

Coefficient bounds for certain classes of analytic functions

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Abstract. We determine coefficient estimates for α -spiral functions of order ρ with respect to N -symmetric points ($|\alpha| < \pi/2$, $0 \leq \rho < 1$ and N is a positive integer). Sharp coefficient bounds are also obtained for functions of the form $f(z)^{-t}$, where t is a positive integer and $f(z)$ is an α -spiral function of order ρ . Using this we deduce coefficient estimates for inverses of univalent α -spiral and meromorphic univalent α -spiral functions with vanishing early coefficients.

1. Introduction. Denote by \mathcal{H} the class of functions

$$(1.1) \quad f(z) = z + a_2 z^2 + \dots$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. For a fixed positive integer N , let $\mu = e^{2\pi i/N}$ be the N th root of unity. Then for a function $f \in \mathcal{H}$ given by (1.1), we define

$$f_N(z) = \frac{1}{N} \sum_{j=0}^{N-1} \mu^{-j} f(\mu^j z).$$

Using the fact that

$$\sum_{j=0}^{N-1} \mu^{jk} = \begin{cases} N & \text{if } k \text{ is a multiple of } N, \\ 0 & \text{otherwise,} \end{cases}$$

it is easily seen that $f_N(z)$ has the form

$$(1.2) \quad f_N(z) = z + a_{N+1} z^{N+1} + a_{2N+1} z^{2N+1} + \dots$$

A function $f \in \mathcal{H}$ is said to be α -spiral of order ρ with respect to N -symmetric points ($|\alpha| < \pi/2$, $0 \leq \rho < 1$) or N - α -spiral of order ρ , in

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symbols $f \in \mathcal{S}_N(\alpha, \varrho)$, if

$$\operatorname{Re} \left\{ e^{i\alpha} \frac{zf'(z)}{f_N(z)} \right\} > \varrho \cos \alpha, \quad z \in U,$$

where $f_N(z)$ is given by (1.2).

The class $\mathcal{S}_N(\alpha, \varrho)$ introduced above includes various subclasses studied earlier by a number of workers. For example, $\mathcal{S}_2(0, 0)$ is the class of starlike functions with respect to symmetric points introduced by Sakaguchi [12] and studied further by Robertson [11]. Similarly, $\mathcal{S}_N(0, 0)$ is the class of starlike functions with respect to N -symmetric points introduced and studied by Chand and Singh [3]. For $N = 1$, $\mathcal{S}_N(\alpha, \varrho)$ reduces to the class $\mathcal{S}(\alpha, \varrho)$ of α -spiral functions of order ϱ introduced and studied by Libera [6]. Špaček [16] has shown that the family of α -spiral functions of order 0 is univalent in U . For $\alpha = 0$, $\mathcal{S}_N(0, \varrho) (\equiv \mathcal{S}_N(\varrho))$ leads to the class of starlike functions of order ϱ with respect to N -symmetric points. This was introduced and studied by Singh and Singh [15] who claimed to have obtained sharp coefficient estimates for this class. Unfortunately, the claim of Singh and Singh [15] about their coefficient estimates being sharp is not correct. In fact, it can be easily checked that the function given by them is not extremal for the coefficient bounds stated.

In Section 2 of the present paper, we obtain sharp coefficient estimates for the class $\mathcal{S}_N(\alpha, \varrho)$ which, for $\alpha = 0$, lead to correct and sharp coefficient estimates for the class $\mathcal{S}_N^*(\varrho)$ introduced by Singh and Singh [15]. For $N = 1$, these estimates reduce to sharp coefficient estimates obtained for the class $\mathcal{S}_N^*(\varrho)$ introduced by Singh and Singh [15]. For $N = 1$, these estimates also reduce to sharp coefficient estimates obtained by Libera [6] for the class $\mathcal{S}(\alpha, \varrho)$.

Let $\mathcal{S}(\alpha, \varrho)^{-1}$ denote the class of functions

$$(1.3) \quad F(w) = w + A_2 w^2 + \dots$$

which are inverse to functions in $\mathcal{S}(\alpha, \varrho)$.

Likewise, let Σ_α denote the class of functions

$$(1.4) \quad g(z) = z + b_0 + b_1 z^{-1} + \dots$$

that are regular and univalent in $V = \{z : 1 < |z| < \infty\}$ and satisfy

$$(1.5) \quad \operatorname{Re} \left\{ e^{i\alpha} \frac{zg'(z)}{g(z)} \right\} > 0, \quad z \in V.$$

We observe that $\Sigma_0 \equiv \Sigma^*$, the class of meromorphic univalent starlike functions. Moreover, let Σ_α^{-1} represent the class of functions which are inverse to functions in Σ_α .

