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Weak and strong topologies and integral equations in Banach spaces

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Abstract. The Schauder–Tikhonov theorem in locally convex topological spaces and an extension of Krasnosel'skiĭ's fixed point theorem due to Nashed and Wong are used to establish existence of L^α and C solutions to Volterra and Hammerstein integral equations in Banach spaces.

1. Introduction. This paper establishes existence of solutions to the Volterra integral equation

$$y(t) = h(t) + \int_0^t k(t,s) f(s, y(s))\, ds \quad \text{a.e. on } [0,T], \quad T > 0 \text{ is fixed,} \tag{1.1}$$

and the Hammerstein integral equation

$$y(t) = h(t) + \int_0^1 k(t,s) f(s, y(s))\, ds \quad \text{a.e. on } [0,1]. \tag{1.2}$$

Here y takes values in a real Banach space B.

In Section 2 existence of $L^\alpha([0,a],B)$ (with $\alpha > 1$, $a = T$ or 1) solutions will be established for (1.1) and (1.2) where B is a reflexive Banach space. In [6], C. Corduneanu first studied the Volterra equation in this setting. Our results extend and complement those in [6]. Also, our technique discusses naturally the interval of existence $[0,T]$. The method also extends so that we can examine the Hammerstein equation in the above setting. Throughout this section our analysis will rely on the Schauder–Tikhonov fixed point theorem in locally convex spaces.

Section 3 establishes existence of $C([0,a],B)$ solutions to (1.1) and (1.2); here B will be a real Banach space. We will assume that f has the splitting $f(t,u) = f_1(t,u) + f_2(t,u)$ where f_1 is a nonlinear contraction (to be

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described later) on bounded sets and f_2 is completely continuous. The technique used will rely on an extension of Krasnosel'skiĭ's fixed point theorem [10] due to Nashed and Wong [16].

Some very interesting existence results for (1.1) and (1.2), in the case $B = \mathbb{R}$, may be found in [3–5, 13, 14]. For example, in [14] the Hammerstein equation (1.2), with $B = \mathbb{R}$, is examined and existence of $C[0,1]$ solutions is established if the nonlinearity f satisfies a "sublinear" type growth condition. The Volterra equation (1.1), with $B = \mathbb{R}$, is discussed in [13]. Gripenberg, Londen and Staffans' basic idea is to show (1.1) has a (local) solution. They then discuss "continuation" of solutions. However, the interval of existence from a construction point of view is only briefly discussed.

For the remainder of this section we gather together some preliminaries that will be needed in Sections 2 and 3. Let (Ω, Σ, μ) be a finite measure space. A Banach space B has the *Radon–Nikodym* (*R–N*) *property* with respect to (Ω, Σ, μ) if for each μ-continuous vector measure $\nu : \Sigma \to B$ of bounded variation there exists $g \in L^1(\mu, B)$ such that $\nu(E) = \int_E g\, d\mu$ for all $E \in \Sigma$.

THEOREM 1.1 [9]. *If B is a reflexive Banach space then B has the R–N property.*

THEOREM 1.2 [2]. *Let (Ω, Σ, μ) be a finite measure space. Suppose $K \subseteq L^\alpha(\mu, B)$, $1 < \alpha < \infty$, is bounded with $K(A) = \{\int_A g\, d\mu : g \in K\}$ relatively weakly compact in B for each $A \in \Sigma$. If B and B^* have the R–N property then K is relatively weakly compact.*

THEOREM 1.3 [9]. *Let (Ω, Σ, μ) be a finite measure space*, $1 < \alpha < \infty$, *and B a Banach space. Then $(L^\alpha(\mu, B))^* = L^\beta(\mu, B^*)$ where $1/\alpha + 1/\beta = 1$ iff B^* has the R–N property with respect to μ.*

Remark. In fact, for $\phi \in (L^\alpha(\mu, B))^*$ there exists $g \in L^\beta(\mu, B^*)$ with

$$\phi(f) = \int_\Omega \langle f, g\rangle\, d\mu \quad \text{for all } f \in L^\alpha(\mu, B).$$

Here $\langle f, g\rangle(t) = g(t)(f(t))$ for $t \in \Omega$.

THEOREM 1.4 [7, 11, 17]. *A subset of a reflexive Banach space is weakly compact iff it is closed in the weak topology and bounded in the norm topology.*

THEOREM 1.5 [7, 11, 17]. *A convex subset of a normed space is closed iff it is weakly closed.*

THEOREM 1.6 (Schauder–Tikhonov) [3]. *Let K be a closed convex subset of a locally convex topological Hausdorff space E. Assume that $g : K \to K$*

is continuous and that $g(K)$ is relatively compact in E. Then g has at least one fixed point in K.

THEOREM 1.7 [17]. *Let B_1, B_2 be Banach spaces and $u : [a,b] \to B_1$ be Bochner integrable. If $\Gamma : B_1 \to B_2$ is a bounded linear operator then $\Gamma u : [a,b] \to B_2$ is integrable and $\int_E \Gamma u(t)\,dt = \Gamma \int_E u(t)\,dt$ for each measurable $E \subseteq [a,b]$.*

An operator T_1 is a *nonlinear contraction* on B (a Banach space) into B if for all $y_1, y_2 \in B$ we have

$$\|T_1(y_1) - T_1(y_2)\| \le \phi(\|y_1 - y_2\|)$$

where ϕ is a real-valued continuous function satisfying $\phi(x) < x$ for $x > 0$.

