb{B}_j$ is affable, each of the sums $\sum_{h=0}^{i} c_h b_{h+1}$, for $i = 0, 1, \ldots, j-2$, is a $\mathbb{B}_j$-representation, so that by Lemma 1,

$$\begin{gathered} c_0 b_1 < b_2, \\ c_0 b_1 + c_1 b_2 < b_3, \\ \vdots \\ c_0 b_1 + c_1 b_2 + \ldots + c_{j-2} b_{j-1} < b_j, \end{gathered}$$

and (ii) holds.

Now suppose n is given by a $\mathbb{B}_{j-1}$-representation as in (7). Then the $j-1$ inequalities in the hypothesis of (ii) hold, by definition of $\mathbb{B}_{j-1}$-representation. So, the $j-1$ inequalities in the conclusion of (ii) hold. These are precisely the inequalities that must be satisfied for the sum in (8) to be a $\mathbb{B}_j$-representation.

Part 2: (iii)$\Rightarrow$(ii) and (iv). Suppose $\mathbb{B}_j$ is a finite basis and $c_0, c_1, \ldots, c_{j-2}$ are nonnegative integers satisfying $\sum_{h=0}^{i} c_h b_h < b_{i+1}$ for $i = 0, 1, \ldots, j-1$. As a first step in an induction argument, assume that $b_0 c_0 < b_1$. By definition of basis, $b_0 = 1$ and $b_1 \geq 2$, and by hypothesis,

$$b_2 \geq f_{\mathbb{B}_1}(b_1 - 1) + 1 = \begin{cases} 3 & \text{if } b_1 = 2, \\ b_1^2 - b_1 + 1 & \text{if } b_1 \geq 3. \end{cases}$$

If $b_1 = 2$, then $c_0 b_0 < b_1$ implies $c_0 = 1$, so that $c_0 b_1 < b_1 + 1 \leq b_2$, as desired. Otherwise, $b_1 \geq 3$, so that $c_0 \leq b_1 - 1$, and $c_0 b_1 \leq b_1^2 - b_1 < b_2$, as desired. As a first step toward proving (iv), if $c_0 b_0 < c_0' b_0$, then clearly $c_0 b_1 < c_0' b_1$.

We shall now use a bipartite induction hypothesis.

HYPOTHESIS I. *If* $h \leq j-1$ *and the* $h-2$ *inequalities*

$$\begin{gathered} c_0 b_0 < b_1, \\ c_0 b_0 + c_1 b_1 < b_2, \\ \vdots \\ c_0 b_0 + c_1 b_1 + \ldots + c_{h-3} b_{h-3} < b_{h-2} \end{gathered}$$

hold, then also the following $h-2$ *inequalities hold*:

$$\begin{gathered} c_0 b_1 < b_2, \\ c_0 b_1 + c_1 b_2 < b_3, \\ \vdots \\ c_0 b_1 + c_1 b_2 + \ldots + c_{h-3} b_{h-2} < b_{h-1}. \end{gathered}$$

HYPOTHESIS II. *If* $c_0', c_1', \ldots, c_{h-3}'$ *are nonnegative integers such that* $c_0' b_0 + c_1' b_1 + \ldots + c_{h-3}' b_{h-3}$ *is a* $\mathbb{B}_{h-3}$*-representation, and the* $h-2$ *inequalities*

$$\begin{gathered} c_0 b_0 < c_0' b_0, \\ c_0 b_0 + c_1 b_1 < c_0' b_0 + c_1' b_1, \\ \vdots \\ c_0 b_0 + c_1 b_1 + \ldots + c_{h-3} b_{h-3} < c_0' b_0 + c_1' b_1 + \ldots + c_{h-3}' b_{h-3} \end{gathered}$$

hold, then also the following $h-2$ inequalities hold:

$$\begin{gathered} c_0b_1 < c_0'b_1, \\ c_0b_1 + c_1b_2 < c_0'b_1 + c_1'b_2, \\ \vdots \\ c_0b_1 + c_1b_2 + \ldots + c_{h-3}b_{h-2} < c_0'b_1 + c_1'b_2 + \ldots + c_{h-3}'b_{h-2}. \end{gathered}$$

Now suppose that the $h-1$ inequalities

$$\begin{gathered} c_0b_0 < b_1, \\ c_0b_0 + c_1b_1 < b_2, \\ \vdots \\ c_0b_0 + c_1b_1 + \ldots + c_{h-3}b_{h-3} < b_{h-2}, \\ c_0b_0 + c_1b_1 + \ldots + c_{h-3}b_{h-3} + c_{h-2}b_{h-2} < b_{h-1} \end{gathered}$$

have been shown to hold. There are $h-1$ inequalities to be proved. The first $h-2$ hold by Hypothesis I, and we now wish to see that the remaining inequality holds, namely

$$c_0b_1 + c_1b_2 + \ldots + c_{h-2}b_{h-1} < b_h. \tag{10}$$

Let $d_0b_0 + d_1b_1 + \ldots + d_{h-2}b_{h-2}$ be the $\mathbb{B}_{h-2}$-representation of $b_{h-1} - 1$. Then

$$c_0b_0 + c_1b_1 + \ldots + c_{h-2}b_{h-2} \le d_0b_0 + d_1b_1 + \ldots + d_{h-2}b_{h-2}. \tag{11}$$

