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Qualitative investigation of nonlinear differential equations describing infiltration of water

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Abstract. A nonlinear differential equation of the form $(q(x)k(u)u')' = F(x,u,u')$ arising in models of infiltration of water is considered, together with the corresponding differential equation with a positive parameter λ, $(q(x)k(u)u')' = \lambda F(x,u,u')$. The theorems about existence, uniqueness, boundedness of solution and its dependence on the parameter are established.

1. Introduction. To describe the mathematical model of unsteady infiltration in water percolation and seepage, the Boussinesq equation is used [4]. The simplest case is that of a horizontal base without accretion, when the flow is the same in all vertical parallel planes. In this case, the corresponding mathematical model assumes the most common form of the Boussinesq equation:

$$(hh_x)_x = mh_t/K. \tag{1}$$

The corresponding equation, when the impervious base has a constant slope, is as follows:

$$(hh_x)_x = Ih_x + mh_t/K. \tag{2}$$

This equation can be reduced to (1) by a transformation of the independent variables

$$x' = x - IKt/m, \quad t' = t.$$

In the case of accretion, the flow on a horizontal base obeys

$$(hh_x)_x = mh_t/K + \varepsilon/K \tag{3}$$

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and on an inclined base of constant slope,

$$(hh_x)_x = Ih_x + mh_t/K + \varepsilon/K. \tag{4}$$

Similar to (2), (3) and (4) can also be reduced to (1) by a suitable transformation. In an axisymmetric unsteady flow, as in the single well problem, $h = h(r,t)$. Boussinesq's equation then becomes

$$(rhh_r)_r = mrh_t/K. \tag{5}$$

Under different cases, their particular similarity solutions are reduced to solving the following second order nonlinear differential equations with unknown function $f = f(\alpha)$ [1, 2, 4]:

$$\alpha(ff')' + ff' + \alpha^2 f'/2 = 0; \tag{6}$$

$$\alpha(ff')' + ff' = n\alpha^2 f' - (1+2n)\alpha f; \tag{7}$$

$$\alpha^2(ff')' + (1+4\nu)\alpha ff' + 2\nu^2 f^2 = \alpha^{3-\nu} f'/(\nu-2) \tag{8}$$

and

$$\alpha(ff')' + ff' = n(\alpha^2 f' - 2\alpha f). \tag{9}$$

Therefore, in [7–9], the authors investigated the following second order nonlinear differential equations:

$$(k(u)u')' = f(x)u', \quad x > 0; \tag{10}$$

$$(q(t)k(u)u')' = f(t)h(u)u', \quad t > 0; \tag{11}$$

and

$$(q(t)k(u)u')' = F(t,u)u', \quad t > 0. \tag{12}$$

In this paper, we shall consider the more general second order nonlinear differential equations arising in models of water infiltration:

$$(q(x)k(u)u')' = F(x,u,u'), \quad x > 0, \tag{13}$$

and

$$(q(x)k(u)u')' = \lambda F(x,u,u'), \quad x > 0. \tag{14}$$

Obviously, (10), (11) and (12) are special cases of (13). We obtain qualitative results on (13) and (14), such as existence, uniqueness, boundedness and dependence on parameters. Our theorems imply all results in [7–9].

2. Definition of solution and equivalence. Let q, k and F satisfy the following assumptions ($\alpha > 0$, $\mathbb{R}_+ = (0,\infty)$, $\mathbb{R}_- = (-\infty,0)$ and $\mathbb{R} = (-\infty,\infty)$):

$$q \in C^0(\overline{\mathbb{R}}_+); \quad q(x) > 0,\ x \in \mathbb{R}_+; \quad \int_0^\alpha (1/q(x))\,dx < \infty; \tag{H_1}$$

$k \in C^0(\mathbb{R}); \quad k(u)u > 0,\ u \in \mathbb{R} - \{0\};$

$$\text{(H}_2\text{)} \qquad \int_0^{\alpha} (k(u)/u)\, du < \infty, \qquad \int_{-\alpha}^{0} (k(u)/u)\, du < \infty,$$

$$\int_{-\infty}^{-\alpha} (k(u)/u)\, du = \infty, \qquad \int_{\alpha}^{\infty} (k(u)/u)\, du = \infty;$$

$$\text{(H}_3\text{)} \qquad F \in C^0(\overline{\mathbb{R}}_+ \times \mathbb{R}^2); \qquad f_1(x)h_1(u) \le F(x,u,u')/u' \le f_2(x)h_2(u),$$

where $f_i \in C^0(\overline{\mathbb{R}}_+)$, $f_i(x) > 0$, $f_i(x)$ is decreasing, either $h_1(u) = h_2(u) \equiv 1$, or $h_i \in C^0(\mathbb{R})$, $h_i(u)u > 0$ for $u \in \mathbb{R} - \{0\},\ i = 1,2$.

Remark 1. It follows from (H_2) that $k(0) = 0$. Similarly, if it is not the case that $h_1(u) = h_2(u) \equiv 1$, then $h_i(0) = 0,\ i = 1,2$, and so $F(x,0,u') = 0$ for $x \in \overline{\mathbb{R}}_+$ and $u' \in \mathbb{R}$.

In what follows, we shall investigate the differential equation (13) on $\overline{\mathbb{R}}_+$ with $u(0) = 0$ under the assumptions above.

Definition. By a *solution* of (13) we mean a function $u \in C^0(\overline{\mathbb{R}}_+) \cap C^1(\mathbb{R}_+)$ such that $u(0) = 0$,

$$\lim_{x \to 0^+} q(x)k(u(x))u'(x) = 0,$$

$q(x)k(u(x))u'(x) \in C^1(\mathbb{R}_+)$ and (13) is satisfied in $\mathbb{R}_+$.

Remark 2. From (H_3), it follows that $F(x,u,0) \equiv 0$ for $x \in \overline{\mathbb{R}}_+$ and $u \in \mathbb{R}$.

Remark 3. Obviously, $u(0) \equiv 0$ for $x \in \overline{\mathbb{R}}_+$ is a solution of (13).

Lemma 1. *Let $u(x)$ be a nontrivial solution of* (13). *Then either $u'(x) > 0$ in $\mathbb{R}_+$ or $u'(x) < 0$ in $\mathbb{R}_+$.*

Proof. First, $u'(x)$ is not equivalent to 0, since otherwise, $u(x) \equiv 0$.