In [9], MacGregor determined coefficient estimates for starlike functions whose power series representations in U are of the form

$$(1.6) \quad f(z) = z + \sum_{n=p+1}^{\infty} a_n z^n.$$

Boyd [2] and Srivastava [17] extended MacGregor's result to the classes of starlike functions of order ϱ (i.e. $\mathcal{S}^*(\varrho)$) and $\mathcal{S}(\alpha, \varrho)$ respectively.

Poole [10] found coefficient estimates for the function $f(z)^{-t}$ when $f \in \mathcal{S}^*$ is of the form (1.6) and t is a positive integer. He also obtained coefficient estimates for inverses of functions in \mathcal{S}^* and Σ^* with vanishing early coefficients. Recently, the authors of [1] generalized the results of the above workers by obtaining coefficient bounds for functions of the form $f(z)^t$ when t is a positive integer and $f \in \mathcal{S}(\alpha, \varrho)$ is of the form (1.6).

Lately, a number of papers (see e.g. [7], [8], [13], [14]) have appeared in which the coefficient bounds for inverse functions have been determined for the class \mathcal{S} and its various subclasses. A number of techniques have been used to tackle these problems.

In Section 3 of the present paper, we obtain sharp coefficient estimates for functions of the form

$$f(z)^{-t} = z^{-t} + \sum_{n=-t+p}^{\infty} a_n^{(-t)} z^n$$

when $f(z)$, given by (1.6), belongs to $\mathcal{S}(\alpha, \varrho)$ and t is a positive integer. We apply these inequalities to deduce coefficient estimates for functions in $\mathcal{S}(\alpha, 0)^{-1}$ and $\Sigma(\alpha, 0)^{-1}$. The results thus obtained include those of Poole [10] and Juneja and Rajasekaran [5].

We conclude this section by indicating the possibility of extending the notion of N -symmetric points for general families of functions considered in [4] and [18] respectively. In fact, Jakubowski [4] discussed a more general family $\mathcal{S}_{m,M}^*(\alpha, \varrho)$ of functions $f \in \mathcal{H}$ such that

$$\left| \left(e^{i\alpha} \frac{z f'(z)}{f(z)} - \varrho \cos \alpha - i \sin \alpha \right) [(1 - \varrho) \cos \alpha]^{-1} - m \right| < M.$$

Also, he considered a similar definition for a family of meromorphic univalent functions of the form

$$(1.7) \quad F(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k, \quad 0 < |z| < 1,$$

and proved a sharp estimate for the modulus of the coefficients of functions belonging to these families.

In [18], Stankiewicz and Waniurski considered the family $\Sigma_n^*(A, B)$ of meromorphic univalent functions F of the form (1.7) such that

$$-\frac{zF'(z)}{F(z)} \prec \frac{1 + Az}{1 + Bz} \quad (-1 \leq A \leq 1, -A < B \leq 1)$$

and obtained sharp coefficient estimates for this family.

2. Coefficient bounds for functions in $\mathcal{S}_N(\alpha, \varrho)$. In this section we solve the coefficient problem for functions in $\mathcal{S}_N(\alpha, \varrho)$. While doing so, we rectify an error of a result in [15].

THEOREM 2.1. *If $f(z) = z + \sum_{n=2}^\infty a_n z^n \in \mathcal{S}_N(\alpha, \varrho)$, then*

$$(2.8) \quad |a_{mN+1}| \leq \prod_{j=0}^{m-1} \left\{ \left| \frac{2(1-\varrho) \cos \alpha e^{-i\alpha}}{N} + j \right| / (j+1) \right\}$$

and

$$(2.9) \quad |a_{mN+p}| \leq \frac{1}{m!} \cdot \frac{N}{mN+p} \prod_{j=0}^m \left\{ \left| \frac{2(1-\varrho) \cos \alpha e^{-i\alpha}}{N} + j \right| \right\},$$

for $m = 1, 2, \dots$ and $p = 2, \dots, N$. These estimates are sharp for the function f given by

$$(2.10) \quad f'(z) = \left\{ 1 + \frac{2(1-\varrho) \cos \alpha e^{-i\alpha} z}{1-z} \right\} / (1-z^N)^{2N-1(1-\varrho) \cos \alpha \exp(-i\alpha)}.$$