Case 1: $d_{h-2} = c_{h-2}$. In this case, (11) implies

$$c_0b_0 + c_1b_1 + \ldots + c_{h-3}b_{h-3} \le d_0b_0 + d_1b_1 + \ldots + d_{h-3}b_{h-3},$$

which by Hypothesis II yields

$$c_0b_1 + c_1b_2 + \ldots + c_{h-3}b_{h-2} \le d_0b_1 + d_1b_2 + \ldots + d_{h-3}b_{h-2}.$$

We add $c_{h-2}b_{h-1} = d_{h-2}b_{h-1}$ to both sides to obtain

$$\begin{aligned} c_0b_1 + c_1b_2 + \ldots + c_{h-2}b_{h-1} &\le d_0b_1 + d_1b_2 + \ldots + d_{h-2}b_{h-1} \\ &= f_{\mathbb{B}_{h-1}}(b_{h-1} - 1) < b_h, \end{aligned}$$

so that (10) holds.

Case 2: $d_{h-2} > c_{h-2}$. Since $c_0b_0 + c_1b_1 + \ldots + c_{h-3}b_{h-3} < b_{h-2}$, we have $c_0b_1 + c_1b_2 + \ldots + c_{h-3}b_{h-2} < b_{h-1}$, by Hypothesis I. Then

$$\begin{aligned} &c_0b_1 + c_1b_2 + \ldots + c_{h-3}b_{h-2} \\ &\qquad \le (d_{h-2} - c_{h-2})b_{h-1} \\ &\qquad \le d_0b_1 + d_1b_2 + \ldots + d_{h-3}b_{h-2} + (d_{h-2} - c_{h-2})b_{h-1}, \end{aligned}$$

from which (10) follows as at the end of Case 1.

Case 3: $d_{h-2} < c_{h-2}$. We rewrite (11) as

$$c_0b_0+c_1b_1+\ldots+c_{h-3}b_{h-3}+(c_{h-2}-d_{h-2})b_{h-2} < d_0b_0+d_1b_1+\ldots+d_{h-3}b_{h-3}.$$

This implies $b_{h-2} < d_0b_0 + d_1b_1 + \ldots + d_{h-3}b_{h-3}$, but this violates the premise that the sum $d_0b_0+d_1b_1+\ldots+d_{h-2}b_{h-2}$, and therefore also $d_0b_0+d_1b_1+\ldots+d_{h-3}b_{h-3}$, is a $\mathbb{B}_{h-2}$-representation. Therefore Case 3 does not occur.

A proof of (ii) is now finished, and we continue with a proof of (iv). Suppose $c'_0, c'_1, \ldots, c'_{j-2}$ are nonnegative integers and

$$c_0b_0 + c_1b_1 + \ldots + c_{j-2}b_{j-2} < c'_0b_0 + c'_1b_1 + \ldots + c'_{j-2}b_{j-2} < b_{j-1},$$

where both sums are $\mathbb{B}_{j-1}$-representations.

Case 1.1: $c'_{j-2} = c_{j-2}$. Here $c_0b_0+c_1b_1+\ldots+c_{j-3}b_{j-3} < c'_0b_0+c'_1b_1+\ldots+c'_{j-3}b_{j-3}$, which by Hypothesis II yields

$$c_0b_1 + c_1b_2 + \ldots + c_{j-3}b_{j-2} < c'_0b_1 + c'_1b_2 + \ldots + c'_{j-3}b_{j-2}.$$

We add $c_{j-2}b_{j-1} = c'_{j-2}b_{j-1}$ to both sides to obtain

$$c_0b_1 + c_1b_2 + \ldots + c_{j-2}b_{j-1} < c'_0b_1 + c'_1b_2 + \ldots + c'_{j-2}b_{j-1}. \tag{12}$$

This proof of (12) for Case 1.1 is obviously very similar to that for Case 1 above. Cases 2.1 and 3.1 are similar to the previous Cases 2 and 3, and corresponding proofs of (12) are omitted. Now suppose $1 \le m < n \le b_{j-1}$. Write $\mathbb{B}_j$-representations for m and n:

$$m = c_0b_0 + \ldots + c_{j-2}b_{j-2},$$
$$n = \begin{cases} c'_0b_0 + \ldots + c'_{j-2}b_{j-2} & \text{if } n < b_{j-1}, \\ b_{j-1} & \text{otherwise} \end{cases}$$

and let

$$h = \max\{i : c_i \ne 0,\ i \le j-2\},$$
$$k = \begin{cases} \max\{i : c'_i \ne 0,\ i \le j-2\} & \text{if } n < b_{j-1}, \\ j-1 & \text{otherwise.} \end{cases}$$

Case 1.2: $h = k$. In this case, $f_{\mathbb{B}_j}(m) < f_{\mathbb{B}_j}(n)$ by (12).