Next, let us prove that $u'(x)$ cannot have more than one root. If not, assume $0 \le x_1 < x_2$ are such that $u'(x_1) = u'(x_2) = 0$ and $u'(x) \ne 0$ in (x_1, x_2); without loss of generality, let $u'(x) > 0$ in (x_1, x_2). Then $u(x)$ is increasing in (x_1, x_2), and for $x > \varepsilon > 0$,

$$\int_{\varepsilon}^{x} F(s, u(s), u'(s))\, ds = q(x)k(u(x))u'(x) - q(\varepsilon)k(u(\varepsilon))u'(\varepsilon).$$

Hence (by letting $\varepsilon \to 0$),

$$\text{(15)} \qquad q(x)k(u(x))u'(x) = \int_0^x F(s, u(s), u'(s))\, ds, \qquad x \in \mathbb{R}_+.$$

In the following, we consider three cases: $u(x) > 0$, $u(x) < 0$ and $u(x_1) < 0 < u(x_2)$. If $u(x) > 0$ in (x_1, x_2), then by (H_3),

$$\int_{x_1}^{x_2} F(s, u(s), u'(s))\, ds \geq \int_{x_1}^{x_2} f_1(s) h_1(u(s)) u'(s)\, ds \geq f_1(x_2) \int_{u(x_1)}^{u(x_2)} h_1(s)\, ds.$$

By the mean value theorem [5],

$$\int_{u(x_1)}^{u(x_2)} h_1(s)\, ds = h_1(\xi)(u(x_2) - u(x_1)),$$

where $\xi \in (u(x_1), u(x_2))$. Hence,

$$\int_{x_1}^{x_2} F(s, u(s), u'(s))\, ds > 0.$$

But, from (15),

$$\int_{x_1}^{x_2} F(s, u(s), u'(s))\, ds = q(x) k(u(x)) u'(x)|_{x=x_1}^{x_2}.$$

Noting that $u'(x_1) = 0$ and $u'(x_2) = 0$, we have

$$\int_{x_1}^{x_2} F(s, u(s), u'(s))\, ds = 0.$$

This is a contradiction.

The case of $u(x) < 0$ in (x_1, x_2) can be treated quite analogously.

If $u(x_1) < 0 < u(x_2)$, then there exists a unique $\overline{x} \in (x_1, x_2)$ such that $u(\overline{x}) = 0$. In this case, $u'(x) > 0$ and $u(x) > 0$ in $(\overline{x}, x_2)$; hence, from the above proof,

$$\int_{\overline{x}}^{x_2} F(s, u(s), u'(s))\, ds > 0;$$

but, from (15) and noting that $u(\overline{x}) = u'(x_2) = 0$, we have

$$\int_{\overline{x}}^{x_2} F(s, u(s), u'(s))\, ds = q(x) k(u(x)) u'(x)|_{x=\overline{x}}^{x_2} = 0,$$

again a contradiction.

Finally, let us prove that there cannot exist a root of $u'(x)$. If not, assume $x_0 > 0$ is such that $u'(x_0) = 0$ and $u'(x) \neq 0$ in $(0, x_0)$. Without loss of generality, let $u'(x) > 0$ in $(0, x_0)$. Then $u(x) > 0$ in this interval and

$$\int_0^{x_0} F(s, u(s), u'(s))\, ds \ge \int_0^{x_0} f_1(s) h_1(u(s)) u'(s)\, ds \ge f_1(x_0) \int_0^{u(x_0)} h_1(s)\, ds > 0.$$

On the other hand,

$$\int_0^{x_0} F(s, u(s), u'(s))\, ds = q(x_0) k(u(x_0)) u'(x_0) = 0,$$

a contradiction. So $u'(x) \neq 0$. Since $u \in C^1(\mathbb{R}_+)$ the proof is complete.

Remark 4. It follows from Lemma 1 that $u \in \mathcal{A}_+$ or $u \in \mathcal{A}_-$ for any nontrivial solution u of (13), where

$$\mathcal{A}_+ = \{u \in C^0(\overline{\mathbb{R}}_+) : u(0) = 0,\ u \text{ is strictly increasing on } \overline{\mathbb{R}}_+\},$$
$$\mathcal{A}_- = \{u \in C^0(\overline{\mathbb{R}}_+) : u(0) = 0,\ u \text{ is strictly decreasing on } \overline{\mathbb{R}}_+\}.$$

Set

$$W_\varepsilon(u) = \int_0^u k(s)\, ds, \qquad u \in \mathcal{A}_\varepsilon,\ \varepsilon \in \{+, -\}.$$

Obviously, W_+ is strictly increasing on $\mathcal{A}_+$ and W_- is strictly decreasing on $\mathcal{A}_-$.

THEOREM 1. *If u is a solution of* (13), $u \neq 0$, *then u is a solution of the functional-integrodifferential equation*

$$u(x) = W_\varepsilon^{-1}\Big(\int_0^x \frac{1}{q(s)} \int_0^s F(t, u(t), u'(t))\, dt\, ds\Big) \tag{16}$$

in the corresponding set $\mathcal{A}_\varepsilon$. Conversely, if $u \in \mathcal{A}_\varepsilon$, $\varepsilon \in \{+, -\}$ is a solution of (16) *then u is a solution of* (13) *and $u \neq 0$. Here W_ε^{-1} denotes the inverse function of W_ε.*

Proof. Let $u \neq 0$ be a solution of (13). Then $u \in \mathcal{A}_+ \cup \mathcal{A}_-$ by Remark 4 and (15) holds. If $u \in \mathcal{A}_\varepsilon$ then

$$W_\varepsilon(u(x)) = \int_0^x \frac{1}{q(s)} \int_0^s F(t, u(t), u'(t))\, dt\, ds$$

for $x \in \overline{\mathbb{R}}_+$ and u is a solution of (16) in $\mathcal{A}_\varepsilon$.

Conversely, noting that W_ε is monotonic and continuously differentiable, we have

$$u'(x) = \frac{1}{k(u(x))}\Big(\int_0^x \frac{1}{q(s)} \int_0^s F(t,u(t),u'(t))\,dt\,ds\Big)'$$
$$= \frac{1}{k(u(x))q(x)} \int_0^x F(t,u(t),u'(t))\,dt$$

or

$$q(x)k(u(x))u'(x) = \int_0^x F(t,u(t),u'(t))\,dt.$$

Hence $q(x)k(u(x))u'(x) \in C^1(\mathbb{R}_+)$ and (13) holds. Consequently, u is a solution of (13).

Remark 5. It follows from Theorem 1 that solving (13) is equivalent to solving (16) in $\mathcal{A}_\varepsilon$.

3. Existence. We further suppose:

(H$_4$)
$$\int_0^\alpha (k(u)/H_i(u))\,du < \infty, \qquad \int_{-\alpha}^0 (k(u)/T_i(u))\,du < \infty,$$
$$\int_\alpha^\infty (k(u)/H_i(u)) = \infty, \qquad \int_{-\infty}^{-\alpha} (k(u)/T_i(u))\,du = \infty,$$

where $\alpha > 0$, $H_i(u) = \int_0^u h_i(s)\,ds$ for $u \in \mathcal{A}_+$ and $T_i(u) = \int_u^0 h_i(s)\,ds$ for $u \in \mathcal{A}_-$, $i = 1,2$.

Set

$$P_i(u) = \int_0^u (k(s)/H_i(s))\,ds, \qquad u \in \mathcal{A}_+;$$
$$V_i(u) = \int_u^0 (k(s)/T_i(s))\,ds, \qquad u \in \mathcal{A}_-;$$
$$k_1(x) = \int_0^x (f_1(s)/q(s))\,ds, \qquad k_2(x) = f_2(0)\int_0^x (1/q(s))\,ds,$$
$$l_1(x) = f_1(0)\int_0^x (1/q(s))\,ds, \qquad l_2(x) = \int_0^x (f_2(s)/q(s))\,ds,$$
$$\underline{\varphi}_+(x) = P_1^{-1}(k_1(x)), \qquad \overline{\varphi}_+(x) = P_2^{-1}(k_2(x)),$$
$$\underline{\varphi}_-(x) = V_1^{-1}(l_1(x)), \qquad \overline{\varphi}_-(x) = V_2^{-1}(l_2(x))$$

for $x \in \overline{\mathbb{R}}_+$, $i = 1, 2$. Obviously, from (H_4),

$$\lim_{u\to\infty} P_i(u) = \infty, \qquad \lim_{u\to-\infty} V_i(u) = \infty$$

and $P_i(u)$ is increasing and $V_i(u)$ decreasing, $i = 1, 2$.

