C O L L O Q U I U M   M A T H E M A T I C U M

VOL. LXVIII 1995 FASC. 1

ON BOUNDEDNESS PROPERTIES OF CERTAIN MAXIMAL OPERATORS

BY

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It is known that the weak type $(1,1)$ for the Hardy–Littlewood maximal operator can be obtained from the weak type $(1,1)$ over Dirac deltas. This theorem is due to M. de Guzmán. In this paper, we develop a technique that allows us to prove such a theorem for operators and measure spaces in which Guzmán's technique cannot be used.

1. Introduction. Let M be the Hardy-Littlewood maximal operator, i.e.

$$Mf(x,v)=\sup_{Q\ni x}\frac{1}{|Q|}\int\limits_Q |f(y)|\,dy$$

($|E|$ represents the Lebesgue measure of the set E). It is known that the inequality

$$|\{x\in\mathbb{R}^n : Mf(x)>\lambda\}|\le\frac{C}{\lambda}\int\limits_{\mathbb{R}^n}|f(x)|\,dx$$

can be obtained from the following condition: for every finite family $\{a_k\}_{k=1}^N$ we have

$$\Big\{x\in\mathbb{R}^n : M\Big(\sum_{k=1}^N\delta_{a_k}\Big)(x)>\lambda\Big\}\le\frac{C}{\lambda}N,$$

where the action of M over a Dirac delta is defined by $M\delta_a(x)=\sup_{x\in Q}|Q|^{-1}\chi_Q(a)$. This result is due to M. de Guzmán (see [3]). Applications of it are shown in [5].

The same kind of theorem can be proved if the Lebesgue measure is substituted by a measure $w(x)dx$, $w(x)>0$ a.e. (see [4]). However, the technique developed for the proof cannot be used for more general measure spaces.

The main purpose of this paper is to prove similar results for general maximal operators in general measure spaces (see Theorem 3), including as particular cases various operators that appear in Harmonic Analysis.

1991 *Mathematics Subject Classification*: Primary 42B25.

For U, V arbitrary sets, we consider positive Borel measures $d\gamma(x,u)$ and $d\beta(x,v)$ defined on $\mathbb{R}^n \times U$ and $\mathbb{R}^n \times V$, respectively, and we suppose that $\mathbb{R}^n \times U$ and $\mathbb{R}^n \times V$ have some topological structure. We denote by $L(\mathbb{R}^n \times U, d\gamma)$, as usual, the set of measurable functions f in $\mathbb{R}^n \times U$ such that $\int_{\mathbb{R}^n \times U} |f(x,u)|\, d\gamma$ is finite.

Let now Φ and Ψ be set functions from cubes in $\mathbb{R}^n$ to Borel sets in $\mathbb{R}^n \times U$ and $\mathbb{R}^n \times V$ respectively, satisfying the following conditions:

I. If Q_1, Q_2 are cubes with $Q_1 \cap Q_2 = \emptyset$, then $\Phi(Q_1) \cap \Phi(Q_2) = \emptyset$ and $\Psi(Q_1) \cap \Psi(Q_2) = \emptyset$.

II. (i) If $Q_1 \subset Q_2$, then $\Phi(Q_1) \subset \Phi(Q_2)$.
(ii) $\Psi(Q(x,r)) \subset \Psi(Q(x,s))$ if $0 < r \le s$ and $x \in \mathbb{R}^n$, where $Q(x,r)$ is the cube centered at x and with side length r.

III. For any $x \in \mathbb{R}^n$,

$$\bigcup_{r>0} \Phi(Q(x,r)) = \mathbb{R}^n \times U \quad \text{and} \quad \bigcup_{r>0} \Psi(Q(x,r)) = \mathbb{R}^n \times V.$$

In this situation, we define

$$Tf(x,v) = \sup \frac{1}{|Q|} \int_{\Phi(Q)} |f(y,u)|\, d\gamma(y,u),$$

where the supremum is taken over all cubes Q such that $(x,v) \in \Psi(Q)$.