Case 2.2: $h < k$ (the case $h > k$ is similar and omitted). Here, $m \le b_k < n$ or $m < b_k \le n$, so that $f_{\mathbb{B}_j}(m) \le f_{\mathbb{B}_j}(b_k) = b_{k+1} \le f_{\mathbb{B}_j}(n)$, with strict inequality in at least one place, and a proof of (iv) is finished.

Part 3: (i)⇒(iii). Suppose $2 \le i \le j$. Then $b_{i-1} - 1$ has a $\mathbb{B}_{i-2}$-representation $c_0b_0 + c_1b_1 + \ldots + c_{i-2}b_{i-2}$. By Lemma 1 (reading $i-1$ for j), we have $i-1$ inequalities:

$$c_0 b_0 < b_1,$$
$$c_0 b_0 + c_1 b_1 < b_2,$$
$$\vdots$$
$$c_0 b_0 + c_1 b_1 + \ldots + c_{i-2} b_{i-2} < b_{i-1}.$$

By (i), the representation for $f_{\mathbb{B}_{i-1}}(b_{i-1}-1)$ as $c_0 b_1 + c_1 b_2 + \ldots + c_{i-2} b_{i-1}$ is a $\mathbb{B}_{i-1}$-representation, and by (ii), already proved to follow from (i), we have $f_{\mathbb{B}_{i-1}}(b_{i-1}-1) < b_i$, and (iii) holds.

Part 4: (iv)$\Rightarrow$(iii). For $i = 2, 3, \ldots, j$, if (iv) holds then $f_{\mathbb{B}_{i-1}}(b_{i-1} - 1) = f_{\mathbb{B}_j}(b_{i-1}-1) < f_{\mathbb{B}_j}(b_{i-1}) = b_i$, so that (iii) holds. ■

Definitions. We extend the notion of affability given earlier: an infinite basis $\mathbb{B} = (b_0, b_1, \ldots)$ is an *affable basis* if the sum $\sum_{h=0}^{k} c_h b_{h+1}$ is a $\mathbb{B}$-representation whenever the sum $\sum_{h=0}^{k} c_h b_h$ is a $\mathbb{B}$-representation. The notion of shift-function is extended also:

if $n = \sum_{h=0}^{k} c_h b_h$ is a $\mathbb{B}$-representation, then $f_{\mathbb{B}}(n) = \sum_{h=0}^{k} c_h b_{h+1}$.

Theorem 2. *Let* $\mathbb{B} = (b_0, b_1, \ldots)$ *be a basis, and let* $\mathbb{B}_j = (b_0, b_1, \ldots, b_j)$ *for* $j \geq 2$. *The following statements are equivalent*:

(i) $\mathbb{B}$ *is an affable basis.*
(ii) $\mathbb{B}_j$ *is an affable finite basis for all* $j \geq 2$.
(iii) $b_j \geq f_{\mathbb{B}_{j-1}}(b_{j-1} - 1) + 1$ *for all* $j \geq 2$.
(iv) $f_{\mathbb{B}}$ *is strictly increasing on* $\mathbb{N}$.

Proof. This follows easily from Theorem 1. ■

Definitions. Suppose $S = (s_n)$ is a sequence (possibly finite) of numbers in $\mathbb{N}$. The *counting array of* S is the array $C(S)$ with terms $a(i,j)$ given by

$a(i,j)$ is the index n for which s_n is the jth occurrence of i in S.

Note that if S is the paraphrase of an infinite basis $\mathbb{B}$, then

$$C(S) = A(\mathbb{B}) \quad \text{and} \quad \mathbb{B} = (a(1,1), a(1,2), a(1,3), \ldots).$$

The following notation will be helpful: if A is a numeration system or a counting array, then

$\#n$ is the number of terms of A that are $\leq n$
and do not lie in column 1 of A.

Lemma 3.1. *A sequence* $S = (s_n)$ *is a fractal sequence if and only if the counting array* $C(S)$, *with terms* $a(i,j)$, *satisfies*

$$\#a(i, j+1) = a(i,j) \tag{13}$$

for all i *and* j *in* $\mathbb{N}$.