Proof. For $z \in U$ and $f \in \mathcal{S}_N(\alpha, \varrho)$, we define

$$(2.11) \quad w(z) = \frac{e^{i\alpha}(zf'(z)/f_N(z) - 1)}{e^{i\alpha}zf'(z)/f_N(z) + (e^{-i\alpha} - 2\varrho \cos \alpha)} = \sum_{n=1}^\infty w_n z^n.$$

Since

$$\operatorname{Re} \left\{ e^{i\alpha} \frac{zf'(z)}{f_N(z)} \right\} > \varrho \cos \alpha,$$

we have $|w(z)| < 1$. We rewrite

$$f_N(z) = z + \sum_{n=2}^\infty a_n \delta_n z^n,$$

where

$$\delta_n \equiv \delta(n, N) = \begin{cases} 1 & \text{if } n = Np + 1, p = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases}$$

Now replacing $f(z)$, $f_N(z)$, $w(z)$ by their power series expansions in the equation (which follows from (2.11))

$$zf'(z) - f_N(z) = \{zf'(z) + (e^{-2i\alpha} - 2\varrho \cos \alpha e^{-i\alpha})f_N(z)\}w(z),$$

we obtain

$$(2.12) \quad \sum_{n=2}^{\infty} (n - \delta_n) a_n z^n = \left\{ \sum_{n=1}^{\infty} [n + (e^{-2i\alpha} - 2\rho \cos \alpha e^{-i\alpha}) \delta_n] a_n z^n \right\} \left(\sum_{n=1}^{\infty} w_n z^n \right),$$

where $a_1 = 1$. Comparing the coefficients of z^n on both sides of (2.12) we have

$$(2.13) \quad (n - \delta_n) a_n = \sum_{k=1}^{n-1} \{k + (e^{-2i\alpha} - 2\rho \cos \alpha e^{-i\alpha}) \delta_k\} a_k w_{n-k}, \quad n \geq 2.$$

This shows that, for $n \geq 2$, the coefficients a_n on the left side depend upon the coefficients a_1, \dots, a_{n-1} . Therefore, for suitable coefficients d_k , (2.12) may be rewritten as

$$\sum_{k=2}^n (k - \delta_k) a_k z^k + \sum_{k=n+1}^{\infty} d_k z^k = \left\{ \sum_{k=1}^{n-1} [k + (e^{-2i\alpha} - 2\rho \cos \alpha e^{-i\alpha}) \delta_k] a_k z^k \right\} w(z).$$

Taking into account that $|w(z)| < 1$, and then multiplying each side of this equation by its conjugate, integrating with respect to θ around 0 to 2π , and taking the limit as $r \rightarrow 1^{-1}$, we obtain

$$\begin{aligned} \sum_{k=2}^n (k - \delta_k)^2 |a_k|^2 &\leq |1 + e^{-2i\alpha} - 2\rho \cos \alpha e^{-i\alpha}|^2 \\ &\quad + \sum_{k=2}^{n-1} |k + (e^{-2i\alpha} - 2\rho \cos \alpha e^{-i\alpha}) \delta_k|^2 |a_k|^2 \end{aligned}$$

and so

$$(n - \delta_n)^2 |a_n|^2 \leq |1 + e^{-2i\alpha} - 2\rho \cos \alpha e^{-i\alpha}|^2 + \sum_{k=2}^{n-1} \{|k + (e^{-2i\alpha} - 2\rho \cos \alpha e^{-i\alpha}) \delta_k|^2 - (k - \delta_k)^2\} |a_k|^2,$$

which after simplification reduces to

$$(2.14) \quad (n - \delta_n)^2 |a_n|^2 \leq 4(1 - \rho) \cos^2 \alpha \left\{ (1 - \rho) + \sum_{k=2}^{n-1} (k - \rho) \delta_k |a_k|^2 \right\},$$

where we have used the fact that $\delta_k = (\delta_k)^2$. When $n = N + 1$, this gives

$$|a_{N+1}| \leq \left| \frac{2(1 - \rho) \cos \alpha e^{-i\alpha}}{N} \right|,$$

which shows that (2.8) is true for $m = 1$.