In this work we prove (Theorem 3) that the weighted weak type $(1,1)$ of T is equivalent to the weighted weak type $(1,1)$ for T acting over finite sums of Dirac deltas. The proof is through a simple induction argument using some ideas developed in [4]. This kind of proof seems to be more natural than the possible proof using dyadic cubes and the ideas developed in [6] and [8].

This operator was considered in [8], in order to prove boundedness properties for maximal operators M_Ω associated with a general domain $\Omega \subset \mathbb{R}^{n+1}_+$.

Previously, related operators where considered in [6] (see Example D below). By using a technical lemma (Lemma 2), we give a unified result for operators in [6] and [8].

In the same way, we can also obtain (Theorem 5) a discrete characterization of the weighted weak type $(1,q)$ for the fractional maximal operator M_α, $0 \le \alpha < n$, defined by

$$M_\alpha f(x,v) = \sup_Q \frac{1}{|Q|^{1-\alpha/n}} \int_{\Phi(Q)} |f(y,u)|\, d\gamma(y,u) \chi_{\Psi(Q)}(x,v).$$

Particular examples are the following:

A. If $\mathbb{R}^n \times U = \mathbb{R}^n \times V = \mathbb{R}^n$, $d\gamma(x,u) = dx$, where dx is the Lebesgue measure on $\mathbb{R}^n$, and $\Phi(Q) = \Psi(Q) = Q$, then T is the Hardy–Littlewood maximal operator.

B. If $\mathbb{R}^n \times U = \mathbb{R}^n$, $V = [0,\infty)$, $d\gamma(x,u) = dx$, $\Phi(Q) = Q$ and $\Psi(Q) = \widetilde{Q} = \{(x,t) : 0 \le t \le \text{side length of } Q\}$, then T is the operator

$$\mathcal{M}f(x,t) = \sup\left\{\frac{1}{|Q|}\int\limits_Q |f(y)|\,dy : (x,t) \in \widetilde{Q}\right\}$$

introduced by Fefferman–Stein in [2].

C. If $U = [0,\infty)$, $\mathbb{R}^n \times V = \mathbb{R}^n$, $\Phi(Q) = \widetilde{Q}$ and $\Psi(Q) = Q$, then T is the maximal operator

$$Cf(x,t) = \sup\left\{\frac{1}{|Q|}\int\limits_{\widetilde{Q}} |f(y,t)| : x \in Q\right\}$$

closely related to tent spaces ([1]).

D. If condition **II** for set functions Φ and Ψ is replaced by a stronger condition

II*. If $Q_1 \subset Q_2$, then $\Phi(Q_1) \subset \Phi(Q_2)$ and $\Psi(Q_1) \subset \Psi(Q_2)$,

we obtain the maximal operator T defined in [6].

2. Technical lemmas. In [6] a characterization of the weighted weak type $(1,1)$ of the operator T was obtained:

THEOREM 1. *Let $w(x,u)$ be a positive function on $\mathbb{R}^n \times U$, $f \in L(\mathbb{R}^n \times U, d\gamma)$, Φ and Ψ set functions as in the introduction satisfying conditions* **I**, **II*** *and* **III**, *and T such that*

$$Tf(x,v) = \sup_Q \frac{1}{|Q|}\int\limits_{\Phi(Q)} |f(y,u)|\,d\gamma(y,u)\chi_{\Psi(Q)}(x,v).$$

The following conditions are equivalent:

(i) *T is bounded from $L^1(\mathbb{R}^n \times U, wd\gamma)$ into $L^{1,\infty}(\mathbb{R}^n \times V, d\beta)$.*

(ii) *There exists a constant $C > 0$ such that for any cube Q in $\mathbb{R}^n$,*

$$\beta(\Psi(Q))/|Q| \le Cw(x,u), \quad \gamma\text{-a.e. } (x,u) \in \Phi(Q).$$

Later, in [8], where condition **II*** was replaced by **II**, it was proved that the weighted weak type $(1,1)$ for the associated operator is equivalent to statement (ii)* of Lemma 2, given below. This lemma shows that for Φ and Ψ satisfying **I**, **III** and the strongest condition **II***, statements (ii) and (ii)* are equivalent.