Proof. It is proved in [3, Theorem 2] that S is a fractal sequence if and only if $C(S)$ is an interspersion. In [3, Lemma 2], it is proved (in different notation) that $C(S)$ is an interspersion if and only if the number of terms of $C(S)$ that lie in column 1 and are not greater than $a(i,j+1)$ is $a(i,j+1)-a(i,j)$. Equivalently, the number of terms of $C(S)$ that lie outside column 1 and are not greater than $a(i,j+1)$ is $a(i,j)$. ∎

DEFINITIONS. A finite sequence $S=(s_0,s_1,\ldots,s_k)$ is a *prefractal sequence* if the following properties hold:

(PF1) if $i+1=s_n$ for some $n\le k$, then $i=s_m$ for some $m<n$, for all i in $\mathbb{N}$;

(PF2) if $\Lambda(S)$ is the sequence obtained from S by deleting the first occurrence of n for each n in S, then $\Lambda(S)$ is an initial segment of S.

A *prefractal basis* is a finite basis $\mathbb{B}=(b_0,b_1,\ldots,b_j)$ such that the first b_j terms of $S(\mathbb{B})$ form a prefractal sequence. For example, if $S=(1,1,1,2,1,3,2,1,4,3,2,5,1)$, the first 13 terms in (5), then $\Lambda(S)=(1,1,1,2,1,3,2,1)$, and this is the initial eight-term segment of S; thus S is a prefractal sequence, and $(1,2,3,5,8,13)$ is a prefractal basis.

LEMMA 3.2. *A finite sequence $T=(t_0,t_1,\ldots,t_k)$ satisfying* (PF1) *is a prefractal sequence if and only if* (13) *holds for all i and j such that $a(i,j)$ and $a(i,j+1)$ are terms of $C(T)$.*

Proof. The counting array $C(T)$ consists of terms $a(i,j)$ which are the numbers $1,\ldots,t_k$. The proof is now similar to that of Lemma 3.1, since all the inequalities needed from [3] and [2] remain intact in the case where the only terms being considered are $1,\ldots,t_k$. ∎

LEMMA 3.3. *If $1\le j_2<j_1$ and $1\le x\le b_{j_2}-1$, then $f_{\mathbb{B}_{j_1}}(x)=f_{\mathbb{B}_{j_2}}(x)$.*

Proof. If $1\le x\le b_{j_2}-1$, then the $\mathbb{B}_{j_1}$-representation of x and the $\mathbb{B}_{j_2}$-representation of x are identical. Thus, the shift-functions defined by (8) have identical values at x. ∎

The next theorem shows that the lower bound for b_j in Theorem 2(iii) for an affable basis is also a lower bound for b_j for a fractal basis. The theorem also gives an upper bound for b_j.

THEOREM 3. *Let $\mathbb{B}=(b_0,b_1,\ldots)$ be a proper basis, and let $\mathbb{B}_j=(b_0,b_1,\ldots,b_j)$ for $j\ge 2$. The following statements are equivalent*:

(i) $\mathbb{B}$ *is a fractal basis.*
(ii) $\#a(i,j+1)=a(i,j)$ *for all i and j in* $\mathbb{N}$.
(iii) $f_{\mathbb{B}_{j-1}}(b_{j-1}-1)+1\le b_j\le f_{\mathbb{B}_{j-1}}(b_{j-1})$ *for all $j\ge 2$.*
(iv) $a(i,j+1)=f_{\mathbb{B}}(a(i,j))$ *for all i and j in* $\mathbb{N}$.

P r o o f. A proof is given in four parts: (i)$\Leftrightarrow$(ii), (iii)$\Rightarrow$(iv), (iv)$\Rightarrow$(ii), and (ii)$\Rightarrow$(iii).

P a r t 1: (i)$\Leftrightarrow$(ii). This is an immediate consequence of Lemma 3.2.

P a r t 2: (iii)$\Rightarrow$(iv). Suppose, to the contrary, that (iii) holds but (iv) fails. Let i be the least index for which

$$a(i,j+1) \neq f_{\mathbb{B}}(a(i,j)) \tag{14}$$

for some j, and assume that j is the least index such that (14) holds for the stipulated i. Write x for $f_{\mathbb{B}}(a(i,j))$. This number must occur somewhere in the array $A(\mathbb{B})$, and then only in column $j+1$ or else row 1.

C a s e 1: *x in column* $j+1$. There is some h for which $a(i,j+1) = f_{\mathbb{B}}(a(h,j))$, and $h > i$, so that x must occur after $f_{\mathbb{B}}(a(h,j))$ in column $j+1$. But now $a(i,j) < a(h,j)$ while $f_{\mathbb{B}}(a(i,j)) > f_{\mathbb{B}}(a(h,j))$, contrary to Theorem 2(iv).

