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The value of the Estermann zeta functions at $s = 0$

by

MAKOTO ISHIBASHI (Kagoshima)

1. Introduction. The Estermann zeta function $E_u(s, a/q)$ is defined by the Dirichlet series

$$E_u\left(s, \frac{a}{q}\right) = \sum_{n=1}^{\infty} \sigma_u(n) e\left(\frac{an}{q}\right) n^{-s}, \qquad \mathrm{Re}(s) > \mathrm{Re}(u) + 1,$$

where $e(\alpha) = e^{2\pi i \alpha}, a, q$ are integers with $q \geq 1, (a, q) = 1$, and $\sigma_u(n) = \sum_{d|n} d^u$. It is known that $E_u(s, a/q)$ can be continued analytically to the whole complex plane up to a double pole at $s = 1$ ([1]). This function naturally occurs in the study of the exponential sums of the type

$$D_u\left(x, \frac{a}{q}\right) = {\sum_{n \leq x}}' \sigma_u(n) e\left(\frac{an}{q}\right) \qquad ([4]\text{–}[6]),$$

where $\sum'$ means that if x is an integer, then the term with $n = x$ in the sum is to be halved. We can easily get the explicit formula for these sums by applying Perron's formula, i.e.

$$D_u\left(x, \frac{a}{q}\right) = \frac{1}{q}(\log x + 2\gamma - 1 - 2q \log x)x$$
$$+ E_u\left(0, \frac{a}{q}\right) + \frac{1}{2\pi i} \int_{(-\varepsilon)} E_u\left(s, \frac{a}{q}\right) x^s s^{-1}\, ds$$

where γ is Euler's constant, and the integral is taken along the vertical line with $\mathrm{Re}(s) = -\varepsilon$, $\varepsilon > 0$.

In this paper we shall evaluate the constants $E_u(0, a/q)$ in terms of the cotangent function in the case u is an integer and determine the $\mathbb{Q}$-linear relations between $E_u(0, a/q)$, where $\mathbb{Q}$ denotes the rational number field.

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2. Constants $E_j(0,a/q)$. Let $B_j(x)$ be the jth Bernoulli polynomial. The next lemma shows a relation between the values of $B_j(x)$ and the jth derivative of $\cot\pi x$ at $x=a/q$.

LEMMA ([2]). *Let $q\ge 2$, $1\le a\le q$, $(a,q)=1$. Then*

$$(j+1)\left(\frac{i}{2}\right)^{j+1}\cot^{(j)}\left(\frac{\pi a}{q}\right)=q^j\sum_{k=1}^{q-1}e\left(-\frac{\pi a}{q}\right)B_{j+1}\left(\frac{k}{q}\right) \tag{1}$$

for $j=0,1,\ldots$

Now this lemma implies

THEOREM 1. *We have*

$$E_j\left(0,\frac{a}{q}\right)=\begin{cases}\dfrac{B_{j+1}}{2(j+1)}, & j\ \text{odd},\\ \left(-\dfrac{i}{2}\right)^{j+1}\displaystyle\sum_{k=1}^{q-1}\frac{k}{q}\cot^{(j)}\left(\frac{\pi ak}{q}\right)+\frac{1}{4}\delta_{j,0}, & j\ \text{even},\end{cases} \tag{2}$$

for $q\ge 2$, where $\delta_{j,0}=1$ for $j=0$ and 0 otherwise. For $q=1$,

$$E_j(0,1)=\frac{(-1)^{j+1}B_{j+1}}{2(j+1)}.$$

Proof. For $\mathrm{Re}(s)>j+1$, substituting for $\sigma_j(n)$, we obtain

$$\begin{aligned}E_j\left(s,\frac{a}{q}\right)&=\sum_{n=1}^{\infty}\sigma_j(n)e\left(\frac{an}{q}\right)n^{-s}=\sum_{m,n=1}^{\infty}e\left(\frac{amn}{q}\right)m^{j-s}n^{-s}\\&=\sum_{k,l=1}^{q}e\left(\frac{akl}{q}\right)\sum_{m\equiv k\ (q)}\sum_{n\equiv l\ (q)}m^{j-s}n^{-s}\\&=q^{j-2s}\sum_{k,l=1}^{q}e\left(\frac{akl}{q}\right)\zeta(s-j,k/q)\zeta(s,l/q),\end{aligned}$$

