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A counterexample to a conjecture of Drużkowski and Rusek

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Abstract. Let $F = X + H$ be a cubic homogeneous polynomial automorphism from $\mathbb{C}^n$ to $\mathbb{C}^n$. Let p be the nilpotence index of the Jacobian matrix JH. It was conjectured by Drużkowski and Rusek in [4] that $\deg F^{-1} \le 3^{p-1}$. We show that the conjecture is true if $n \le 4$ and false if $n \ge 5$.

1. Introduction. In [1] and [7] it was shown that it suffices to prove the Jacobian Conjecture for cubic homogeneous polynomial maps from $\mathbb{C}^n$ to $\mathbb{C}^n$, i.e. maps of the form $F = (F_1, \ldots, F_n)$ with $F_i = X_i + H_i$, where each H_i is either zero or a homogeneous polynomial of degree 3. In [2] it was shown that it even suffices to consider cubic linear polynomial maps, i.e. maps such that each H_i is of the form $H_i = l_i^3$, where l_i is a linear form.

A crucial result (cf. [1] and [6]) asserts that the degree of the inverse of a polynomial automorphism F is bounded by $(\deg F)^{n-1}$ (where $\deg F = \max \deg F_i$). In [4] Drużkowski and Rusek proved that for cubic homogeneous (resp. cubic linear) automorphisms this degree estimate could be improved in some special cases; more precisely, if $\operatorname{ind} JH$ denotes the index of nilpotency of JH then they showed that $\deg F^{-1} \le 3^{\operatorname{ind} JH - 1}$ if $\operatorname{ind} JH \le 2$ and also if H is cubic linear and $\operatorname{ind} JH \le 3$. This led them to the following conjecture:

Conjecture 1.1 (D–R) ([4], 1985). *If* $F = X+H$ *is a cubic homogeneous polynomial automorphism*, *then* $\deg F^{-1} \le 3^{p-1}$, *where* $p = \operatorname{ind} JH$.

Recently, in [3], Drużkowski showed that Conjecture D–R is true in case all coefficients of H are real numbers ≤ 0 (in which case the map F is stably tame, a result obtained by Yu in [8]).

In the present paper we show that the conjecture is true if $n \le 4$ and false if $n \ge 5$.

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2. The counterexample for $n \geq 5$. Let $n \geq 5$ and consider the polynomial ring $\mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_n]$.

THEOREM 2.1. *For each $n \geq 5$ the polynomial automorphism*

$$F = (X_1 + 3X_4^2X_2 - 2X_4X_5X_3, X_2 + X_4^2X_5, X_3 + X_4^3, X_4 + X_5^3, X_5, \ldots, X_n)$$

is a counterexample to Conjecture D–R.

Proof. Put $H = F - X$. Then one easily verifies that $(JH)^3 = 0$ and $(JH)^2 \neq 0$. Thus $\operatorname{ind} JH = 3$. So if Conjecture D–R is true, then $\deg F^{-1} \leq 9$. However, the inverse $G = (G_1, \ldots, G_n)$ of F is given by the following formulas:

$$\begin{aligned} G_1 &= X_1 - 3(X_4 - X_5^3)^2(X_2 - (X_4 - X_5^3)^2X_5) \\ &\quad + 2(X_4 - X_5^3)X_5(X_3 - (X_4 - X_5^3)^3), \\ G_2 &= X_2 - (X_4 - X_5^3)^2X_5, \\ G_3 &= X_3 - (X_4 - X_5^3)^3, \\ G_4 &= X_4 - X_5^3, \\ G_i &= X_i \quad \text{for all } 5 \leq i \leq n. \end{aligned}$$

So looking at the highest power of X_5 appearing in G_1, one easily verifies that $\deg G_1 = 13 > 9$. ∎

3. The case $n \leq 4$. The main result of this section is

PROPOSITION 3.1. *Conjecture D–R is true if $n \leq 4$.*

To prove this result we need the following theorem (cf. [5]):