Now we assume that (2.8) is true for $m = 1, \dots, q - 1$. For $m = q$, since the contribution on the right hand side of (2.14) comes only from $a_{N+1}, a_{2N+1}, \dots, a_{(q-1)N+1}$, we have

$$\begin{aligned} q^2 N^2 |a_{qN+1}|^2 &\leq 4(1 - \varrho) \cos^2 \alpha \left\{ (1 - \varrho) \right. \\ &\quad \left. + \sum_{j=1}^{q-1} (jN + 1 - \varrho) \prod_{k=0}^{j-1} \left(\frac{|[2(1 - \varrho) \cos \alpha e^{-i\alpha} / N] + k|}{k + 1} \right)^2 \right\} \\ &= \left\{ qN \prod_{j=0}^{q-1} \left(\frac{|[2(1 - \varrho) \cos \alpha e^{-i\alpha} / N] + j|}{j + 1} \right) \right\}^2, \end{aligned}$$

where the last equality can easily be proved by induction. This completes the proof of (2.8).

When $n = N + p, p = 2, \dots, N$, (2.14) gives

$$(N + p)^2 |a_{N+p}|^2 \leq 4(1 - \varrho) \cos^2 \alpha \{ (1 - \varrho) + (N + 1 - \varrho) |a_{N+1}|^2 \},$$

from which we obtain

$$|a_{N+p}| \leq \frac{N}{N + p} \prod_{j=0}^1 \left| \frac{2(1 - \varrho) \cos \alpha e^{-i\alpha}}{N} + j \right|.$$

Since for $n = mN + p, p = 2, \dots, N$, the contribution on the right hand side of (2.14) comes only from $a_{N+1}, a_{2N+1}, \dots, a_{mN+1}$, and so we can use the induction hypothesis as in the proof of (2.8) to obtain

$$|a_{mN+p}| \leq \frac{1}{m!} \cdot \frac{N}{mN + p} \prod_{j=0}^m \left| \frac{2(1 - \varrho) \cos \alpha e^{-i\alpha}}{N} + j \right|,$$

which is (2.9). This completes the proof of the theorem. ■

Remark 2.1. In [15] it was shown that if $\mathcal{S}_N^*(\varrho)$, then

$$(2.15) \quad (n - \delta_n)^2 |a_n|^2 \leq 4(1 - \varrho)^2 + 2 \sum_{k=2}^{n-1} \{k + (1 - \varrho)(k - 2\varrho)\} \delta_k |a_k|^2.$$

It was further claimed that the estimate (2.15) is sharp for the function (2.10) when $\alpha = 0$. However, it is easy to check that this claim is not correct for any $n > 2N$. In fact, when $\alpha = 0$, for the function f given by (2.10),

$$\begin{aligned} (2N)^2 |a_{2N+1}|^2 &= \left\{ (2 - 2\varrho) \left(\frac{2 - 2\varrho}{N} + 1 \right) \right\}^2 \\ &< 4(1 - \varrho)^2 + 2\{(N + 1) + (1 - \varrho)(N + 1 - 2\varrho)\} \frac{(2 - 2\varrho)^2}{N^2} \end{aligned}$$

and

$$\begin{aligned}
 & (2N+p)^2 |a_{2N+p}|^2 \\
 &= \left\{ \frac{N}{2!} \prod_{j=0}^2 \left(\frac{2-2\varrho}{N} + j \right) \right\}^2 \\
 &< 4(1-\varrho)^2 + 2\{(N+1) + (1-\varrho)(N+1-2\varrho)\} \frac{(2-2\varrho)^2}{N^2} \\
 &\quad + 2\{2N+1 + (1-\varrho)(2N+1-\varrho)\} \frac{\{((2-2\varrho)/N)((2-2\varrho)/N+1)\}^2}{(2!)^2},
 \end{aligned}$$

where $p = 2, \dots, N$, and so on.

On the other hand, our estimate (2.14), for $\alpha = 0$, leads to

$$(2.16) \quad (n - \delta_n)^2 |a_n|^2 \leq 4(1-\varrho) \left\{ (1-\varrho) + \sum_{k=2}^{n-1} (k-\varrho) \delta_k |a_k|^2 \right\}$$

and it is easy to see that equality occurs for every n in (2.16) for the function f given by (2.10) when $\alpha = 0$.