LEMMA 2. *Under assumptions* (H_1)–(H_4), *if* $u \in \mathcal{A}_\varepsilon$ *is a solution of* (16), $\varepsilon \in \{+, -\}$, *then*

$$\underline{\varphi}_\varepsilon(x) \le u(x) \le \overline{\varphi}_\varepsilon(x), \qquad x \in \mathbb{R}_+, \tag{17}$$

and for $0 < x_1 < x_2$,

$$\begin{aligned} u(x_2) - u(x_1) &\ge \frac{H_1(\underline{\varphi}_+(x_1))(k_1(x_2) - k_1(x_1))}{\max\{k(u) : \underline{\varphi}_+(x_1) \le u \le \overline{\varphi}_+(x_2)\}}, \qquad u \in \mathcal{A}_+, \\ u(x_1) - u(x_2) &\ge \frac{T_2(\overline{\varphi}_-(x_1))(l_2(x_1) - l_2(x_2))}{\max\{-k(u) : \underline{\varphi}_-(x_2) \le u \le \overline{\varphi}_-(x_1)\}}, \qquad u \in \mathcal{A}_-. \end{aligned} \tag{18}$$

Proof. Let $u \in \mathcal{A}_+$ be a solution of (16). Then

$$\begin{aligned} f_1(x) \int_0^{u(x)} h_1(s)\,ds &\le q(x)k(u(x))u'(x) \\ &= \int_0^x F(s, u(s), u'(s))\,ds \le f_2(0) \int_0^{u(x)} h_2(s)\,ds. \end{aligned}$$

Hence,

$$f_1(x)/q(x) \le P_1'(u(x)), \qquad f_2(0)/q(x) \ge P_2'(u(x)), \tag{19}$$

and integrating (19) from 0 to x we obtain

$$k_1(x) \le P_1(u(x)), \qquad k_2(x) \ge P_2(u(x)).$$

Consequently, $\underline{\varphi}_+(x) \le u(x) \le \overline{\varphi}_+(x)$ for $x \in \overline{\mathbb{R}}_+$.

Let $0 < x_1 < x_2$. Then

$$\begin{aligned} &W_+(u(x_2)) - W_+(u(x_1)) \\ &\quad = \int_{x_1}^{x_2} \frac{1}{q(x)} \int_0^x F(s, u(s), u'(s))\,ds\,dx \ge \int_{x_1}^{x_2} \frac{f_1(x)}{q(x)} \int_0^{u(x)} h_1(s)\,ds\,dx \\ &\quad \ge \int_{x_1}^{x_2} \frac{f_1(x)}{q(x)} \int_0^{\underline{\varphi}_+(x)} h_1(s)\,ds\,dx \ge H_1(\underline{\varphi}_+(x_1))(k_1(x_2) - k_1(x_1)) \end{aligned}$$

and since $W_+(u(x_2)) - W_+(u(x_1)) = k(\xi)(u(x_2) - u(x_1))$, where $\xi \in (u(x_1), u(x_2)) \subset (\underline{\varphi}_+(x_1), \overline{\varphi}_+(x_2))$, we see that (18) is true for $u \in \mathcal{A}_+$.

The case of $u \in \mathcal{A}_-$ can be treated quite analogously.

Set $\mathcal{K}_\varepsilon = \{u \in \mathcal{A}_\varepsilon : \underline{\varphi}_\varepsilon(x) \le u(x) \le \overline{\varphi}_\varepsilon(x)$ for $x \in \overline{\mathbb{R}}_+$, u satisfies (18)$\}$ and define $T_\varepsilon : \mathcal{K}_\varepsilon \to C^0(\overline{\mathbb{R}}_+)$ by

$$(T_\varepsilon u)(x) = W_\varepsilon^{-1}\Big(\int_0^x \frac{1}{q(s)} \int_0^s F(t,u(t),u'(t))\,dt\,ds\Big), \quad u \in \mathcal{K}_\varepsilon,\ \varepsilon \in \{+,-\}.$$

LEMMA 3. *$T_\varepsilon : \mathcal{K}_\varepsilon \to \mathcal{K}_\varepsilon$ for each $\varepsilon \in \{+,-\}$.*

Proof. We prove $T_+ : \mathcal{K}_+ \to \mathcal{K}_+$ (the proof of $T_- : \mathcal{K}_- \to \mathcal{K}_-$ is very similar and will be omitted). Let $u \in \mathcal{K}_+$. Setting

$$\alpha(x) = \int_0^x \frac{1}{q(s)} \int_0^s F(t,u(t),u'(t))\,dt\,ds - W_+(\underline{\varphi}_+(x)),$$
$$\beta(x) = \int_0^x \frac{1}{q(s)} \int_0^s F(t,u(t),u'(t))\,dt\,ds - W_+(\overline{\varphi}_+(x))$$

for $x \in \mathbb{R}_+$, we have

$$\begin{aligned}
\alpha'(x) &= \frac{1}{q(x)} \int_0^x F(t,u(t),u'(t))\,dt - k(\underline{\varphi}_+(x))\underline{\varphi}_+'(x)\\
&= \frac{1}{q(x)}\Big[\int_0^x F(t,u(t),u'(t))\,dt - f_1(x)H_1(\underline{\varphi}_+(x))\Big]\\
&\ge \frac{1}{q(x)}\Big[\int_0^x f_1(t)h_1(u(t))u'(t)\,dt - f_1(x)H_1(\underline{\varphi}_+(x))\Big]\\
&\ge \frac{1}{q(x)}\Big[f_1(x)\int_0^{u(x)} h_1(s)\,ds - f_1(x)\int_0^{\underline{\varphi}_+(x)} h_1(s)\,ds\Big]\\
&= \frac{f_1(x)}{q(x)} \int_{\underline{\varphi}_+(x)}^{u(x)} h_1(s)\,ds \ge 0,\\
\beta'(x) &= \frac{1}{q(x)} \int_0^x F(t,u(t),u'(t))\,dt - k(\overline{\varphi}_+(x))\overline{\varphi}_+'(x)\\
&\le -\frac{f_2(0)}{q(x)} \int_{u(x)}^{\overline{\varphi}_+(x)} h_2(s)\,ds \le 0
\end{aligned}$$

for $x \in \mathbb{R}_+$. Since $\alpha(0) = \beta(0) = 0$, we have $\alpha(x) \ge 0$ and $\beta(x) \le 0$ for $x \in \mathbb{R}_+$, and consequently,

$$\underline{\varphi}_+(x) \le (T_+u)(x) \le \overline{\varphi}_+(x), \quad x \in \overline{\mathbb{R}}_+. \tag{20}$$