LEMMA 2. *For Φ, Ψ and w as in Theorem* 1, *statement* (ii) *in this theorem is equivalent to* (ii)* *There exist* $C_2 > 0$ *and* $m > 0$ *such that for any cube* $Q = Q(y, r)$ *in* $\mathbb{R}^n$,

$$\beta(S_m(Q))/|Q| \le C_2 w(x,u), \qquad \gamma\text{-}a.e.\ (x,u) \in \Phi(Q),$$

where $S_m(Q) = \bigcup_{\xi \in Q(y,mr)} \Psi(Q(\xi, r))$.

Proof. (ii)* $\Rightarrow$(ii) immediately follows from the fact that $\Psi(Q) \subset S_m(Q)$ for $m > 0$.

(ii)$\Rightarrow$(ii)*. Let $Q = Q(y,r)$. If $\xi \in Q$, then $Q(\xi, r) \subset Q^* = Q(y, 2r)$; by **II***, $\Psi(Q(\xi,r)) \subset \Psi(Q^*)$, and so $S_1(Q) \subset \Psi(Q^*)$. Then

$$\frac{\beta(S_1(Q))}{|Q|} \le 3^n \frac{\beta(\Psi(Q^*))}{|Q^*|} \le C_2 w(x,u), \qquad \gamma\text{-a.e. } (x,u) \in \Phi(Q^*).$$

3. Main results. Let T be the maximal operator defined in the introduction.

The action of T over one Dirac delta can be defined by

$$T\delta_{(a,u)}(x,v) = \sup_Q \frac{1}{|Q|} \chi_{\Phi(Q)}(a,u) \chi_{\Psi(Q)}(x,v)$$

for $(a,u) \in \mathbb{R}^n \times U$ and $(x,v) \in \mathbb{R}^n \times V$.

Following [5], we say that the operator T is of *weak type* $(1,1)$ *over finite sums of Dirac deltas* if there is a set $A \subset \mathbb{R}^n \times U$ with $\gamma(\mathbb{R}^n \times U \setminus A) = 0$, and also a constant $C > 0$ such that for every finite family $\{(a_k, u_k)\}_{k=1}^N \subset A$ and $\lambda > 0$,

$$\beta\Big(\Big\{(x,v) \in \mathbb{R}^n \times V : T\Big(\sum_{k=1}^N \delta_{(a_k,u_k)}\Big)(x,v) > \lambda\Big\}\Big) \le \frac{C}{\lambda} \sum_{k=1}^N w(a_k, u_k).$$

We are going to show that the operator T has a weak boundedness property if and only if T has the same property over finite sums of Dirac deltas.

THEOREM 3. *Let* T *be as before, with* Φ, Ψ *set functions satisfying conditions* **I**, **II** *and* **III**, w *a positive function defined on* $\mathbb{R}^n \times U$ *and* $d\gamma(x,u)$ *and* $d\beta(x,v)$ *positive Borel measures on* $\mathbb{R}^n \times U$ *and* $\mathbb{R}^n \times V$, *respectively. The following statements are equivalent*:

(a) T *is bounded from* $L^1(\mathbb{R}^n \times U, w d\gamma)$ *into* $L^{1,\infty}(\mathbb{R}^n \times V, d\beta)$.

(b) T *is of weak type* $(1,1)$ *over finite sums of Dirac deltas.*

(c) *There are* $m > 0$ *and* $C > 0$ *such that for any cube* $Q = Q(y,r)$ *in* $\mathbb{R}^n$,

$$\beta(S_m(Q))/|Q| \le C w(x,u), \qquad \gamma\text{-}a.e.\ (x,u) \in \Phi(Q),$$

where $S_m(Q) = \bigcup_{\xi \in Q(y,mr)} \Psi(Q(\xi,r))$ (*the constants in* (b) *and* (c) *are not necessarily the same*).