THEOREM 1.8 (Krasnosel'skiĭ–Nashed–Wong) [16]. *Let $C \subseteq B$ (a Banach space) be a closed convex subset and T_1, T_2 be operators on B with $T_1(x) + T_2(y) \in C$ for all $x, y \in C$. Suppose that*

(i) *$T_2 : B \to B$ is continuous and compact ($T_2(B)$ is relatively compact),*
(ii) *$T_1 : B \to B$ is a nonlinear contraction.*

Then there exists $y \in C$ with $T_1(y) + T_2(y) = y$.

Remark. If $T_2 = 0$ in Theorem 1.8 then in fact there exists a unique (cf. [1]) $y \in C$ with $T_1(y) = y$.

THEOREM 1.9 (Arzelà–Ascoli) [15]. *Let B be a Banach space. A subset M of $C([a,b], B)$ is relatively compact iff M is bounded, equicontinuous and the set $\{u(t) : u \in M\}$ is relatively compact in B for each $t \in [a,b]$.*

2. Solutions in L^α, $\alpha > 1$. Throughout this section B will be a reflexive Banach space. We begin by first examining the Hammerstein integral equation

$$y(t) = h(t) + \int_0^1 k(t,s) f(s, y(s))\,ds \quad \text{a.e. on } [0,1]. \tag{2.1}$$

THEOREM 2.1. *Suppose $1 < \alpha < \infty$ and β is the conjugate of α. Let $f : [0,1] \times B \to B$ and $Fu(t) = f(t, u(t))$. Assume that*

(2.2) $h \in L^\alpha([0,1], B)$,
(2.3) $k : [0,1] \times [0,1] \to \mathbb{R}$ *with $(t,s) \to k(t,s)$ measurable and $\int_0^1 \int_0^1 |k(t,s)|^\alpha\,ds\,dt < \infty$,*
(2.4) $F : L^\alpha([0,1], B) \to L^\beta([0,1], B)$ *is weakly continuous,*
(2.5) *there exists a nondecreasing continuous function $\psi : [0,\infty) \to [0,\infty)$ with $\int_0^1 \|f(s,u(s))\|^\beta\,ds \le \psi(\int_0^1 \|u(s)\|^\alpha\,ds)$ for any $u \in L^\alpha([0,1], B)$,*

$$2^{\alpha-1}\Big(\int_0^1\int_0^1 |k(t,s)|^\alpha\,ds\,dt\Big)\limsup_{x\to\infty}\frac{\psi^{\alpha/\beta}(x)}{x}<1. \tag{2.6}$$

Then (2.1) *has a solution* $y\in L^\alpha([0,1],B)$.

Remark. As an example of how to apply Theorem 2.1 let $\alpha=\beta=2$, and let $0\neq b_0\in B$ be fixed. Also suppose $f(t,u)=b_0+u$ and

$$2\Big(\int_0^1\int_0^1 |k(t,s)|^\alpha\,ds\,dt\Big)<1.$$

Now (2.5) is satisfied with $\psi(x)=\|b_0\|^2+2\|b_0\|\sqrt{x}+x$ since

$$\begin{aligned}\int_0^1 \|f(s,u(s))\|^2\,ds &\le \int_0^1 (\|b_0\|^2+2\|b_0\|\|u(s)\|+\|u(s)\|^2)\,ds\\ &\le \|b_0\|^2+2\|b_0\|\Big(\int_0^1\|u(s)\|^2\,ds\Big)^{1/2}+\int_0^1\|u(s)\|^2\,ds\\ &=\psi\Big(\int_0^1\|u(s)\|^2\,ds\Big)\quad\text{for any } u\in L^2([0,1],B).\end{aligned}$$

In addition, (2.4) is true since if $y_n\rightharpoonup y$ in $L^2([0,1],B)$ then $f(t,y_n)=b_0+y_n\rightharpoonup b_0+y=f(t,y)$ in $L^2([0,1],B)$. Here $\rightharpoonup$ denotes weak convergence. Finally, (2.6) is satisfied with the above ψ and so (2.1) has a solution in $L^2([0,1],B)$.