Let $0 < x_1 < x_2$. Then

$$\begin{aligned}
&W_+((T_+u)(x_2)) - W_+((T_+u)(x_1))\\
&\quad = \int_{x_1}^{x_2} \frac{1}{q(x)} \int_0^x F(s,u(s),u'(s))\,ds\,dx\\
&\quad \ge \int_{x_1}^{x_2} \frac{1}{q(x)} \int_0^x f_1(s)h_1(u(s))u'(s)\,ds\\
&\quad \ge \int_{x_1}^{x_2} \frac{f_1(x)}{q(x)} \int_0^{u(x)} h_1(s)\,ds\,dx \ge \int_{x_1}^{x_2} \frac{f_1(x)}{q(x)}\,dx \int_0^{u(x_1)} h_1(s)\,ds\\
&\quad \ge (k_1(x_2)-k_1(x_1)) \int_0^{\underline{\varphi}_+(x_1)} h_1(s)\,ds = H_1(\underline{\varphi}_+(x_1))(k_1(x_2)-k_1(x_1))
\end{aligned}$$

and

$$\begin{aligned}
W_+((T_+u)(x_2)) - W_+((T_+u)(x_1)) &= k(\xi)[(T_+u)(x_2)-(T_+u)(x_1)]\\
&\le [(T_+u)(x_2)-(T_+u)(x_1)]\max\{k(u) : \underline{\varphi}_+(x_1) \le u \le \overline{\varphi}_+(x_2)\}
\end{aligned}$$

(here $\xi \in ((T_+u)(x_1),(T_+u)(x_2)) \subset (\underline{\varphi}_+(x_1),\overline{\varphi}_+(x_2))$), thus

$$\begin{aligned}
(21)\qquad &(T_+u)(x_2)-(T_+u)(x_1)\\
&\ge H_1(\underline{\varphi}_+(x_1))(k_1(x_2)-k_1(x_1))[\max\{k(u) : \underline{\varphi}_+(x_1) \le u \le \overline{\varphi}_+(x_2)\}]^{-1}.
\end{aligned}$$

From (20) and (21) it follows that $T_+u \in \mathcal{K}_+$, therefore, $T_+ : \mathcal{K}_+ \to \mathcal{K}_+$.

THEOREM 2. *Let assumptions* (H_1)–(H_4) *be satisfied. Then a solution* $u \in \mathcal{A}_\varepsilon$ *of* (13) *exists for each* $\varepsilon \in \{+,-\}$.

Proof. By Lemma 2, $u \in \mathcal{A}_\varepsilon$ is a solution of (13) if and only if u is a fixed point of the operator T_ε. We shall prove that under assumptions (H_1)–(H_4) a fixed point of T_+ exists. The existence of a fixed point of T_- can be proved similarly.

Let X be the Fréchet space of C^0-functions on $\overline{\mathbb{R}}_+$ with the topology of uniform convergence on compact subintervals of $\overline{\mathbb{R}}_+$. Then $\mathcal{K}_+$ is a bounded closed convex subset of X and $T_+ : \mathcal{K}_+ \to \mathcal{K}_+$ (see Lemma 3) is a continuous operator. It follows from the inequalities $(0 \le x_1 < x_2)$

$$\begin{aligned}
0 \le W_+((T_+u)(x_2)) - W_+((T_+u)(x_1)) &= \int_{x_1}^{x_2} \frac{1}{q(x)} \int_0^x F(s,u(s),u'(s))\,ds\,dx\\
\le \int_{x_1}^{x_2} \frac{1}{q(x)} \int_0^x f_2(s)h_2(u(s))u'(s)\,ds\,dx &\le f_2(0)H_2(\overline{\varphi}_+(x_2)) \int_{x_1}^{x_2} \frac{1}{q(x)}\,dx
\end{aligned}$$

and from the Arzelà–Ascoli theorem [3] that $T_+(\mathcal{K}_+)$ is a relatively compact subset of X. According to the Tikhonov–Schauder fixed point theorem [6] there exists a fixed point u_+ of T_+.

4. Boundedness

THEOREM 3. *Let assumptions* (H_1)–(H_4) *be satisfied. Then any nontrivial solution of* (13) *on* $\overline{\mathbb{R}}_+$ *is bounded if and only if* $\int_0^\infty (1/q(s))\,ds < \infty$.

P r o o f. We prove this for $\varepsilon = +$ (the case $\varepsilon = -$ is similar).

Sufficiency. If $\int_0^\infty (1/q(x))\,dx < \infty$ then any solution of (13) is bounded by Lemma 2.

Necessity. Let $\int_0^\infty (1/q(x))\,dx = \infty$ and $u \in \mathcal{A}_+$ be a solution of (13). Then $u \neq 0$ and

$$W_+(u(x)) = \int_0^x \frac{1}{q(s)} \int_0^s F(t,u(t),u'(t))\,dt \geq \int_0^x \frac{1}{q(s)} \int_0^s f_1(t)h_1(u(t))u'(t)\,dt.$$

Since

$$\Big(\int_0^s f_1(t)h_1(u(t))u'(t)\,dt\Big)' = f_1(s)h_1(u(s))u'(s) > 0$$

in $\mathbb{R}_+$ and we have $\int_0^s f_1(t)h_1(u(t))u'(t)\,dt > 0$ in $\mathbb{R}_+$, it follows that $\lim_{x\to\infty} W_+(u(x)) = \infty$. So $\lim_{x\to\infty} u(x) = \infty$.

5. Uniqueness

THEOREM 4. *Let assumptions* (H_1)–(H_4) *be satisfied and suppose that for* $0 \leq x_1 < x_2$ *and* $u_2(x) > u_1(x)$,

$$(\mathrm{H}_5)\qquad \begin{aligned} &\int_{x_1}^{x_2} [F(s,u_2(s),u_2'(s)) - F(s,u_1(s),u_1'(s))]\,ds > 0, \qquad u_i \in \mathcal{A}_+,\\ &\int_{x_1}^{x_2} [F(s,u_2(s),u_2'(s)) - F(s,u_1(s),u_1'(s))]\,ds < 0, \qquad u_i \in \mathcal{A}_-. \end{aligned}$$

Then there exist solutions $u_\varepsilon, v_\varepsilon \in \mathcal{A}_\varepsilon$ *of* (13) *for each* $\varepsilon \in \{+,-\}$ *such that* $u_\varepsilon(x) \leq v_\varepsilon(x)$ *for* $x \in \overline{\mathbb{R}}_+$. *Moreover*,

$$u_\varepsilon(x) \leq u(x) \leq v_\varepsilon(x), \qquad x \in \overline{\mathbb{R}}_+, \tag{22}$$

for any solution $u \in \mathcal{A}_\varepsilon$ *of* (13) *and*

$$u(x) \neq v(x), \qquad x > 0, \tag{23}$$

for any two different solutions u, v *of* (13).