C a s e 2: *x in row* 1. Here, $x = b_k$ for some $k > 1$, so that $x = f_{\mathbb{B}}(b_{k-1})$. But also, $f_{\mathbb{B}}(a(i,j)) = x$, so that $a(i,j) = b_{k-1}$, since, by Theorem 2, $f_{\mathbb{B}}$ is strictly increasing. But this implies $i = 1$, a contradiction, since we have equality in (14) when $i = 1$, by definition of $f_{\mathbb{B}}$.

P a r t 3: (iv)$\Rightarrow$(ii). For any i and j in $\mathbb{N}$, let $S_1 = \{1, 2, \ldots, a(i,j)\}$ and $S_2 = \{m : m \leq a(i,j+1)$ and m is not in column 1 of $A(\mathbb{B})\}$. By (iv), the mapping $f_{\mathbb{B}}$ is a one-to-one correspondence from S_1 onto S_2. Therefore, $\#a(i,j+1) = a(i,j)$.

P a r t 4: (ii)$\Rightarrow$(iii). Suppose, to the contrary, that (iii) fails. Let k be the least index not less than 2 for which $b_k < f_{\mathbb{B}_{k-1}}(b_{k-1}-1)+1$ or $b_k > f_{\mathbb{B}_{k-1}}(b_{k-1})$. Let $\widehat{b}_h = b_h$ for $h = 0, 1, \ldots, k-1$, and define inductively

$$\widehat{b}_{k+h} = f_{\widehat{\mathbb{B}}_{k+h-1}}(\widehat{b}_{k+h-1}-1)+1 \quad \text{and} \quad \widehat{\mathbb{B}}_{k+h} = (\widehat{b}_0, \widehat{b}_1, \ldots, \widehat{b}_{k+h})$$

for $h = 0, 1, \ldots$ The basis $\widehat{\mathbb{B}} = (\widehat{b}_0, \widehat{b}_1, \ldots)$ satisfies (iii) (with notation modified in an obvious way), so that by Parts 2 and 3 of this proof, already proved, property (ii) holds for the array $A(\widehat{\mathbb{B}})$. That is, $\#\widehat{a}(i,j+1) = \widehat{a}(i,j)$ for all i and j in $\mathbb{N}$, where $\widehat{a}$ denotes terms of $A(\widehat{\mathbb{B}})$.

C a s e 1: $b_k < f_{\mathbb{B}_{k-1}}(b_{k-1}-1)+1$. The number $b_{k-1}-1$ is in $A(\mathbb{B})$, which is to say that it is $a(i,j)$ for some (i,j). The inequality $b_{k-1}-1 < b_{k-1}$ can therefore be written as

$$a(i,j) < a(1,k). \tag{15}$$

Now $a(h,j+1) = \widehat{a}(h,j+1)$ for $h = 1, \ldots, i-1$, and this accounts for the first $i-1$ terms of column $j+1$ of array $A(\mathbb{B})$. The greatest of these, $a(i-1,j+1)$, is the greatest number that has $\mathbb{B}$-residue less than b_k. Additionally, the number $b_{k-1}+b_k=a(1,k)+a(1,k+1)$ is the least number not less than b_k whose $\mathbb{B}$-residue is b_{k-1}; hence $a(i,j+1)=a(1,k)+a(1,k+1)$, so that

$$a(i,j+1)>a(1,k+1). \tag{16}$$

But since (ii) holds in $A(\mathbb{B})$, the inequalities (15) and (16) are incompatible, and we conclude that $b_k \geq f_{\mathbb{B}_{k-1}}(b_{k-1}-1)+1$.

Case 2: $b_k > f_{\mathbb{B}_{k-1}}(b_{k-1})$. Let i be the index for which $a(i,1)=b_{k-1}+1$. Then in $A(\widehat{\mathbb{B}})$ we have $f_{\mathbb{B}_{k-1}}(\widehat{a}(i,1))=f_{\mathbb{B}_{k-1}}(b_{k-1}+1)$, or equivalently, $\widehat{a}(i,2)=f_{\mathbb{B}_{k-1}}(b_{k-1})+b_1$. Now $a(h,2)=\widehat{a}(h,2)$ for $h=1,\ldots,i-1$, and this accounts for the first $i-1$ terms of column 2 of $A(\widehat{\mathbb{B}})$. Since $\widehat{a}(i,2)$ and b_1 both have $\mathbb{B}$-residue b_1, their difference, $f_{\mathbb{B}_{k-1}}(b_{k-1})$, also has $\mathbb{B}$-residue b_1, so that

$$a(i,2) \leq f_{\mathbb{B}_{k-1}}(b_{k-1}) = a(1,k+1).$$

We now have $a(i,1)>a(1,k)$ and $a(i,2)\leq a(1,k+1)$, contrary to (ii). Therefore, $b_k \leq f_{\mathbb{B}_{k-1}}(b_{k-1})$. ■