Proof of Theorem 2.1. Consider the set S of real numbers $x\ge 0$ which satisfy the inequality

$$x\le 2^{\alpha-1}\int_0^1\|h(t)\|^\alpha\,dt+2^{\alpha-1}\Big(\int_0^1\int_0^1|k(t,s)|^\alpha\,ds\,dt\Big)\psi^{\alpha/\beta}(x).$$

Then S is bounded above, i.e. there exists a constant M_1 with

$$x\le M_1\quad\text{for all } x\in S. \tag{2.7}$$

If (2.7) were not true then there would exist a sequence $0\neq x_n\in S$ with $x_n\to\infty$ as $n\to\infty$ and

$$1\le\frac{2^{\alpha-1}\int_0^1\|h(t)\|^\alpha\,dt}{x_n}+2^{\alpha-1}\Big(\int_0^1\int_0^1|k(t,s)|^\alpha\,ds\,dt\Big)\frac{\psi^{\alpha/\beta}(x_n)}{x_n}.$$

Thus

$$1\le 2^{\alpha-1}\Big(\int_0^1\int_0^1|k(t,s)|^\alpha\,ds\,dt\Big)\limsup_{x_n\to\infty}\frac{\psi^{\alpha/\beta}(x_n)}{x_n},$$

which contradicts (2.6). Thus (2.7) is true. Choose $M_0 > M_1$. Then

$$2^{\alpha-1}\int_0^1 \|h(t)\|^\alpha\,dt + 2^{\alpha-1}\Big(\int_0^1\int_0^1 |k(t,s)|^\alpha\,ds\,dt\Big)\psi^{\alpha/\beta}(M_0) < M_0 \tag{2.8}$$

for otherwise $M_0 \in S$ and this would contradict (2.7).

Our strategy will be to apply the Schauder–Tikhonov theorem to $L^\alpha([0,1],B)$ endowed with the weak topology. Let

$$K = \Big\{y \in L^\alpha([0,1],B) : \int_0^1 \|y(s)\|^\alpha\,ds \le M_0\Big\}.$$

Now K is convex and norm closed. Hence K is weakly closed by Theorem 1.5. A solution to (2.1) will be a fixed point of the operator $N : L^\alpha([0,1],B) \to L^\alpha([0,1],B)$ defined by

$$Ny(t) = h(t) + \int_0^1 k(t,s)f(s,y(s))\,ds.$$

We *claim* that $N : K \to K$ is weakly continuous and $N(K)$ is relatively weakly compact in $L^\alpha([0,1],B)$. If this is true then the Schauder–Tikhonov theorem (Theorem 1.6) implies that N has a fixed point in K, i.e. (2.1) has a solution $y \in L^\alpha([0,1],B)$.

It remains to prove the claim. First we show $N : K \to K$. To see this notice that for a.e. $t \in [0,1]$ we have

$$\begin{aligned}\|Ny(t)\|^\alpha &\le 2^{\alpha-1}\|h(t)\|^\alpha + 2^{\alpha-1}\int_0^1 |k(t,s)|^\alpha\,ds\Big(\int_0^1 \|f(s,y(s))\|^\beta\,ds\Big)^{\alpha/\beta}\\ &\le 2^{\alpha-1}\|h(t)\|^\alpha + 2^{\alpha-1}\int_0^1 |k(t,s)|^\alpha\,ds\,\psi^{\alpha/\beta}\Big(\int_0^1 \|y(s)\|^\alpha\,ds\Big)\\ &\le 2^{\alpha-1}\|h(t)\|^\alpha + 2^{\alpha-1}\int_0^1 |k(t,s)|^\alpha\,ds\,\psi^{\alpha/\beta}(M_0)\end{aligned}$$

and so

$$\begin{aligned}\int_0^1 \|Ny(t)\|^\alpha\,dt &\le 2^{\alpha-1}\int_0^1 \|h(s)\|^\alpha\,ds\\ &\quad + 2^{\alpha-1}\psi^{\alpha/\beta}(M_0)\int_0^1\int_0^1 |k(t,s)|^\alpha\,ds\,dt < M_0\end{aligned}$$

from (2.8). Consequently, $N : K \to K$. Next we show $N(K)$ is relatively weakly compact in $L^\alpha([0,1],B)$. Clearly, since $N(K)\subseteq K$, we see that $N(K)$

is bounded in $L^\alpha([0,1],B)$. Notice as well that

$$N(K)(A)=\Big\{\int\limits_A g\,dt : g\in N(K)\Big\}$$

is relatively weakly compact in B for every subset A of $[0,1]$. This follows immediately from Theorem 1.4 and

$$\|(Ny)(A)\|\le\int\limits_0^1\|Ny(t)\|\,dt\le\Big(\int\limits_0^1\|Ny(t)\|^\alpha\,dt\Big)^{1/\alpha}\le M_0^{1/\alpha};$$

here $y\in K$ and A is any measurable subset of $[0,1]$. Thus $N(K)(A)$ is relatively weakly compact in B. This, together with Theorem 1.2 (due to Brooks and Dinculeanu), implies that $N(K)$ is relatively weakly compact in $L^\alpha([0,1],B)$. Finally, it remains to show that $N:L^\alpha([0,1],B)\to L^\alpha([0,1],B)$ is weakly continuous, i.e.

$$\text{if } y_n\rightharpoonup y \text{ in } L^\alpha([0,1],B) \text{ then } Ny_n\rightharpoonup Ny \text{ in } L^\alpha([0,1],B);$$

hence (y_n) is a net in $L^\alpha([0,1],B)$. Let $\phi\in(L^\alpha([0,1],B)^*$. Then there exists $g\in L^\beta([0,1],B^*)$ with (see Theorem 1.3)

$$\phi(Ny_n-Ny)=\int\limits_0^1 g(t)\Big(\int\limits_0^1 k(t,s)[f(s,y_n(s))-f(s,y(s))]\,ds\Big)\,dt.$$