Remark 2.2. Singh and Singh [15] have defined the class $\mathcal{C}_N(\varrho)$ as follows: $f \in \mathcal{C}_N(\varrho)$ if and only if $zf' \in \mathcal{S}_N^*(\varrho)$. For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_N(\varrho)$, they have obtained the estimates

$$\begin{aligned}
 (2.17) \quad & \{mN(mN+1)\}^2 |a_{mN+1}|^2 \\
 & \leq 4(1-\varrho)^2 + 2 \sum_{k=1}^{m-1} \{(2-\varrho)(kN+1) - 2\varrho(1-\varrho)\} (kN+1)^2 |a_{kN+1}|^2
 \end{aligned}$$

for $m = 1, 2, \dots$, and

$$\begin{aligned}
 (2.18) \quad & (mN+p)^2 |a_{mN+p}|^2 \\
 & \leq 4(1-\varrho)^2 + \sum_{k=1}^m \{(2-\varrho)(kN+1) - 2\varrho(1-\varrho)\} (kN+1)^2 |a_{kN+1}|^2
 \end{aligned}$$

for $m = 0, 1, \dots$ and $p = 2, \dots, N$.

In view of (2.16), the results (2.17) and (2.18) can be modified to

$$\begin{aligned}
 & \{mN(mN+1)\}^2 |a_{mN+1}|^2 \\
 & \leq 4(1-\varrho) \left\{ (1-\varrho) + \sum_{k=1}^{m-1} (kN+1-\varrho)(kN+1)^2 |a_{kN+1}|^2 \right\}
 \end{aligned}$$

for $m = 1, \dots$, and

$$(mN + p)^4 |a_{mN+p}|^2 \leq 4(1 - \varrho) \left\{ (1 - \varrho) + \sum_{k=1}^{m-1} (kN + 1 - \varrho)(kN + 1)^2 |a_{kN+1}|^2 \right\}$$

for $m = 0, 1, \dots$ and $p = 2, \dots, N$.

Now, putting $\alpha = 0$, Theorem 2.1 gives the following result, which is a refinement of the results in [3] and [15].

COROLLARY 2.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}_N^*(\varrho)$, then*

$$|a_{mN+1}| \leq \prod_{j=0}^{m-1} \left\{ \left(\frac{2 - 2\varrho}{N} + j \right) / (j + 1) \right\}$$

for $m = 1, 2, \dots$, and

$$|a_{mN+p}| \leq \frac{1}{m!} \cdot \frac{N}{mN + p} \prod_{j=0}^m \left(\frac{2 - 2\varrho}{N} + j \right)$$

for $m = 0, 1, \dots$ and $p = 2, \dots, N$. These estimates are sharp.

In view of the fact that $f \in \mathcal{C}_N(\varrho)$ if and only if $zf' \in \mathcal{S}_N^*(\varrho)$, we have

COROLLARY 2.2. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{C}_N(\varrho)$, then*

$$|a_{mN+1}| \leq \frac{1}{mN + 1} \prod_{j=0}^{m-1} \left\{ \left(\frac{2 - 2\varrho}{N} + j \right) / (j + 1) \right\}$$

for $m = 1, 2, \dots$, and

$$|a_{mN+p}| \leq \frac{1}{m!} \cdot \frac{N}{(mN + p)^2} \prod_{j=0}^m \left(\frac{2 - 2\varrho}{N} + j \right)$$

for $m = 0, 1, \dots$ and $p = 2, \dots, N$. These inequalities are sharp.

3. Some other coefficient bounds

THEOREM 3.1. *Suppose $f(z) = z + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{S}(\alpha, \varrho)$, and for integral $t \geq 1$, let*

$$f(z)^{-t} = z^{-t} + \sum_{\gamma=-t+p}^{\infty} a_{\gamma}^{(-t)} z^{\gamma}, \quad 0 < |z| < 1.$$

Then

$$(3.19) \quad |a_{\gamma}^{(-t)}| \leq \frac{mp}{\gamma + t} \prod_{j=0}^{m-1} \left(\frac{|(2t/p)(1 - \varrho) \cos \alpha e^{-i\alpha} - j|}{j + 1} \right)$$

for $-t + mp \leq \gamma \leq -t + (m + 1)p - 1$, $m = 1, \dots, M + 1$, where $M = [t(1 - \varrho)/p]$. (Here $[x]$ denotes the largest integer not exceeding x .) The result is sharp for $\gamma = -t + mp$.

The following two lemmas will be useful in the proof of this theorem.