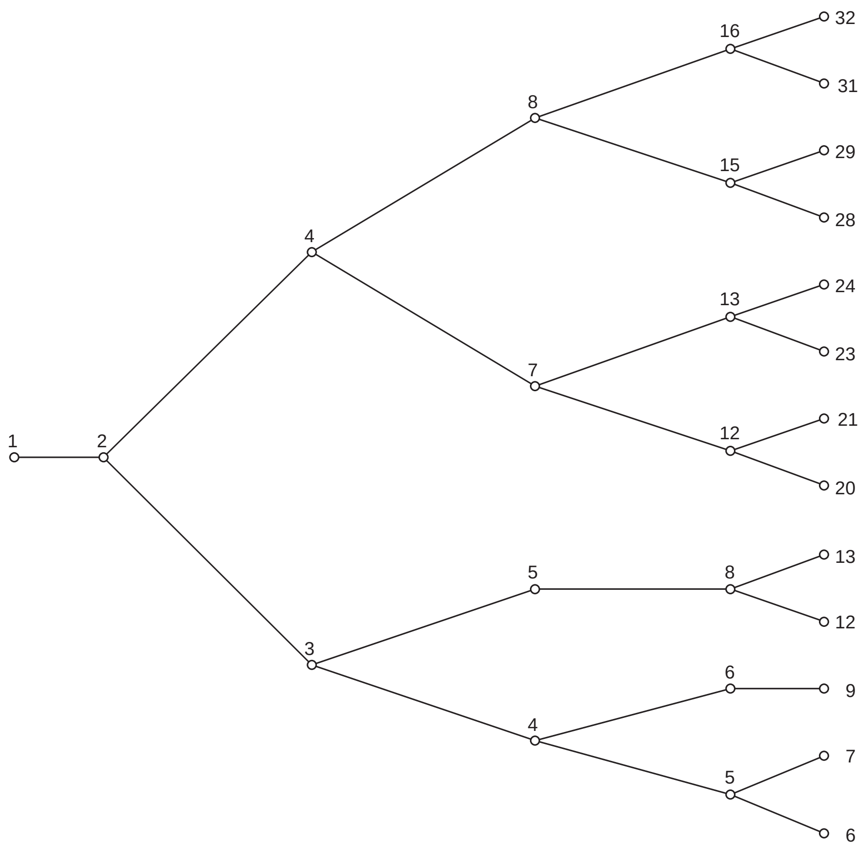

Fig. 1. The first six terms of the fractal bases in which $b_1 = 2$

Corollary 3.1. *With reference to statement* (iii) *in Theorem* 3, *the number of allowable* b_j *is not greater than* b_1.

Proof. By Theorem 3, the greatest allowable b_j is the number $f_{\mathbb{B}_{j-1}}(b_{j-1})$, which we abbreviate as M. In the array $A(\mathbb{B})$, the consecutive integers

$$M-1, M-2, \ldots, M-[f_{\mathbb{B}_{j-1}}(b_{j-1})-f_{\mathbb{B}_{j-1}}(b_{j-1}-1)-1]$$

all lie in column 1, since none of them is of the form $f_{\mathbb{B}_{j-1}}(x)$. It follows easily from the definition of $\mathbb{B}_{j-1}$-representation and Theorem 3(iii), that the maximum number of consecutive integers for which this is possible is b_1. ∎

Corollary 3.2. *Let* $\mathbb{B}=(b_0,b_1,\ldots)$ *be a proper basis. Suppose* $\mathbb{B}_{j_1}=(b_0,b_1,\ldots,b_{j_1})$ *is a prefractal basis for all* $j_1\geq 1$, *and* $b_j=f_{\mathbb{B}_{j-1}}(b_{j-1})$ *for* $j=j_1+1,j_1+2,\ldots$ *Then* $\mathbb{B}$ *is a fractal basis. If the* $\mathbb{B}_{j_1-1}$*-representation of* b_{j_1} *is given by*

$$b_{j_1}=\gamma_{j_1-p}b_{j_1-p}+\gamma_{j_1-p+1}b_{j_1-p+1}+\ldots+\gamma_{j_1-1}b_{j_1-1}, \tag{17}$$

then the row sequences of $A(\mathbb{B})$ *satisfy the homogeneous linear recurrence inherited from* (17):

$$a(i,j_1+q)$$
$$=\gamma_{j_1-p}a(i,j_1+q-p)+\gamma_1 a(i,j_1+q-p+1)+\ldots+\gamma_{j_1-1}a(i,j_1+q-1),$$
$$i\geq 1,\ q\geq 0.$$