Theorem 1.7 and changing the order of integration yield

$$\begin{aligned}\phi(Ny_n-Ny)&=\int\limits_0^1\int\limits_0^1 k(t,s)g(t)(f(s,y_n(s))-f(s,y(s)))\,ds\,dt\\&=\int\limits_0^1\int\limits_0^1 k(t,s)g(t)(f(s,y_n(s))-f(s,y(s)))\,dt\,ds\\&=\int\limits_0^1\Big(\int\limits_0^1 k(t,s)g(t)\,dt\Big)(f(s,y_n(s))-f(s,y(s)))\,ds\\&=\int\limits_0^1 g_1(s)(f(s,y_n(s))-f(s,y(s)))\,ds\end{aligned}$$

where $g_1(s)=\int_0^1 k(t,s)g(t)\,dt$. This, together with (2.4) and $g_1\in L^\alpha([0,1],B^*)$ (note (2.6) and $g\in L^\beta([0,1],B^*)$), implies that $N:L^\alpha([0,1],B)\to L^\alpha([0,1],B)$ is weakly continuous.

The Schauder–Tikhonov theorem guarantees that N has a fixed point in K. ∎

Essentially the same reasoning as in Theorem 2.1 immediately establishes an existence result for the Volterra integral equation

$$y(t) = h(t) + \int_0^t k(t,s) f(s, y(s))\, ds \quad \text{a.e. on } [0,T]. \tag{2.9}$$

THEOREM 2.2. *Suppose* $1 < \alpha < \infty$ *and* β *is the conjugate of* α. *Let* $f : [0,T] \times B \to B$ *where* B *is a reflexive Banach space and* $Fu(t) = f(t, u(t))$. *Assume that*

(2.10) $h \in L^\alpha([0,T], B)$,

(2.11) $k : [0,T] \times [0,T] \to \mathbb{R}$ *with* $(t,s) \to k(t,s)$ *measurable and* $\int_0^T \int_0^t |k(t,s)|^\alpha\, ds\, dt < \infty$,

(2.12) $F : L^\alpha([0,T], B) \to L^\beta([0,T], B)$ *is weakly continuous*,

(2.13) *there exists a nondecreasing continuous function* $\psi : [0,\infty) \to [0,\infty)$ *with* $\int_0^t \|f(s, u(s))\|^\beta\, ds \le \psi(\int_0^t \|u(s)\|^\alpha\, ds)$ *for* $t \in [0,T]$ *and any* $u \in L^\alpha([0,T], B)$,

$$2^{\alpha-1}\Big(\int_0^T \int_0^t |k(t,s)|^\alpha\, ds\, dt\Big) \limsup_{x\to\infty} \frac{\psi^{\alpha/\beta}(x)}{x} < 1. \tag{2.14}$$

Then (2.9) *has a solution* $y \in L^\alpha([0,T], B)$.

However, it is possible to improve this result.

THEOREM 2.3. *Let* $1 < \alpha < \infty$ *and* β *be the conjugate of* α. *Suppose* $f : [0,T] \times B \to B$ *and* $Fu(t) = f(t, u(t))$. *Assume that* (2.10)–(2.13) *hold. In addition, assume that*

$$2^{\alpha-1}\Big(\int_0^T \|h(s)\|^\alpha\, ds + \int_0^T \int_0^t |k(t,s)|^\alpha\, ds\, dt\Big) < \int_0^\infty \frac{du}{1 + \psi^{\alpha/\beta}(u)}. \tag{2.15}$$

Then (2.9) *has a solution* $y \in L^\alpha([0,T], B)$.

Proof. Let

$$I(z) = \int_0^z \frac{du}{1 + \psi^{\alpha/\beta}(u)}$$

and

$$a(t) = I^{-1}\Big(2^{\alpha-1} \int_0^t \|h(s)\|^\alpha\, ds + 2^{\alpha-1} \int_0^t \int_0^s |k(s,x)|^\alpha\, dx\, ds\Big). \tag{2.16}$$

Now let

$$K = \Big\{y \in L^\alpha([0,T], B) : \int_0^t \|y(s)\|^\alpha\, ds \le a(t)\Big\}.$$

The set K is convex and weakly closed. Also, a solution to (2.9) will be a fixed point of the operator $N : L^\alpha([0,T],B) \to L^\alpha([0,T],B)$ defined by

$$Ny(s) = h(s) + \int_0^s k(s,x) f(x,y(x))\,dx.$$

We *claim* that $N : K \to K$. To see this notice for a.e. $s \in [0,T]$ that

$$\begin{aligned}\|Ny(s)\|^\alpha &\le 2^{\alpha-1}\|h(s)\|^\alpha + 2^{\alpha-1}\int_0^s |k(s,x)|^\alpha\,dx\Big(\int_0^s \|f(x,y(x))\|^\beta\,dx\Big)^{\alpha/\beta}\\ &\le 2^{\alpha-1}\|h(s)\|^\alpha + 2^{\alpha-1}\int_0^s |k(s,x)|^\alpha\,dx\,\psi^{\alpha/\beta}\Big(\int_0^s \|y(x)\|^\alpha\,dx\Big)\\ &\le \Big(2^{\alpha-1}\|h(s)\|^\alpha + 2^{\alpha-1}\int_0^s |k(s,x)|^\alpha\,dx\Big)(1+\psi^{\alpha/\beta}(a(s))).\end{aligned}$$