LEMMA 3.1. *If p , t and q are positive integers, then for $0 \leq \varrho < 1$, we have*

$$\begin{aligned} & 4t(1 - \varrho) \cos^2 \alpha \left\{ t(1 - \varrho) \right. \\ & \quad \left. + \sum_{m=1}^{q-1} [t(1 - \varrho) - mp] \prod_{j=0}^{m-1} \left(\frac{|(2t/p)(1 - \varrho) \cos \alpha e^{-i\alpha} - j|}{j + 1} \right)^2 \right\} \\ & = (qp)^2 \left\{ \prod_{j=0}^{q-1} \left(\frac{|(2t/p)(1 - \varrho) \cos \alpha e^{-i\alpha} - j|}{j + 1} \right) \right\}^2 \end{aligned}$$

whenever $t(1 - \varrho) - (q - 1)p \geq 0$.

The above lemma can be easily proved by induction on q .

LEMMA 3.2. *If p , t and q are positive integers and $0 \leq \varrho < 1$, then*

$$(\gamma + t)^2 \geq \frac{(qp)^2(-\gamma - \varrho t)}{t(1 - \varrho) - qp}$$

for $\gamma \geq -t + qp$ and $t(1 - \varrho) - (q - 1)p \geq 0$.

The proof of Lemma 3.2 is straightforward and hence omitted.

PROOF OF THEOREM 3.1. For $0 < |z| < 1$, let

$$q(z) = \frac{e^{i\alpha}}{-t} \cdot \frac{z[f(z)^{-t}]'}{f(z)^{-t}} = e^{i\alpha} \frac{zf'(z)}{f(z)}$$

and $q(0)$ can be defined so that $q(z)$ is continuous at $z = 0$. Moreover, let

$$(3.20) \quad w(z) = \frac{q(z) - e^{i\alpha}}{q(z) + e^{-i\alpha} - 2\varrho \cos \alpha} = \sum_{n=p}^{\infty} w_n z^n.$$

Since $\operatorname{Re} q(z) > \varrho \cos \alpha$, we have $|w(z)| < 1$ in $|z| < 1$. Equating the coefficients of the same powers on both sides of (3.20) we get

$$e^{i\alpha} \{ [f(z)^{-t}]' + tf(z)^{-t} \} = \{ e^{i\alpha} z [f(z)^{-t}]' - (e^{-i\alpha} - 2\varrho \cos \alpha) tf(z)^{-t} \} w(z)$$

or

$$\begin{aligned}
 (3.21) \quad e^{i\alpha} \sum_{n=t+p}^{\infty} (n+t)a_n^{(-t)} z^n \\
 &= \left(\sum_{n=p}^{\infty} w_n z^n \right) \left\{ -2t(1-\varrho) \cos \alpha z^{-t} \right. \\
 &\quad \left. + \sum_{n=-t+p}^{\infty} [ne^{i\alpha} - t(e^{-i\alpha} - 2\varrho \cos \alpha)] a_n^{(-t)} z^n \right\},
 \end{aligned}$$

from which we obtain the relations

$$(3.22) \quad e^{i\alpha} \gamma a_{\gamma-t}^{(-t)} = -2t(1-\varrho) \cos \alpha w_\gamma, \quad \gamma = p, p+1, \dots, 2p-1.$$

Since $|w(z)| < 1$, we have $\sum_{n=p}^{\infty} |w_n|^2 \leq 1$ and therefore

$$(3.23) \quad \sum_{n=p}^{2p-1} |w_n|^2 \leq 1.$$

Now (3.22) and (3.23) yield

$$(3.24) \quad \sum_{n=p}^{2p-1} \gamma^2 |a_{\gamma-t}|^2 \leq 4t^2(1-\varrho)^2 \cos^2 \alpha.$$

We rewrite (3.21) in the form

$$\begin{aligned}
 (3.25) \quad \sum_{n=-t+p}^k e^{i\alpha} (n+t) a_n^{(-t)} z^n + \sum_{n=k+1}^{\infty} d_n z^n \\
 &= w(z) \left\{ -2t(1-\varrho) \cos \alpha z^{-t} \right. \\
 &\quad \left. + \sum_{n=-t+p}^{k-p} [ne^{i\alpha} - t(e^{-i\alpha} - 2\varrho \cos \alpha)] a_n^{(-t)} z^n \right\}.
 \end{aligned}$$