Proof. Let $u \in \mathcal{A}_+$ be a solution of (13). Define sequences $\{u_n\} \subset \mathcal{A}_+$ and $\{v_n\} \subset \mathcal{A}_+$ by the recurrence formulas

$$u_0 = \underline{\varphi}_+, \quad u_{n+1} = T_+(u_n), \quad v_0 = \overline{\varphi}_+, \quad v_{n+1} = T_+(v_n) \tag{24}$$

for $n \in \mathbb{N}$. Then $u_0(x) \le u(x) \le v_0(x)$ in $\overline{\mathbb{R}}_+$ by Lemma 2 and $u_0(x) \le u_1(x) \le v_0(x)$, $u_0(x) \le v_1(x) \le v_0(x)$ in $\overline{\mathbb{R}}_+$ by Lemma 3. Since $\alpha_1, \alpha_2 \in \mathcal{A}_+$, $\underline{\varphi}_+(x) \le \alpha_1(x) \le \alpha_2(x) \le \overline{\varphi}_+(x)$ for $x \in \overline{\mathbb{R}}_+$ implies

$$\begin{aligned}
(T_+\alpha_2)(x) &- (T_+\alpha_1)(x) \\
&= W_+^{-1}\Big(\int_0^x \frac{1}{q(s)} \int_0^s F(t, \alpha_2(t), \alpha_2'(t))\,dt\,ds\Big) \\
&\qquad\qquad - W_+^{-1}\Big(\int_0^x \frac{1}{q(s)} \int_0^s F(t, \alpha_1(t), \alpha_1'(t))\,dt\,ds\Big) \\
&= \frac{1}{k(\xi)} \int_0^x \frac{1}{q(s)} \int_0^s [F(t, \alpha_2(t), \alpha_2'(t)) - F(t, \alpha_1(t), \alpha_1'(t))]\,dt\,ds > 0,
\end{aligned}$$

where $\xi \in (\underline{\varphi}_+(x_1), \overline{\varphi}_+(x_2))$ and $T_+ : \mathcal{K}_+ \to \mathcal{K}_+$ by Lemma 3, we have

$$\begin{aligned}
u_0(x) \le u_1(x) \le \ldots \le u_n(x) \le \ldots \le u(x) \le \ldots& \\
\ldots \le v_n(x) \le \ldots \le v_1(x) \le v_0(x)&
\end{aligned}$$

for $x \in \overline{\mathbb{R}}_+$ and $n \in \mathbb{N}$. Therefore, the two limits $\lim_{n\to\infty} u_n(x) = u_+(x)$ and $\lim_{n\to\infty} v_n(x) = v_+(x)$ exist for all $x \ge 0$. We have $u_+(x) \le u(x) \le v_+(x)$ on $\overline{\mathbb{R}}_+$ and using the Lebesgue dominated convergence theorem [6] we see that $u_+, v_+ \in \mathcal{K}_+$ are solutions of (16), and thus also solutions of (13) by Theorem 1. Let $u, v \in \mathcal{A}_+$ be different solutions of (13). First, suppose that there exists a $x_0 > 0$ such that $u(x) < v(x)$ for $x \in (0, x_0)$ and $u(x_0) = v(x_0)$. Then

$$\begin{aligned}
0 &= W_+(v(x_0)) - W_+(u(x_0)) \\
&= \int_0^{x_0} \frac{1}{q(s)} \int_0^s [F(t, v(t), v'(t)) - F(t, u(t), u'(t))]\,dt\,ds.
\end{aligned}$$

On the other hand, by (H_5),

$$\int_0^{x_0} \frac{1}{q(s)} \int_0^s [F(t, v(t), v'(t)) - F(t, u(t), u'(t))]\,dt\,ds > 0,$$

a contradiction.

Now, assume that there exist $0 < x_1 < x_2$ such that $u(x_1) = v(x_1)$, $u(x_2) = v(x_2)$ and $u(x) \ne v(x)$ for $x \in (x_1, x_2)$; without loss of generality, let $u(x) < v(x)$ for $x \in (x_1, x_2)$. Then $u'(x_1) \le v'(x_1)$, $u'(x_2) \ge v'(x_2)$ and

$$0 \ge q(x_2)k(u(x_2))(v'(x_2) - u'(x_2)) - q(x_1)k(u(x_1))(v'(x_1) - u'(x_1)) = \int_{x_1}^{x_2} [F(s, v(s), v'(s)) - F(s, u(s), u'(s))]\, ds,$$

contrary to (H_5). So, the proof is complete.

THEOREM 5. *Let assumptions* (H_1)–(H_4) *be satisfied. Moreover*, *assume that*

(H_6) (i) *there exist* $\varepsilon_0, \varepsilon > 0$ *such that*

$$\Big| \int_0^u [F(w_1(s), s, 1/w_1'(s))w_1'(s) - F(w_2(s), s, 1/w_2'(s))w_2'(s)]\, ds\Big| \le L|w_1(u) - w_2(u)| \min\{|H_1(u)|, |H_2(u)|\}$$

for $(x, u_i) \in [0, \varepsilon] \times [-\varepsilon_0, \varepsilon_0]$ $(i = 1, 2)$, *where* w_i *is the inverse function of* u_i, $u_i \in \mathcal{A}_\varepsilon$, *and* $L > 0$ *is a constant*;

(ii) *the modulus of continuity* $\gamma(X) = \sup\{|q(x_1) - q(x_2)| : x_1, x_2 \in [0, \varepsilon],\ |x_1 - x_2| \le X\}$ *of* q *on* $[0, \varepsilon]$ *satisfies* $\lim_{x\to 0+} \sup \gamma(x)/x < \infty$;

and

(H_7) *there exist two positive constants* K_0 *and* ε_0 *such that*

$$|F(w_2(t), t, 1/w_2'(t))w_2'(t) - F(w_1(t), t, 1/w_1'(t))w_1'(t)| \le K_0|w_2(t) - w_1(t)| \quad \textit{for } 0 < |t| < \varepsilon_0.$$

Then equation (13) *admits a unique solutions in* $\mathcal{A}_\varepsilon$, $\varepsilon = \{+, -\}$.

Proof. Assume $u_1, u_2 \in \mathcal{A}_+$ are solutions of (13) and assume $u_1 \ne u_2$. First, we prove $u_1(x) = u_2(x)$ on an interval $[0, a]$, $a > 0$. Setting $A_i = \lim_{x\to\infty} u_i(x)$, $i = 1, 2$, we see that $0 < A_i \le \infty$ and the $w_i : [0, A_i) \to \overline{\mathbb{R}}_+$ are continuous strictly increasing functions,

$$w_i'(u) = k(u)q(w_i(u))\Big[\int_0^{w_i(u)} F(s, u_i(s), u_i'(s))\, ds\Big]^{-1}, \quad u \in (0, A_i),\ i = 1, 2.$$