Thus for $t \in [0,T]$ we have

$$\begin{aligned}&\int_0^t \|Ny(s)\|^\alpha\,ds\\ &\quad\le \int_0^t \Big(2^{\alpha-1}\|h(s)\|^\alpha + 2^{\alpha-1}\int_0^s |k(s,x)|^\alpha\,dx\Big)(1+\psi^{\alpha/\beta}(a(s)))\,ds\\ &\quad= \int_0^t a'(s)\,ds = a(t)\end{aligned}$$

since (2.16) implies

$$\int_0^{a(s)} \frac{du}{1+\psi^{\alpha/\beta}(u)} = 2^{\alpha-1}\Big(\int_0^s \|h(x)\|^\alpha\,dx + \int_0^s\int_0^z |k(z,x)|^\alpha\,dx\,dz\Big).$$

Consequently, $Ny \in K$ and so $N : K \to K$. Essentially the same reasoning as in Theorem 2.1 shows that $N(K)$ is relatively weakly compact in $L^\alpha([0,T],B)$ and $N : K \to K$ is weakly continuous. The Schauder–Tikhonov theorem now guarantees a fixed point of N in K. ∎

3. Solutions in C. Throughout this section, B will be a real Banach space. We consider first the Volterra integral equation

$$y(t) = h(t) + \int_0^t k(t,s) f(s,y(s))\,ds, \qquad t \in [0,T]. \tag{3.1}$$

We will assume that $f : [0,T] \times B \to B$ is a L^β-Carathéodory function; here $\beta \ge 1$. By this we mean that

(i) the map $t \to f(t,z)$ is measurable (Bochner) for all $z \in B$,

(ii) the map $z \to f(t,z)$ is continuous for almost all $t \in [0,T]$,

(iii) for each $r > 0$ there exists $\mu_r \in L^\beta([0,T],\mathbb{R})$ such that $\|z\| \le r$ implies $\|f(t,z)\| \le \mu_r(t)$ for almost all $t \in [0,T]$.

THEOREM 3.1. *Let $1 \le \alpha \le \infty$ and β be the conjugate of α. Suppose $f : [0,T] \times B \to B$ has the decomposition $f = f_1 + f_2$ where f_1 and f_2 are L^β-Carathéodory functions. Assume that*

(3.2) $h \in C([0,T],B)$,

(3.3) *$k(t,s) \in L^\alpha([0,T],\mathbb{R})$ for each $t \in [0,T]$ and the map $t \to k(t,s)$ is continuous from $[0,T]$ to $L^\alpha([0,T],\mathbb{R})$,*

(3.4) *there exists a nondecreasing continuous function $\Phi : [0,\infty) \to [0,\infty)$ with $\int_0^t \|k(t,s)f(s,u(s))\|\,ds \le \Phi(\int_0^t \|u(s)\|\,ds)$ for $t \in [0,T]$ and any $u \in C([0,T],B)$,*

(3.5) $$T < \int_0^\infty \frac{du}{\Phi(u)+h_0} \quad \textit{where } h_0 = \sup_{[0,T]} \|h(t)\|.$$

Let

$$J(z) = \int_0^z \frac{du}{\Phi(u)+h_0}$$

and notice that $J : [0,\infty) \to [0,\infty)$ is strictly increasing. Define

(3.6) $$M_1 = J^{-1}(T) \quad \textit{and} \quad M_0 = h_0 + \Phi(M_1).$$

In addition, suppose that

(3.7) *for each $t \in [0,T]$ the set $\{\int_0^t k(t,s)f_2(s,u(s))\,ds : u \in C([0,T],B)$ with $\|u(s)\| \le M_0$ for all $s \in [0,T]\}$ is relatively compact,*

and

(3.8) *there exists a continuous $Q : [0,T] \to [0,\infty)$ such that*

$$\sup_{[0,T]} \Big\| e^{-Q(t)} \int_0^t k(t,s)[f_1(s,u(s)) - f_1(s,v(s))]\,ds \Big\| \le \phi\big(\tfrac{1}{2} \sup_{[0,T]} e^{-Q(t)} \|u(t)-v(t)\|\big)$$

for all $u,v \in C([0,T],B)$ with $\|u(s)\|, \|v(s)\| \le M_0$ for all $s \in [0,T]$; here ϕ is a real-valued nondecreasing continuous function satisfying $\phi(x) < x$ for $x > 0$.