Taking into account that $|w(z)| < 1$, and then multiplying each side of (3.25) by its conjugate, integrating around $|z| = r < 1$ and letting $r \rightarrow 1^-$ yields the inequality

$$\begin{aligned}
 (3.26) \quad \sum_{n=-t+p}^{k-p} (n+t)^2 |a_n^{(-t)}|^2 \\
 \leq 4t(1-\varrho) \cos^2 \alpha \left\{ t(1-\varrho) + \sum_{n=-t+p}^{k-p} (-n-\varrho t) |a_n^{(-t)}|^2 \right\}.
 \end{aligned}$$

Putting $k = -t + (q+1)p - 1$, (3.25) can be rewritten as

$$\begin{aligned} & \sum_{n=-t+p}^{-t+(q+1)p-1} (n+t)^2 |a_n^{(-t)}|^2 \\ & \leq 4t(1-\varrho) \cos^2 \alpha \left\{ t(1-\varrho) + \sum_{n=-t+p}^{-t+qp-1} (-n-\varrho t) |a_n^{(-t)}|^2 \right\} \\ & = 4t(1-\varrho) \cos^2 \alpha \left\{ t(1-\varrho) + \sum_{m=1}^{q-1} \sum_{n=-t+mp}^{-t+(m+1)p-1} (-n-\varrho t) |a_n^{(-t)}|^2 \right\}. \end{aligned}$$

Now, by Lemma 3.2 and (3.24),

$$\begin{aligned} \sum_{n=-t+p}^{-t+2p-1} (-n-\varrho t) |a_n^{(-t)}|^2 &= \frac{t(1-\varrho)-p}{p^2} \sum_{n=-t+p}^{-t+2p-1} \frac{p^2(-n-\varrho t)}{t(1-\varrho)-p} |a_n^{(-t)}|^2 \\ &\leq \frac{[t(1-\varrho)-p]}{p^2} \sum_{n=-t+p}^{-t+2p-1} (n+t)^2 |a_n^{(-t)}|^2, \\ &\leq \frac{[t(1-\varrho)-p]}{p^2} \cdot 4t^2(1-\varrho)^2 \cos^2 \alpha, \\ &= [t(1-\varrho)-p] (2t/p)(1-\varrho) \cos \alpha e^{i\alpha} |a_n^{(-t)}|^2. \end{aligned}$$

Now we assume

$$\begin{aligned} & \sum_{n=-t+mp}^{-t+(m+1)p-1} (-n-\varrho t) |a_n^{(-t)}|^2 \\ & \leq [t(1-\varrho)-mp] \prod_{j=0}^{m-1} \left(\frac{|(2t/p)(1-\varrho) \cos \alpha e^{-i\alpha} - j|}{j+1} \right)^2 \end{aligned}$$

for $m = 2, \dots, q-1$. Then from (3.26),

$$\begin{aligned} & \sum_{n=-t+qp}^{-t+(q+1)p-1} (n+t)^2 |a_n^{(-t)}|^2 \\ & \leq 4t(1-\varrho) \cos^2 \alpha \left\{ t(1-\varrho) + \sum_{m=1}^{q-1} \sum_{n=-t+mp}^{-t+(m+1)p-1} (-n-\varrho t) |a_n^{(-t)}|^2 \right\} \\ & = 4t(1-\varrho) \cos^2 \alpha \left\{ t(1-\varrho) + \sum_{m=1}^{q-1} [t(1-\varrho)-mp] \right\} \end{aligned}$$

$$\begin{aligned} & \times \prod_{j=0}^{m-1} \left(\frac{|(2t/p)(1-\varrho)\cos\alpha e^{-i\alpha} - j|}{j+1} \right)^2 \Big\} \\ & = (qp)^2 \prod_{j=0}^{q-1} \left(\frac{|(2t/p)(1-\varrho)\cos\alpha e^{-i\alpha} - j|}{j+1} \right)^2, \end{aligned}$$

by Lemma 3.1, as long as $t(1-\varrho) - (q-1)p \geq 0$. Now we use Lemma 3.2 and this estimate to show that

$$\begin{aligned} & \sum_{n=-t+qp}^{-t+(q+1)p-1} (-n-\varrho t) |a_n^{(-t)}|^2 \\ & = \frac{[t(1-\varrho) - qp]}{(qp)^2} \left\{ \sum_{n=-t+qp}^{-t+(q+1)p-1} \frac{(qp)^2(-n-\varrho t)}{t(1-\varrho) - qp} |a_n^{(-t)}|^2 \right\} \\ & \leq \frac{[t(1-\varrho) - qp]}{(qp)^2} \left\{ \sum_{n=-t+qp}^{-t+(q+1)p-1} (n+t)^2 |a_n^{(-t)}|^2 \right\} \\ & = [t(1-\varrho) - qp] \prod_{j=0}^{q-1} \left(\frac{|(2t/p)(1-\varrho)\cos\alpha e^{-i\alpha} - j|}{j+1} \right)^2. \end{aligned}$$