Hence,

$$w_i(u) = \int_0^u k(s)q(w_i(s))\Big[\int_0^{w_i(s)} F(t, u_i(t), u_i'(t))\, dt\Big]^{-1} ds, \quad u \in (0, A_i),\ i = 1, 2,$$

and thus for $u \in [0, \min(A_1, A_2)]$ we have

$$\begin{aligned}
&w_1(u) - w_2(u)\\
&= \int_0^u k(s)[q(w_1(s)) - q(w_2(s))]\Big[\int_0^{w_2(s)} F(t, u_2(t), u_2'(t))\,dt\Big]^{-1} ds\\
&\quad + \int_0^u k(s)q(w_1(s))\frac{\int_0^{w_2(s)} F(t, u_2(t), u_2'(t))\,dt - \int_0^{w_1(s)} F(t, u_1(t), u_1'(t))\,dt}{\int_0^{w_1(s)} F(t, u_1(t), u_1'(t))\,dt \int_0^{w_2(s)} F(t, u_2(t), u_2'(t))\,dt}\,ds\\
&\le \int_0^u k(s)[q(w_1(s)) - q(w_2(s))]\Big[\int_0^{w_2(s)} f_1(t)h_1(u_2(t))u_2'(t)\,dt\Big]^{-1} ds\\
&\quad + \int_0^u k(s)q(w_1(s))\\
&\quad \times \frac{\int_0^s [F(w_2(t), t, 1/w_2'(t))w_2'(t) - F(w_1(t), t, 1/w_1'(t))w_1'(t)]\,dt}{\int_0^{w_1(s)} f_1(t)h_1(u_1(t))u_1'(t)\,dt \int_0^{w_2(s)} f_1(t)h_1(u_2(t))u_2'(t)\,dt}\,ds.
\end{aligned}$$

Let $\varepsilon > 0$ be as in assumption (H_6) and set $a = \min\{u_1(\varepsilon), u_2(\varepsilon)\}$, $X(u) = \max\{|w_1(s) - w_2(s)| : 0 \le s \le u\}$ for $u \in [0, a]$. Suppose $X(u) > 0$ for $u \in (0, a]$. Then (cf. (H_6))

$$|q(w_1(u)) - q(w_2(u))| \le \gamma(X(u)), \qquad u \in [0, a].$$

In this way,

$$\begin{aligned}
&|w_1(u) - w_2(u)|\\
&\qquad \le \int_0^u \frac{k(s)\gamma(X(s))}{f_1(w_2(s))H_1(s)}\,ds + \int_0^u \frac{k(s)q(w_1(s))L|w_1(s) - w_2(s)|}{f_1(w_1(s))f_1(w_2(s))H_1(s)}\,ds\\
&\qquad \le \gamma(X(u))P_1(u)/f_1(\varepsilon)\\
&\qquad\quad + LX(u)P_1(u)\max\{q(x) : 0 \le x \le \varepsilon\}/f_1^2(\varepsilon), \qquad 0 \le u \le a.
\end{aligned}$$

Hence

$$X(u) \le (B\gamma(X(u)) + CX(u))P_1(u), \qquad u \in [0, a],$$

where $B = 1/f_1(\varepsilon)$, $C = B^2 L \max\{q(x) : 0 \le x \le \varepsilon\}$, and therefore

$$\gamma(X(u))P_1(u)/X(u) \ge (1 - CP_1(u))/B, \qquad u \in (0, a]. \tag{25}$$

Now, on the left-hand side of (25) (cf. (H_6)),

$$\lim_{u\to 0+}\gamma(X(u))P_1(u)/X(u)=0;$$

but, on the right-hand side of (25),

$$\lim_{u\to 0+}(1-CP_1(u))/B=1/B>0.$$

This is a contradiction.

Next, assume $[0,c]$ is the maximal interval where $u_1(x)=u_2(x)$. Define

$$\begin{gathered}Y(x)=\max\{|u_2(s)-u_1(s)| : c\le s\le x\},\\ \alpha(x)=\min\{u_1(x),u_2(x)\},\qquad \beta(x)=\max\{u_1(x),u_2(x)\}\end{gathered}$$

for $x\ge c$. Then $Y(c)=0$, $\alpha(c)=\beta(c)$, $0\le\beta(x)-\alpha(x)\le Y(x)$ and $Y(x)>0$ for $x>c$. We have

$$\begin{aligned}&W_+(u_2(x))-W_+(u_1(x))\\ &=\int_c^x\frac{1}{q(s)}\Big[\int_c^s[F(t,u_2(t),u_2'(t))-F(t,u_1(t),u_1'(t))]\,dt\Big]\,ds\\ &=\int_c^x\frac{1}{q(s)}\Big[\int_{u_2(c)}^{u_2(s)}F(w_2(t),t,1/w_2'(t))w_2'(t)\,dt\\ &\quad-\int_{u_1(c)}^{u_1(s)}F(w_1(t),t,1/w_1'(t))w_1'(t)\,dt\Big]\,ds\\ &=\int_c^x\frac{1}{q(s)}\Big\{\int_{u_1(c)}^{u_2(s)}[F(w_2(t),t,1/w_2'(t))w_2'(t)-F(w_1(t),t,1/w_1'(t))w_1'(t)]\,dt\\ &\quad+\int_{u_1(s)}^{u_2(s)}F(w_1(t),t,1/w_1'(t))w_1'(t)\,dt\Big\}\,ds\\ &\le\int_c^x\frac{1}{q(s)}\Big\{K_0\int_{u_1(c)}^{u_2(s)}|w_2(t)-w_1(t)|\,dt+\int_{\alpha(s)}^{\beta(s)}F(w_1(t),t,1/w_1'(t))w_1'(t)\,dt\Big\}\,ds.\end{aligned}$$

Set $\varepsilon=\min\{\varepsilon_0,\beta^{-1}(\varepsilon_0+u_1(c))-c\}$, $m=\min\{u_1'(x):c\le x\le\alpha^{-1}(\beta(c+\varepsilon))\}$, $M=\max\{F(w_1(t),t,1/w_1'(t))w_1'(t):\alpha(c)\le t\le\beta(c+\varepsilon)\}$ and $r=\max\{u_2'(x):c\le x\le c+\varepsilon\}$. For $x\in[c,c+\varepsilon]$ we have

$$\begin{aligned}|w_1(u_2(x))-x|&=|w_1(u_2(x))-w_1(u_1(x))|=w_1'(\xi)|u_2(x)-u_1(x)|\\ &=(1/u_1'(\eta))|u_2(x)-u_1(x)|\le Y(x)/m,\end{aligned}$$

where $\xi=(\alpha(x),\beta(x))$, $\eta=w_1(\xi)\in(w_1(\alpha(x)),w_1(\beta(x)))\subset[c,\alpha^{-1}(\beta(c+\varepsilon))]$. Consequently,

$$|w_1(u)-w_2(u)|\le Y(w_2(u))/m,\qquad u\in[u_1(c),u_2(c+\varepsilon)].$$

Therefore

$$\begin{aligned}|W_+(u_2(x))-W_+(u_1(x))| &\le \int\limits_c^x \frac{1}{q(s)}\Big\{r(K_0/m)\int\limits_c^s Y(t)\,dt+MY(s)\Big\}\,ds\\ &\le \int\limits_c^x \frac{1}{q(s)}[K_0r(s-c)/m+M]Y(s)\,ds\\ &\le (K_0r\varepsilon/m+M)Y(x)\int\limits_c^x \frac{1}{q(s)}\,ds\end{aligned}$$

for $x\in[c,c+\varepsilon]$. Since $|W_+(u_2(x))-W_+(u_1(x))| = k(\xi)|u_2(x)-u_1(x)|$, where $\xi\in(\alpha(x),\beta(x))\subset[\alpha(c),\beta(c+\varepsilon)]$, we have

$$|u_2(x)-u_1(x)|\le[(K_0r\varepsilon/m+M)Y(x)/p]\int\limits_c^x \frac{1}{q(s)}\,ds,$$

where $p=\min\{k(u):\alpha(c)\le u\le\beta(c+\varepsilon)\}$. Hence,

$$Y(x)\le[(K_0r\varepsilon/m+M)Y(x)/p]\int\limits_c^x \frac{1}{q(s)}\,ds,\qquad x\in[c,c+\varepsilon].$$

Then

$$1\le[(K_0r\varepsilon/m+M)/p]\int\limits_c^x \frac{1}{q(s)}\,ds,\qquad c\le x\le c+\varepsilon,$$

which is impossible. This proves $u_1(x)=u_2(x)$ for $x\in\overline{\mathbb{R}}_+$.