Then (3.1) *has a solution $y \in C([0,T],B)$.*

Remarks. (i) Let $k \equiv 1$ and suppose there exists $q \in L^1([0,T],\mathbb{R})$ with

$$\|f_1(t,u) - f_1(t,v)\| \le q(t)\|u-v\|$$

for a.e. $t \in [0,T]$ and all $u, v \in B$ with $\|u\| \le M_0$, $\|v\| \le M_0$. Then (3.8) is satisfied. To see this consider any $u, v \in C([0,T], B)$ with $\|u(s)\|, \|v(s)\| \le M_0$ for $s \in [0,T]$. With $Q(t) = 2\int_0^t q(s)\,ds$ we have

$$\begin{aligned}\sup_{[0,T]} \Big\| e^{-Q(t)} \int_0^t [f_1(s,u(s)) - f_1(s,v(s))]\,ds \Big\| \\ &\le \sup_{t\in[0,T]} e^{-Q(t)} \int_0^t e^{Q(s)} q(s) e^{-Q(s)} \|u(s) - v(s)\|\,ds \\ &\le \|u-v\|_Q \sup_{t\in[0,T]} e^{-Q(t)} \tfrac{1}{2}[e^{Q(t)} - 1] \\ &= \tfrac{1}{2}(1 - e^{-Q(T)})\|u-v\|_Q\end{aligned}$$

where $\|u-v\|_Q = \sup_{[0,T]} e^{-Q(t)}\|u(t) - v(t)\|$. Clearly (3.8) is satisfied with $\phi(x) = (1 - e^{-Q(T)})x$.

(ii) We can replace $\frac{1}{2}$ in (3.8) by 1 if $B = H$, a Hilbert space.

(iii) We can replace $e^{-Q(t)}$ in (3.8) with an arbitrary weight function $w(t)$.

(iv) If $f_2 = 0$ in Theorem 3.1 then in fact (3.1) has a *unique* solution $y \in C([0,T], B)$.

Proof of Theorem 3.1. Consider the modified Volterra equation

$$y(t) = h(t) + \int_0^t k(t,s)[f_1(s, r(y(s))) + f_2(s, r(y(s)))]\,ds, \quad t \in [0,T], \tag{3.9}$$

where $r : B \to \overline{B(0, M_0)} = \{y : \|y\| \le M_0\}$ defined by

$$r(u) = \begin{cases} u, & \|u\| \le M_0, \\ M_0 u/\|u\|, & \|u\| > M_0, \end{cases}$$

is the radial retraction; M_0 is as described in (3.6). Recall the radial retraction r is Lipschitz [8, 12] and in fact

$$\|r(u_1) - r(u_2)\| \le 2\|u_1 - u_2\| \quad \text{for all } u_1, u_2 \in B. \tag{3.10}$$

Remark. If $B = H$, a real Hilbert space, then in fact r is nonexpansive [10, 12].

Let us endow $C([0,T], B)$ with the norm

$$\|u\|_Q = \sup_{t\in[0,T]} e^{-Q(t)} \|u(t)\|. \tag{3.11}$$

A solution to (3.9) is a fixed point of the operator $S : C([0,T],B) \to C([0,T],B)$ defined by

$$Sy(t) = h(t) + \int_0^t k(t,s) f(s, r(y(s)))\, ds \equiv (T_1 y)(t) + (T_2 y)(t)$$

where

$$(T_1 y)(t) = h(t) + \int_0^t k(t,s) f_1(s, r(y(s)))\, ds,$$

$$(T_2 y)(t) = \int_0^t k(t,s) f_2(s, r(y(s)))\, ds.$$

Now $T_1 : C([0,T],B) \to C([0,T],B)$ is a nonlinear contraction since for $u, v \in C([0,T],B)$ we have, with $\|\cdot\|_Q$ as described in (3.11),

$$\begin{aligned}\|T_1(u) - T_1(v)\|_Q &= \sup_{[0,T]} \Big\| e^{-Q(t)} \int_0^t k(t,s)[f_1(s, r(u(s))) - f_1(s, r(v(s)))]\, ds \Big\| \\ &\le \phi\big(\tfrac{1}{2} \sup_{[0,T]} e^{-Q(t)} \|r(u(t)) - r(v(t))\|\big) \\ &\le \phi(\sup_{[0,T]} e^{-Q(t)} \|u(t) - v(t)\|) = \phi(\|u - v\|_Q),\end{aligned}$$

using (3.8), (3.10) and the fact that ϕ is nondecreasing.

Next we show that $T_2 : C([0,T],B) \to C([0,T],B)$ is continuous and compact. To see continuity let $y_n \to y$ in $C([0,T],B)$. Now $\|r(y_n(s))\| \le M_0$ and $\|r(y(s))\| \le M_0$ for all $s \in [0,T]$. Also, there exists $\mu \in L^\beta([0,T],\mathbb{R})$ with $\|f_2(t,u)\| \le \mu(t)$ for a.e. $t \in [0,T]$ and all $\|u\| \le M_0$. In addition, for each $t \in [0,T]$ we have

$$k(t,s) f_2(s, r(y_n(s))) \to k(t,s) f_2(s, r(y(s))) \quad \text{for a.e. } s \in [0,T]$$

and this, together with the Lebesgue dominated convergence theorem, implies $T_2 y_n(s) \to T_2 y(s)$ pointwise on $[0,T]$. Next we show the convergence is uniform and this of course implies $T_2 : C([0,T],B) \to C([0,T],B)$ is continuous. Let $t, t_1 \in [0,T]$ with $t_1 < t$. Then