Proof. This is an obvious consequence of Theorem 3(iv). ∎

In summary, a prefractal basis $(b_0,b_1,\ldots,b_j)$ can always be extended to a prefractal basis $(b_0,b_1,\ldots,b_j,b_{j+1})$, where the number p of allowable values of b_{j+1} satisfies $1\leq p\leq b_1$. Extending inductively, we can in this manner construct any fractal basis as a limit of prefractal bases. If the choice of b_{j+1} is always maximal beginning with the first term after some particular b_{j_1}, then we obtain, in accord with Corollary 3.2, the *homogeneous extension* of $(b_0,b_1,\ldots,b_{j_1})$. Specifically, if b_{j_1} is given by the $\mathbb{B}_{j_1-1}$-representation

$$b_{j_1}=\mathcal{R}(b_0,b_1,\ldots,b_{j_1-1})=\gamma_0 b_0+\gamma_1 b_1+\ldots+\gamma_{j_1-1}b_{j_1-1},$$

then all the row sequences of the limiting fractal basis satisfy the homogeneous recurrence determined by $\mathcal{R}$. We now turn to certain nonhomogeneous linear recurrences, associated with minimal choices of b_{j+1}, as given by (20).

Corollary 3.3. *Suppose* $\mathbb{B}_{j_1}=(b_0,b_1,\ldots,b_{j_1})$ *is a prefractal basis for all* $j_1\geq 1$. *Let the* $\mathbb{B}_{j_1}$*-representation of* $b_{j_1}-1$ *be given by*

$$b_{j_1}-1=\delta_{j_1-p}b_{j_1-p}+\delta_{j_1-p+1}b_{j_1-p+1}+\ldots+\delta_{j_1-1}b_{j_1-1}, \tag{18}$$

where $\delta_{j_1-p} \neq 0$, *so that the order of the homogeneous linear recurrence* $\mathcal{S}$ *given by*

$$(19)\qquad \mathcal{S}(x_1, x_2, \ldots, x_p) = \delta_{j_1-p}x_1 + \delta_{j_1-p+1}x_2 + \ldots + \delta_{j_1-1}x_p$$

is p. *Let* $\mathbb{B}$ *be the fractal basis obtained inductively from* $\mathbb{B}_{j_1}$ *by defining*

$$(20)\qquad b_{j_1+q+1} = f_{\mathbb{B}_{j_1+q}}(b_{j_1+q} - 1) + 1$$

for $q = 0, 1, \ldots$ *Then row* 1 *of* $A(\mathbb{B})$ *satisfies the nonhomogeneous linear recurrence*

$$(21)\qquad \begin{aligned} b_j = a(1, j+1) &= \mathcal{S}(b_{j-p}, b_{j-p+1}, \ldots, b_{j-1}) + 1 \\ &= \mathcal{S}(a(1, j-p+1), a(1, j-p+2), \ldots, a(1, j)) + 1, \end{aligned}$$

for $j = j_1, j_1 + 1, \ldots,$ *and row* i *of* $A(\mathbb{B})$ *satisfies the nonhomogeneous linear recurrence*

$$(22)\qquad a(i, j+1) = \mathcal{S}(a(i, j-p+1), a(i, j-p+2), \ldots, a(i, j)) + Q_i,$$

where Q_i *depends only on* i, *for all* i *in* $\mathbb{N}$, *for* $j = j_1, j_1 + 1, \ldots$

Proof. Equations (18) and (19) give

$$f_{\mathbb{B}_{j_1}}(b_{j_1} - 1) = f_{\mathbb{B}_{j_1}}(\mathcal{S}(b_{j_1-p}, b_{j_1-p+1}, \ldots, b_{j_1-1})),$$

so that by (20) with $q = 0$, we have $b_{j_1+1} = \mathcal{S}(b_{j_1-p+1}, b_{j_1-p+2}, \ldots, b_{j_1}) + 1$. The same method easily completes an induction proof that (21) holds for all $j \geq j_1$, so that (22) is established for $i = 1$.

Assume now that $i \geq 2$ and $j \geq j_1$. Let the $\mathbb{B}$-representation of $a(i, 1)$ be given by $a(i, 1) = \sum_{h=1}^{v} c_{h-1}a(1, h)$, and let $Q_i = \sum_{h=0}^{v-1} c_h$. By Theorem 3,

$$\begin{aligned} a(i, j+1) &= \sum_{h=1}^{v} c_{h-1}a(1, j+h) \\ &= \sum_{h=1}^{v} c_{h-1}(\mathcal{S}(b_{j-p+h-1}, b_{j-p+h}, \ldots, b_{j+h-1}) + 1) \\ &= Q_i + \sum_{h=1}^{v} c_{h-1} \sum_{k=0}^{p-1} \delta_{j-p+k} b_{j-p+k} \\ &= Q_i + \sum_{k=1}^{p} \delta_{j-k} \sum_{h=1}^{v} c_{h-1}a(1, j-k+h) \\ &= Q_i + \sum_{k=1}^{p} \delta_{j-k}a(i, j-k+1) \\ &= \mathcal{S}(a(i, j-p+1), a(i, j-p+2), \ldots, a(i, j)) + Q_i. \ \blacksquare \end{aligned}$$