THEOREM 3.2. *Let K be a field of characteristic zero and $F = X - H$ a cubic homogeneous polynomial map in dimension four such that* $\operatorname{Det}(JF) = 1$. *Then there exists some $T \in GL_4(K)$ such that $T^{-1}FT$ is of one of the following forms*:

$$(1) \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 - a_4x_1^3 - b_4x_1^2x_2 - c_4x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 \\ \quad - h_4x_1x_3^2 - k_4x_2^3 - l_4x_2^2x_3 - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix},$$

$$(2) \quad \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 - h_2x_1x_3^2 - q_2x_3^3 \\ x_3 \\ x_4 - x_1^2x_3 - h_4x_1x_3^2 - q_4x_3^3 \end{pmatrix},$$

$$(3)\quad \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 - c_1x_1^2x_4 + 3c_1x_1x_2x_3 - \frac{16q_4c_1^2 - r_4^2}{48c_1^2}x_1x_3^2 \\ \quad - \frac{1}{2}r_4x_1x_3x_4 + \frac{3}{4}r_4x_2x_3^2 - \frac{r_4q_4}{12c_1}x_3^3 - \frac{r_4^2}{16c_1}x_3^2x_4 \\ x_3 \\ x_4 - x_1^2x_3 + \frac{r_4}{4c_1}x_1x_3^2 - 3c_1x_1x_3x_4 + 9c_1x_2x_3^2 \\ \quad - q_4x_3^3 - \frac{3}{4}r_4x_3^2x_4 \end{pmatrix},$$

$$(4)\quad \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 - e_4x_1x_2^2 - k_4x_2^3 \end{pmatrix},$$

$$(5)\quad \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 + i_3x_1x_2x_4 - j_2x_1x_4^2s_3x_2x_4^2 + i_3^2x_3x_4^2 - t_2x_4^3 \\ x_3 - x_1^2x_2 - \frac{2s_3}{i_3}x_1x_2x_4 - i_3x_1x_3x_4 - j_3x_1x_4^2 - \frac{s_3^2}{i_3^2}x_2x_4^2 \\ \quad - s_3x_3x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix},$$

$$(6)\quad \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 - j_2x_1x_4^2 - t_2x_4^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 - g_3x_1x_2x_4 - j_3x_1x_4^2 - k_3x_2^3 \\ \quad - m_3x_2^2x_4 - p_3x_2x_4^2 - t_3x_4^3 \\ x_4 \end{pmatrix},$$

$$(7)\quad \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 - k_3x_2^3 \\ x_4 - x_1^2x_3 - e_4x_1x_2^2 - f_4x_1x_2x_3 - h_4x_1x_3^2 - k_4x_2^3 \\ \quad - l_4x_2^2x_3 - n_4x_2x_3^2 - q_4x_3^3 \end{pmatrix},$$

$$(8)\quad \begin{pmatrix} x_1 \\ x_2 - \frac{1}{3}x_1^3 \\ x_3 - x_1^2x_2 - e_3x_1x_2^2 + g_4x_1x_2x_3 - k_3x_2^3 + m_4x_2^2x_3 + g_4^2x_2^2x_4 \\ \quad x_4 - x_1^2x_3 - e_4x_1x_2^2 - \frac{2m_4}{g_4}x_1x_2x_3 - g_4x_1x_2x_4 - k_4x_2^3 \\ \quad - \frac{m_4^2}{g_4^2}x_2^2x_3 - m_4x_2^2x_4 \end{pmatrix}.$$

Proof. See [5, Theorem 2.7].

Proof of 3.1. As remarked in the introduction, the case $\operatorname{ind} JH = n$ was proved in [1] and [6]. The case $\operatorname{ind} JH = 2$ was done in [4]. So we may assume that $2 < \operatorname{ind} JH < n$. Therefore only the case $n = 4$ and $\operatorname{ind} JH = 3$ remains. By the classification theorem of Hubbers ([5, Theorem 2.7]) we know that there exists $T \in GL_4(\mathbb{C})$ such that $T^{-1}FT$ has one of the eight forms described above. One easily verifies that in each of the eight cases in which the nilpotency index of JH equals 3, $\deg(T^{-1}FT)^{-1} \leq 9$, so $\deg F^{-1} \leq 9$. ∎

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