where in the final step we have used the Hurwitz zeta function $\zeta(s,x)=\sum_{n=0}^{\infty}(n+x)^{-s}$ $(0<x\le 1)$. Then it follows from the analytic continuation of $\zeta(s,x)$ that

$$\begin{aligned}E_j(0,a/q)&=q^j\sum_{k,l=1}^{q}e\left(\frac{akl}{q}\right)\zeta(-j,k/q)\zeta(0,l/q)\\&=q^j\sum_{k,l=1}^{q-1}e\left(\frac{akl}{q}\right)\zeta(-j,k/q)\zeta(0,l/q)\\&\quad+q^j\Big(\sum_{k=1}^{q-1}\zeta(0,1)\zeta(-j,k/q)+\sum_{l=1}^{q}\zeta(-j,1)\zeta(0,l/q)\Big)\end{aligned}$$

$$= \frac{q^j}{j+1} \sum_{l=1}^{q-1} B_1(l/q) \sum_{k=1}^{q-1} e\left(\frac{akl}{q}\right) B_{j+1}(k/q)$$
$$+ \begin{cases} 1/4, & j=0, \\ \dfrac{B_{j+1}}{2(j+1)}, & j \ge 1, \end{cases}$$

after some computations using

$$\zeta(-j, k/q) = -\frac{1}{j+1} B_{j+1}(k/q), \qquad j \ge 0.$$

Changing the variable of summation k to $q-k$ and using $B_{j+1}(1-x) = (-1)^{j+1} B_{j+1}(x)$, $B_1(l/q) = l/q - 1/2$ and Lemma, we obtain our formula.

3. $\mathbb{Q}$-linear relations. In [2], K. Girstmair gave a unified approach to the determination of all the $\mathbb{Q}$-linear relations between conjugate numbers in a cyclotomic field. Summarizing, his method is as follows: Let $\mathbb{Q}_q = \mathbb{Q}(\zeta)$ be the qth cyclotomic field with $\zeta = e(1/q)$ and let $G = \mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q})$ be its Galois group viewed as $(\mathbb{Z}/q\mathbb{Z})^\times$. We consider $\mathbb{Q}_q$ as a $\mathbb{Q}G$-module, where $\mathbb{Q}G$ denotes the group ring. For $b \in \mathbb{Q}_q$, the $\mathbb{Q}$-linear relations of the numbers $\sigma(b)$, $\sigma \in G$, are determined by the annihilator $W_q(b)$ of b in $\mathbb{Q}G$ defined by

$$W_q(b) = \{\alpha \in \mathbb{Q}G : \alpha \circ b = 0\},$$

where $\alpha \circ b = \sum_{\sigma \in G} a_\sigma \sigma(b)$ for $\alpha = \sum_{\sigma \in G} a_\sigma \sigma \in \mathbb{Q}G$. It is known that any non-zero ideal I in $\mathbb{Q}G$ is generated by the unique idempotent element $\varepsilon_X = \sum_{\chi \in X} \varepsilon_\chi$, written $I = \langle \varepsilon_X \rangle$, where

$$\varepsilon_\chi = |G|^{-1} \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma, \qquad X = \{\chi \in \widehat{G} : \chi(I) \ne 0\}.$$

($\widehat{G}$ denotes the character group of G.) He proves

THEOREM A ([2]). *The ideal $W_q(b)$ is generated by ε_X with*

$$X = \{\chi \in \widehat{G} : y(\chi|b) = 0\},$$

where $y(\chi|b)$ are Leopoldt's character coordinates defined by

$$y(\chi|b)\tau(\overline{\chi}_f|1) = \sum_{\sigma \in G} \chi(\sigma^{-1})\sigma(b) = \sum_{k=1}^{q} \overline{\chi}(k)\sigma_k(b), \tag{3}$$

where f means the conductor of χ, χ_f is the primitive character modulo f attached to χ and $\tau(\chi|k)$ is the k-th Gauss sum.