$$\begin{aligned}&\|T_2 y_n(t) - T_2 y_n(t_1)\| \\ &\qquad \le \|h(t) - h(t_1)\| + \int_0^{t_1} |k(t,s) - k(t_1,s)|\, \|f(s, r(y_n(s)))\|\, ds \\ &\qquad\quad + \int_{t_1}^t |k(t,s)|\, \|f(s, r(y_n(s)))\|\, ds\end{aligned}$$

$$\leq \|h(t)-h(t_1)\| + \Big(\int\limits_0^T |k(t,s)-k(t_1,s)|^\alpha\,ds\Big)^{1/\alpha}\Big(\int\limits_0^T \mu^\beta(s)\,ds\Big)^{1/\beta}$$
$$+ \sup_{t\in[0,T]}\Big(\int\limits_0^T |k(t,s)|^\alpha\,ds\Big)^{1/\alpha}\Big(\int\limits_{t_1}^t \mu^\beta(s)\,ds\Big)^{1/\beta}.$$

A similar bound can be obtained for $\|T_2y(t)-T_2y(t_1)\|$. Thus for any $\varepsilon>0$ there exists $\delta>0$ such that $t,t_1\in[0,T]$ and $|t-t_1|<\delta$ imply

$$\|T_2y_n(t)-T_2y_n(t_1)\|<\varepsilon \quad \text{for all } n \quad \text{and} \quad \|T_2y(t)-T_2y(t_1)\|<\varepsilon. \tag{3.12}$$

Now (3.12), together with the fact that $T_2y_n(s)\to T_2y(s)$ pointwise on $[0,T]$, implies that the convergence is uniform. Consequently, $T_2: C([0,T],B)\to C([0,T],B)$ is continuous. In addition, the Arzelà–Ascoli theorem (Theorem 1.9), together with (3.7) and the ideas used to prove (3.12), implies that $T_2: C([0,T],B)\to C([0,T],B)$ is compact.

The Krasnosel'skiĭ–Nashed–Wong fixed point theorem guarantees a fixed point of S, i.e. (3.9) has a solution $y\in C([0,T],B)$. We now show that y is a solution of (3.1).

Remark. It is worth remarking here that (3.4) and (3.5) are only needed, so far, to define M_0; in fact, we have shown that (3.9) has a solution for any constant M_0.

Now for each $t\in(0,T)$,

$$\|y(t)\| \leq \|h(t)\| + \int\limits_0^t |k(t,s)|\,\|f(s,r(y(s)))\|\,ds$$
$$\leq \|h(t)\| + \Phi\Big(\int\limits_0^t \|r(y(x))\|\,dx\Big) \leq h_0 + \Phi\Big(\int\limits_0^t \|y(x)\|\,dx\Big),$$

using (3.4) and the fact that $\|r(y(x))\|\leq\|y(x)\|$, $x\in[0,T]$; here $h_0=\sup_{[0,T]}\|h(t)\|$. Consequently, integration from 0 to t yields

$$\int\limits_0^{\int_0^t \|y(x)\|\,dx} \frac{du}{\Phi(u)+h_0} \leq t \leq T,$$

so

$$\int\limits_0^t \|y(x)\|\,dx \leq J^{-1}(T) = M_1 \quad \text{for } t\in[0,T].$$

Also, we have

$$\|y(t)\| \leq h_0 + \Phi\Big(\int\limits_0^t \|y(x)\|\,dx\Big) \leq h_0 + \Phi(M_1) = M_0.$$

Thus $f(s,r(y(s)))=f(s,y(s))$, so y is a solution of (3.1). ■

R e m a r k. $\Phi(\int_0^t \|y(x)\|\, dx)$ in (3.4) could be replaced by $\Phi(\int_0^t \|y(x)\|^\sigma\, dx)$ for some constant $\sigma \geq 1$ and existence of a solution to (3.1) is again guaranteed (of course (3.5) has to be appropriately adjusted).

Next we examine the Hammerstein integral equation

$$y(t) = h(t) + \int_0^1 k(t,s) f(s, y(s))\, ds, \qquad t \in [0,1]. \tag{3.13}$$

Throughout, $f : [0,1] \times B \to B$ will be a L^β-Carathéodory function. Also, the following will be satisfied (here $1 \leq \alpha \leq \infty$ and β is the conjugate to α):

(3.14) $h \in C([0,1], B)$,

(3.15) $k(t,s) \in L^\alpha([0,1], \mathbb{R})$ for each $t \in [0,1]$ and the map $t \to k(t,s)$ is continuous from $[0,1]$ to $L^\alpha([0,1], \mathbb{R})$,

(3.16) there exists a nondecreasing continuous function $\theta : [0,\infty) \to [0,\infty)$ with $\int_0^1 \|f(s,u(s))\|^\beta\, ds \leq \theta(\int_0^1 \|u(s)\|^\alpha\, ds)$ for any $u \in C([0,1], B)$,

(3.17) $2^{\alpha-1}\Big(\int_0^1 \int_0^1 |k(t,s)|^\alpha\, ds\, dt\Big) \limsup_{x\to\infty} \dfrac{\theta^{\alpha/\beta}(x)}{x} < 1.$

R e m a r k. (3.17) has an obvious analogue when $\alpha = \infty$.