The uniqueness of solution of (13) in $\mathcal{A}_-$ can be treated analogously.

THEOREM 6. *Suppose that assumptions* (H_1)–(H_6) *are satisfied. Then* (13) *admits a unique solution in* $\mathcal{A}_\varepsilon$, $\varepsilon=\{+,-\}$.

Proof. It is sufficient to prove that under assumptions (H_1)–(H_6), $u_\varepsilon=v_\varepsilon$, $\varepsilon\in\{+,-\}$, where u_ε, v_ε are defined in Theorem 3. If not, for example, $u_+\ne v_+$, without loss of generality, let $u_+(x)<v_+(x)$ in $\mathbb{R}_+$ by Theorem 4. Since assumptions (H_1)–(H_4) and (H_6) imply (see the first part of the proof of Theorem 5) that $u_+(x)=v_+(x)$ on an interval $[0,b]$ ($b>0$), we have a contradiction.

6. Dependence of solution on a parameter. Consider the differential equation (14) depending on a positive parameter λ.

THEOREM 7. *Suppose that assumptions* (H_1)–(H_5) *are satisfied. Then for each* $\varepsilon \in \{+,-\}$ *there exist solutions* $u_\varepsilon(x,\lambda)$, $v_\varepsilon(x,\lambda)$ *of* (14) *such that*

$$u_\varepsilon(x,\lambda) \le u(x,\lambda) \le v_\varepsilon(x,\lambda), \quad x \in \overline{\mathbb{R}}_+, \tag{26}$$

for any solution $u(x,\lambda) \in \mathcal{A}_\varepsilon$ *of* (14) *and*

$$\begin{aligned} &u_+(x,\lambda_1) < u_+(x,\lambda_2), \quad v_+(x,\lambda_1) < v_+(x,\lambda_2),\\ &u_-(x,\lambda_1) > u_-(x,\lambda_2), \quad v_-(x,\lambda_1) > v_-(x,\lambda_2) \end{aligned} \tag{27}$$

for all $x \in \mathbb{R}_+$ *and* $0 < \lambda_1 < \lambda_2$.

Proof. The first part of the statement follows from Theorem 3. Set

$$\begin{aligned} &\underline{\varphi}_+(x,\lambda) = P_1^{-1}(\lambda k_1(x)), \quad \overline{\varphi}_+(x,\lambda) = P_2^{-1}(\lambda k_2(x)),\\ &\underline{\varphi}_-(x,\lambda) = V_1^{-1}(\lambda l_1(x)), \quad \overline{\varphi}_-(x,\lambda) = V_2^{-1}(\lambda l_2(x)) \end{aligned} \tag{28}$$

for $x \in \overline{\mathbb{R}}_+$, $\lambda > 0$. Since (14) can be rewritten in the form

$$(q(x)k(u)u'/\lambda)' = F(x,u,u'), \quad \lambda > 0,$$

we have (see Lemma 2)

$$\begin{aligned} u(x_2) - u(x_1) \ge \lambda H_1(\underline{\varphi}_+(x,\lambda))(k_1(x_2) - k_1(x_1))&\\ \times [\max\{k(u) : \underline{\varphi}_+(x_1,\lambda) \le u \le \overline{\varphi}_+(x_2,\lambda)\}]^{-1}& \end{aligned} \tag{29_+}$$

for any solution $u \in \mathcal{A}_+$ of (14) and $0 < x_1 < x_2$, and

$$\begin{aligned} u(x_1) - u(x_2) \ge \lambda T_2(\overline{\varphi}_-(x_1,\lambda))(l_2(x_1) - l_2(x_2))&\\ \times[\max\{-k(u) : \underline{\varphi}_-(x_2,\lambda) \le u \le \overline{\varphi}_-(x_1,\lambda)\}]^{-1}& \end{aligned} \tag{29_-}$$

for any solution $u \in \mathcal{A}_-$ of (14) and $0 < x_1 < x_2$.

Set $\mathcal{K}_{\lambda,\varepsilon} = \{u \in \mathcal{A}_\varepsilon : \underline{\varphi}_\varepsilon(x,\lambda) \le u(x) \le \overline{\varphi}_\varepsilon(x,\lambda),\ x \in \overline{\mathbb{R}}_+,\ u$ satisfies $(29_\varepsilon)\}$ and define $T_{\lambda,\varepsilon} : \mathcal{K}_{\lambda,\varepsilon} \to C^0(\overline{\mathbb{R}}_+)$ by

$$(T_{\lambda,\varepsilon}u)(x) = W_\varepsilon^{-1}\biggl(\lambda \int_0^x \frac{1}{q(s)} \int_0^s F(t,u(t),u'(t))\,dt\,ds\biggr),$$

where $\varepsilon \in \{+,-\}$, $\lambda > 0$. Then (cf. Lemma 3) $T_{\lambda,\varepsilon} : \mathcal{K}_{\lambda,\varepsilon} \to \mathcal{K}_{\lambda,\varepsilon}$. Next, set

$$\begin{aligned} &u^{(0)}_{\lambda,\varepsilon}(x) = \underline{\varphi}_\varepsilon(x,\lambda), \quad u^{(n+1)}_{\lambda,\varepsilon}(x) = (T_{\lambda,\varepsilon}u^{(n)}_{\lambda,\varepsilon})(x),\\ &v^{(0)}_{\lambda,\varepsilon}(x) = \overline{\varphi}_\varepsilon(x,\lambda), \quad v^{(n+1)}_{\lambda,\varepsilon}(x) = (T_{\lambda,\varepsilon}v^{(n)}_{\lambda,\varepsilon})(x) \end{aligned}$$

for $x \in \overline{\mathbb{R}}_+$, $\lambda > 0$ and $\varepsilon \in \{+,-\}$. Then the limits

$$\lim_{n\to\infty} u^{(n)}_{\lambda,\varepsilon}(x) = u_\varepsilon(x,\lambda), \quad \lim_{n\to\infty} v^{(n)}_{\lambda,\varepsilon}(x) = v_\varepsilon(x,\lambda)$$

exist for $x \in \overline{\mathbb{R}}_+$, $\lambda > 0$ and $\varepsilon \in \{+,-\}$.