He also shows how to compute ε_X explicitly for a special choice of X and obtains, among over things, THEOREM B ([2]). *Let $j \ge 0$ and $q \ge 2$. Then*

$$W_q[i^{j+1}\cot^{(j)}(\pi/q)] = \langle 1 + (-1)^j \sigma_{-1} \rangle,$$

with $X = \{\chi \in \widehat{G} : \chi(\sigma_{-1}) = (-1)^j\}$, where $\sigma_k \in G$ are such that $\sigma_k(\zeta) = \zeta^k$ for $(k,n) = 1$, and $i = \sqrt{-1}$.

Similarly, in the case of the numbers $E_j(0, a/q)$, $1 \le a \le q$, with $(a,q) = 1$ which are conjugate in $\mathbb{Q}_q$, we can prove

THEOREM 2. *Let $q \ge 2$ be a prime power and let j be an even integer. Then*

$$W_q[E_j(0, 1/q)] = \langle 1 + \sigma_{-1} \rangle.$$

Proof. By virtue of Theorem A, we have only to compute the character coordinates of the numbers $E_j(0,1/q)$. From the $\mathbb{Q}G$-linearity of $y(\chi|-)$ and the character coordinates for $i^{j+1}\cot^{(j)}(\pi/q)$ ([2], Theorem 2), we have

$$\begin{aligned} y(\chi|E_j(0,1/q)) &= \left(-\frac{1}{2}\right)^{j+1} \sum_{k=1}^{q-1} \frac{k}{q}\chi(k) y(\chi|i^{j+1}\cot^{(j)}(\pi/q)) \\ &= \frac{-1}{j+1}\left(\frac{q}{f}\right)^{j+1} \prod_{p|q}\left(1 - \frac{\overline{\chi}_f(p)}{p^{j+1}}\right) B_{1,\chi} B_{j+1,\chi_f}, \end{aligned}$$

where p runs through the prime factors of q.

Here $B_{n,\chi}$ denotes the generalized nth Bernoulli number, satisfying the relations

$$B_{n,\chi} = m^{n-1} \sum_{a=1}^{m} \chi(a) B_n(a/m),$$

where m is the modulus of χ. In the case of primitive character it is known [3] that for the principal character $\chi_0, B_{1,\chi_0} \ne 0$ and $B_{j+1,\chi_0} = 0$, for even $j \ge 2$, and for non-principal χ,

$$\begin{cases} B_{j+1,\chi} \ne 0, & j \not\equiv \delta_\chi \pmod 2, \\ B_{j+1,\chi} = 0, & j \equiv \delta_\chi \pmod 2, \end{cases}$$

where

$$\delta_\chi = \begin{cases} 0, & \chi \text{ even}, \\ 1, & \chi \text{ odd}. \end{cases}$$

Further, we see that $B_{1,\chi} \ne 0$ for odd χ if the modulus of χ is a prime power. Hence, we get $X = \{\chi \in \widehat{G} : \chi(\sigma_{-1}) = 1\}$, for $W_q[E_j(0,1/q)]$, which is just the same as Theorem B. This completes the proof.

Theorem 2 implies

THEOREM 3. *For the numbers* $E_j(0,a/q)$, $1 \le a \le q$ $(a,q)=1$, *q a prime power and* $c_a \in \mathbb{Q}$ *we have*

$$\sum_{(a,q)=1} c_a E_j(0,a/q) = 0 \quad \textit{if, and only if,} \quad c_a = c_{q-a} \textit{ and } \sum c_a = 0.$$

We see easily that

COROLLARY 1. *The numbers* $E_j(0,a/q)$, $1 \le a \le q/2$, $(a,q)=1$, *q a prime power*, *j even*, *are linearly independent over* $\mathbb{Q}$.

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1460-1 SHINKO, HAYATO-CHO, AIRA-GUN
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