Consider the set S of real numbers $x \geq 0$ which satisfy the inequality

$$x \leq 2^{\alpha-1} \int_0^1 \|h(t)\|^\alpha\, dt + 2^{\alpha-1}\Big(\int_0^1 \int_0^1 |k(t,s)|^\alpha\, ds\, dt\Big)\theta^{\alpha/\beta}(x).$$

Then S is bounded above (see Theorem 2.1), i.e. there exists a constant M_2 with

$$x \leq M_2 \quad \text{for all } x \in S. \tag{3.18}$$

THEOREM 3.2. *Suppose $f : [0,1] \times B \to B$ has the decomposition $f = f_1 + f_2$ where f_1 and f_2 are L^β-Carathéodory functions. Assume that* (3.14)–(3.17) *hold. Let M_2 be as in* (3.18) *and define*

$$M_3 = \sup_{[0,1]} \|h(t)\| + \sup_{[0,1]} \Big(\int_0^1 |k(t,s)|^\alpha\, ds\Big)^{1/\alpha} \theta^{1/\beta}(M_2). \tag{3.19}$$

In addition, assume that

(3.20) *for each $t \in [0,1]$ the set $\{\int_0^1 k(t,s) f_2(s,u(s))\, ds : u \in C([0,1], B)$ with $\|u(s)\| \leq M_3$ for all $s \in [0,1]\}$ is relatively compact*,

(3.21) *there exists a continuous $Q : [0,1] \to [0,\infty)$ such that*

$$\sup_{[0,1]}\left\|e^{-Q(t)}\int_0^1 k(t,s)[f_1(s,u(s))-f_1(s,v(s))]\,ds\right\| \le \phi\big(\tfrac{1}{2}\sup_{[0,1]} e^{-Q(t)}\|u(t)-v(t)\|\big)$$

for all $u,v\in C([0,1],B)$ *with* $\|u(s)\|,\|v(s)\|\le M_3$ *for all* $s\in[0,1]$; *here* ϕ *is a real-valued nondecreasing continuous function satisfying* $\phi(x)<x$ *for* $x>0$.

Then (3.13) *has a solution* $y\in C([0,1],B)$.

Proof. Consider the modified Hammerstein equation

$$y(t)=h(t)+\int_0^1 k(t,s)f(s,r(y(s)))\,ds,\qquad t\in[0,1], \tag{3.22}$$

where $r:B\to\overline{B(0,M_3)}=\{y:\|y\|\le M_3\}$ is the radial retraction. Essentially the same reasoning as in Theorem 3.1 implies that (3.22) has a solution $y\in C([0,1],B)$.

Now for $t\in(0,1)$ we have

$$\|y(t)\|\le\|h(t)\|+\int_0^1|k(t,s)|\,\|f(s,r(y(s)))\|\,ds. \tag{3.23}$$

We will just consider the case $1\le\alpha<\infty$. The case $\alpha=\infty$ is similar. Hölder's inequality, together with (3.16), yields

$$\begin{aligned}\int_0^1\|y(t)\|^\alpha\,dt &\le 2^{\alpha-1}\int_0^1\|h(t)\|^\alpha\,dt\\ &\quad+2^{\alpha-1}\Big(\int_0^1\int_0^1|k(t,s)|^\alpha\,ds\,dt\Big)\theta^{\alpha/\beta}\Big(\int_0^1\|r(y(s))\|^\alpha\,ds\Big)\\ &\le 2^{\alpha-1}\int_0^1\|h(t)\|^\alpha\,dt\\ &\quad+2^{\alpha-1}\Big(\int_0^1\int_0^1|k(t,s)|^\alpha\,ds\,dt\Big)\theta^{\alpha/\beta}\Big(\int_0^1\|y(s)\|^\alpha\,ds\Big)\end{aligned}$$

since θ is nondecreasing and $\|r(y(s))\|\le\|y(s)\|$, $s\in[0,1]$. This, together with (3.18), yields

$$\int_0^1 \|y(s)\|^\alpha \, ds \le M_2.$$

Returning to (3.23), for $t \in [0,1]$ we have

$$\begin{aligned}\|y(t)\| &\le \sup_{[0,1]} \|h(t)\| + \Big(\int_0^1 |k(t,s)|^\alpha \, ds\Big)^{1/\alpha} \theta^{1/\beta}\Big(\int_0^1 \|r(y(s))\|^\alpha \, ds\Big) \\ &\le \sup_{[0,1]} \|h(t)\| + \sup_{[0,1]} \Big(\int_0^1 |k(t,s)|^\alpha \, ds\Big)^{1/\alpha} \theta^{1/\beta}(M_2) = M_3\end{aligned}$$

since $\int_0^1 \|r(y(s))\|^\alpha \, ds \le \int_0^1 \|y(s)\|^\alpha \, ds \le M_2$. Since $\|y(t)\| \le M_3$ for $t \in [0,1]$, we find that $f(s, r(y(s))) = f(s, y(s))$ and the result follows. ∎

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