Let $0 < \lambda_1 < \lambda_2$ and $\varepsilon = +$ (for $\varepsilon = -$, the proof is similar). Then $\underline{\varphi}_+(x,\lambda_1) < \underline{\varphi}_+(x,\lambda_2)$, $\overline{\varphi}_+(x,\lambda_1) < \overline{\varphi}_+(x,\lambda_2)$ and for each $\alpha_1,\alpha_2 \in \mathcal{A}_+$

with $\alpha_1(x) < \alpha_2(x)$ in $\mathbb{R}_+$ we have

$$\begin{aligned}
&(T_{\lambda_2,+}\alpha_2)(x) - (T_{\lambda_1,+}\alpha_1)(x)\\
&\quad = W_+^{-1}\Big(\lambda_2 \int_0^x \frac{1}{q(s)} \int_0^s F(t,\alpha_2(t),\alpha_2'(t))\,dt\,ds\Big)\\
&\qquad - W_+^{-1}\Big(\lambda_1 \int_0^x \frac{1}{q(s)} \int_0^s F(t,\alpha_1(t),\alpha_1'(t))\,dt\,ds\Big)\\
&\quad = \frac{1}{k'(\xi)} \int_0^x \frac{1}{q(s)} \int_0^s [\lambda_2 F(t,\alpha_2(t),\alpha_2'(t)) - \lambda_1 F(t,\alpha_1(t),\alpha_1'(t))]\,dt\,ds\\
&\quad \ge \frac{\lambda_1}{k'(\xi)} \int_0^x \frac{1}{q(s)} \int_0^s [F(t,\alpha_2(t),\alpha_2'(t)) - F(t,\alpha_1(t),\alpha_1'(t))]\,dt\,ds > 0
\end{aligned}$$

and therefore $u^{(n)}_{\lambda_1,+}(x) < u^{(n)}_{\lambda_2,+}(x)$ and $v^{(n)}_{\lambda_1,+}(x) < v^{(n)}_{\lambda_2,+}(x)$ for $x \in \mathbb{R}_+$, $n \in \mathbb{N}$. Hence

$$u_+(x,\lambda_1) \le u_+(x,\lambda_2), \quad v_+(x,\lambda_1) \le v_+(x,\lambda_2), \quad x \in \overline{\mathbb{R}}_+.$$

If $r(x_0,\lambda_1) = r(x_0,\lambda_2)$ for an $x_0 > 0$, where r is either u_+ or v_+, then $(r_i(x) = r(x,\lambda_i),\ i = 1,2)$

$$\begin{aligned}
r_1(x_0) &= W_+^{-1}\Big(\lambda_1 \int_0^{x_0} \frac{1}{q(s)} \int_0^s F(t,r_1(t),r_1'(t))\,dt\,ds\Big)\\
&< W_+^{-1}\Big(\lambda_2 \int_0^{x_0} \frac{1}{q(s)} \int_0^s F(t,r_2(t),r_2'(t))\,dt\,ds\Big) = r_2(x_0),
\end{aligned}$$

which is a contradiction. So $u_+(x,\lambda_1) < u(x,\lambda_2)$ and $v_+(x,\lambda_1) < v_+(x,\lambda_2)$ for $x \in \mathbb{R}_+$.

THEOREM 8. *Let* $\int_0^\infty (1/q(s))\,ds < \infty$ *and assumptions* (H_1)–(H_6) *be satisfied. Then for* $a \in \mathbb{R} - \{0\}$, *there exists a unique* $\lambda_0 > 0$ *such that* (14) *has a* (*necessarily unique*) *solution* $u(x,\lambda_0)$ *with* $\lim_{x\to\infty} u(x,\lambda_0) = a$.

Proof. By Theorem 6, (14) has a unique solution $u_+(x,\lambda) \in \mathcal{A}_+$ and a unique solution $u_-(x,\lambda) \in \mathcal{A}_-$ for each $\lambda > 0$ and the two finite limits $\lim_{x\to\infty} u_+(x,\lambda)$ (> 0) and $\lim_{x\to\infty} u_-(x,\lambda)$ (< 0) exist by Theorem 4. Define

$$g_+(\lambda) = \lim_{x\to\infty} u_+(x,\lambda), \qquad g_-(\lambda) = \lim_{x\to\infty} u_-(x,\lambda)$$

for $\lambda > 0$. Then $g_+ : (0,\infty) \to (0,\infty)$ and $g_- : (0,\infty) \to (-\infty,0)$. In view of Theorem 7, g_+ is increasing on $(0,\infty)$ and g_- is decreasing on $(0,\infty)$. If for example, $g_+(\lambda_1) = g_+(\lambda_2)$ for some $0 < \lambda_1 < \lambda_2$, then setting

$r_i(x) = u_+(x, \lambda_i)$ for $x \in \overline{\mathbb{R}}_+$ we have $r_1(x) < r_2(x)$ in $\mathbb{R}_+$, hence

$$\begin{aligned} g_+(\lambda_1) &= W_+^{-1}\bigg(\lambda_1 \int_0^\infty \frac{1}{q(s)} \int_0^s F(t, r_1(t), r_1'(t))\, dt\, ds\bigg) \\ &< W_+^{-1}\bigg(\lambda_2 \int_0^\infty \frac{1}{q(s)} \int_0^s F(t, r_2(t), r_2'(t))\, dt\, ds\bigg) = g_+(\lambda_2). \end{aligned}$$

This is a contradiction. Consequently, g_+ is strictly increasing and g_- is strictly decreasing.

To prove our theorem, it is enough to show that g_+ and g_- map $(0, \infty)$ onto $(0, \infty)$ and $(-\infty, 0)$, respectively. We prove, for example, that g_+ maps $(0, \infty)$ onto itself. First, from $\underline{\varphi}_+(x, \lambda) \le u_+(x, \lambda) \le \overline{\varphi}_+(x, \lambda)$ we see that $\lim_{\lambda \to 0+} g_+(\lambda) = 0$ and $\lim_{\lambda \to \infty} g_+(\lambda) = \infty$. Next, assume, on the contrary, that $\lim_{\lambda \to \lambda_{0-}} g_+(\lambda) < \lim_{\lambda \to \lambda_{0+}} g_+(\lambda)$ for $\lambda_0 > 0$. Setting $v_1(x) = \lim_{\lambda \to \lambda_{0-}} u_+(x, \lambda)$ and $v_2(x) = \lim_{\lambda \to \lambda_{0+}} u_+(x, \lambda)$ for $x \ge 0$, we get $v_1 \ne v_2$. Using the Lebesgue dominated convergence theorem as $\lambda \to \lambda_{0-}$ and $\lambda \to \lambda_{0+}$ in the equality $(r_\lambda(x) = u_+(x, \lambda)$ for $(x, \lambda) \in \mathbb{R}_+ \times (0, \infty))$

$$r_\lambda(x) = W_+^{-1}\bigg(\lambda_0 \int_0^x \frac{1}{q(s)} \int_0^s F(t, r_\lambda(t), r_\lambda'(t))\, dt\, ds\bigg),$$

we see that

$$v_i(x) = W_+^{-1}\bigg(\lambda_0 \int_0^x \frac{1}{q(s)} \int_0^s F(t, v_i(t), v_i'(t))\, dt\, ds\bigg), \qquad x > 0,\ i = 1, 2.$$

Therefore v_1 and v_2 are solutions of (14) for $\lambda = \lambda_0$, and consequently $v_1 = v_2$. This is a contradiction.

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