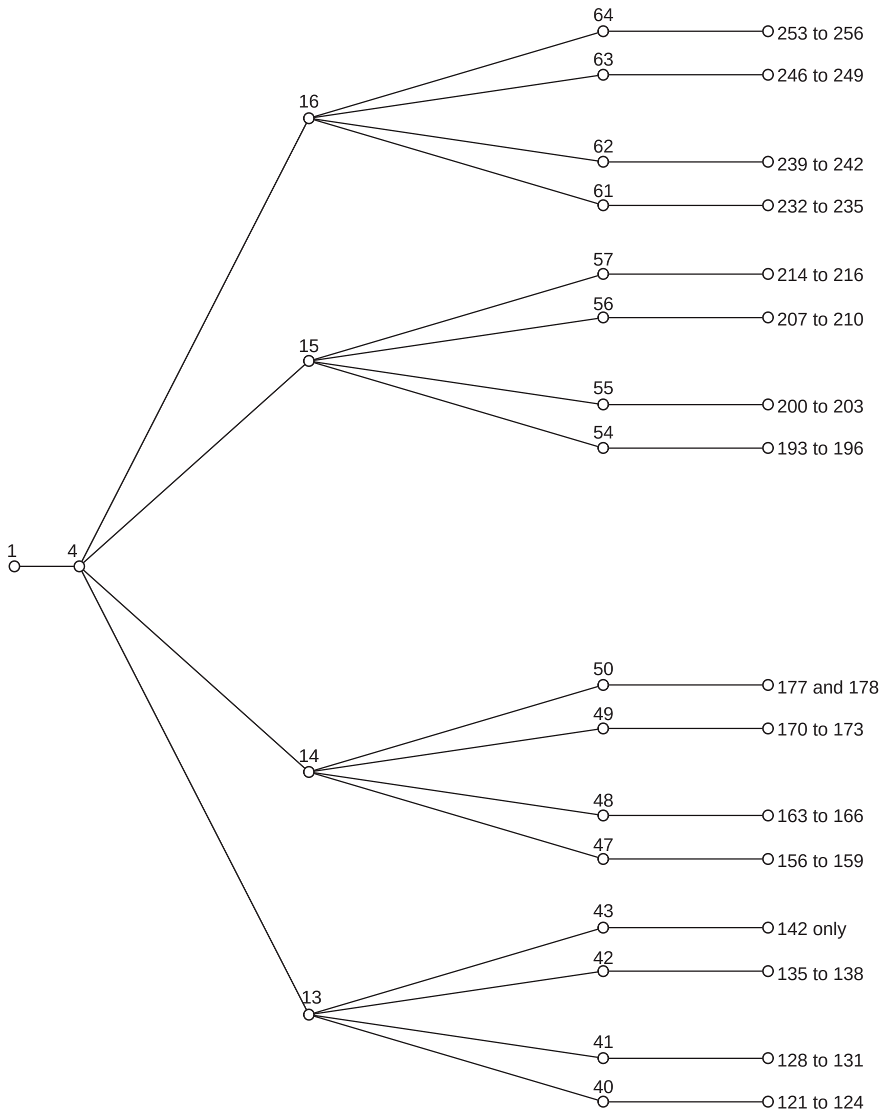

Fig. 2. The first five terms of the fractal bases in which $b_1 = 4$

Figures 1 and 2 lead one to conjecture that every prefractal basis has uncountably many extensions to fractal bases. Another question concerns the inequality in Theorem 3(iii): when is there only one possible choice of b_j, as exemplified by $b_4 = 8$ following $b_3 = 5$ in Figure 1, and also by $b_4 = 142$ following $b_3 = 43$ in Figure 2?

Finally, as you may have already observed, for each choice of $b_1 \geq 2$, the fractal bases with second term b_1 fan themselves out between two extreme cases, one an arithmetic sequence and the other a geometric sequence.

References

[1] A. Fraenkel, *Systems of numeration*, Amer. Math. Monthly 92 (1985), 105–114.
[2] C. Kimberling, *Interspersions and dispersions*, Proc. Amer. Math. Soc. 117 (1993), 313–321.
[3] —, *Fractal sequences and interspersions*, Ars Combin., to appear.
[4] A. M. Yaglom and I. M. Yaglom, *Challenging Mathematical Problems With Elementary Solutions*, Vol. II, Dover, New York, 1967.

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