Thus we have established by induction that

$$\begin{aligned} (3.27) \quad & \sum_{n=-t+mp}^{-t+(m+1)p-1} (n+t)^2 |a_n^{(-t)}|^2 \\ & \leq (mp)^2 \prod_{j=0}^{m-1} \left(\frac{|(2t/p)(1-\varrho)\cos\alpha e^{-i\alpha} - j|}{j+1} \right)^2 \end{aligned}$$

and

$$\begin{aligned} & \sum_{n=-t+mp}^{-t+(m+1)p-1} (-n-\varrho t) |a_n^{(-t)}|^2 \\ & \leq [t(1-\varrho) - mp] \prod_{j=0}^{m-1} \left(\frac{|(2t/p)(1-\varrho)\cos\alpha e^{-i\alpha} - j|}{j+1} \right)^2 \end{aligned}$$

for m such that $t(1-\varrho) - (m-1)p \geq 0$.

The theorem now follows from (3.27). If $n = -t + mp$, the inequality (3.19) is sharp for the function

$$f(z) = z(1-z^p)^{-2(1-\varrho)\cos\alpha\exp(-i\alpha)/p}.$$

This completes the proof. ■

Now take two functions

$$F(w) = w + \sum_{n=2}^{\infty} A_n w^n \in \mathcal{S}(\alpha, 0)^{-1}, \quad G(w) = w + \sum_{n=1}^{\infty} B_n w^{-n} \in \Sigma(\alpha, 0)^{-1}.$$

It has been proved in [5] that

$$A_n = a_{-1}^{(-n)}/n \quad \text{and} \quad B_n = -a_1^{(-n)}/n, \quad \text{for } n = 1, 2, \dots$$

Now using these results and Theorem 3.1, we obtain the following results.

THEOREM 3.2. *If $F(w) = w + \sum_{n=p+1}^{\infty} A_n w^n \in \mathcal{S}(\alpha, 0)^{-1}$, then*

$$|A_n| \leq \frac{mp}{n(n-1)} \prod_{j=0}^{m-1} \left(\frac{|(2n/p) \cos \alpha e^{-i\alpha} - j|}{j+1} \right)$$

for $mp+1 \leq n \leq (m+1)p$, $m = 1, 2, \dots$

THEOREM 3.3. *If $G(w) = w + \sum_{n=p-1}^{\infty} B_n w^{-n} \in \Sigma(\alpha, 0)^{-1}$, then*

$$|B_n| \leq \frac{mp}{n(n+1)} \prod_{j=0}^{m-1} \left(\frac{|(2n/p) \cos \alpha e^{-i\alpha} - j|}{j+1} \right)$$

for $mp-1 \leq n \leq (m+1)p-2$, $m = 1, 2, \dots$

For $\alpha = 0$, Theorems 3.2 and 3.3 give the following results of Poole [10].

COROLLARY 3.1. *If $F(w) = w + \sum_{n=p+1}^{\infty} A_n w^n \in \mathcal{S}^{*-1}$, then*

$$|A_n| \leq \frac{mp}{n(n-1)} \prod_{j=0}^{m-1} \left(\frac{(2n/p) - j}{j+1} \right)$$

for $mp+1 \leq n \leq (m+1)p$, $m = 1, 2, \dots$

COROLLARY 3.2. *If $G(w) = w + \sum_{n=p-1}^{\infty} B_n w^{-n} \in \Sigma^{*-1}$, then*

$$|B_n| \leq \frac{mp}{n(n+1)} \prod_{j=0}^{m-1} \left(\frac{(2n/p) - j}{j+1} \right)$$

for $mp-1 \leq n \leq (m+1)p$, $m = 1, 2, \dots$

Moreover, when $n = mp+1$, Theorem 3.2 gives a result in [5]. Similarly, when $n = mp-1$, Theorem 3.3 gives a result in [5].

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