COROLLARY 4. *In the situation of the above theorem, if condition* **II** *is replaced by* $\mathbf{II}^*$, *then statement* (c) *can be changed to*

(c*) *There is* $C > 0$ *such that for any cube* Q *in* $\mathbb{R}^n$,

$$\beta(\Psi(Q))/|Q| \le Cw(x,u), \qquad \gamma\text{-a.e. } (x,u) \in \Phi(Q).$$

This corollary follows immediately from Theorem 3 and Lemma 2.

Proof of Theorem 3. It was proved in [8] that (a)$\Leftrightarrow$(c). Thus, we only need to prove the equivalence between (b) and (c). It is enough to show that for every fixed set $A \subset \mathbb{R}^n \times U$ the following statements, (b′) and (c′), are equivalent:

(b′) *There exists a constant* $C > 0$ *such that for every finite family* $\{(a_k, u_k)\}_{k=1}^N \subset A$ *and* $\lambda > 0$,

$$\beta\Big(\Big\{(x,v) \in \mathbb{R}^n \times V : T\Big(\sum_{k=1}^N \delta_{(a_k,u_k)}\Big)(x,v) > \lambda\Big\}\Big) \le \frac{C}{\lambda}\sum_{k=1}^N w(a_k,u_k).$$

(c′) *There exist* $m > 0$ *and* $C > 0$ *such that for any cube* $Q = Q(y,r)$ *in* $\mathbb{R}^n$,

$$\beta(S_m(Q))/|Q| \le Cw(x,u), \qquad \forall (x,u) \in \Phi(Q) \cap A.$$

We first show (b′)$\Rightarrow$(c′).

Let $Q = Q(y,r)$, $(a,u) \in A \cap \Phi(Q)$, and $\lambda = 1/|Q|$. We now prove that for each $m > 0$,

$$S_m(Q) \subset E_\lambda = \{(x,v) \in \mathbb{R}^n \times V : T\delta_{(a,u)}(x,v) > c\lambda\},$$

where c is a constant which depends on m. Indeed, if $(x,v) \in S_m(Q)$, there exists $\xi \in Q(y,mr)$ such that $(x,v) \in \Psi(Q(\xi,r))$. Therefore, for $K = 2(m+1)$,

$$T\delta_{(a,u)}(x,v) > \frac{1}{|Q(\xi,Kr)|} \ge \frac{c}{|Q|} = c\lambda,$$

because $Q \subset Q(\xi,Kr)$, and then $\Phi(Q) \subset \Phi(Q(\xi,Kr))$.

So, we have

$$\beta(S_m(Q)) \le \beta(E_\lambda) \le \frac{1}{c\lambda}w(a,u) = \frac{C}{\lambda}w(a,u) = C|Q|w(a,u).$$

We now prove (c′)$\Rightarrow$(b′). We use a simple induction argument, without dyadic cubes as in [6] and [8].

Suppose that (c′) is satisfied for a constant $m \ge 2$. Note that this is not a restriction because, as one can conclude from [8], if there are some m and C satisfying (c′), then for every $m > 0$ it is possible to find a C with the same condition.

Let $(a,u) \in A$; we define

$$E_\lambda = \{(x,v) \in \mathbb{R}^n \times V : T\delta_{(a,u)}(x,v) > \lambda\}.$$

We show that $E_\lambda \subset S_m(Q)$ for some m and Q. If $(x,v) \in E_\lambda$, then there is a cube Q with $|Q| = 1/\lambda$, $(a,u) \in \Phi(Q)$ and $(x,v) \in \Psi(Q)$. If we now take two points of E_λ, (x_1,v_1) and (x_2,v_2), with associated cubes Q_1 and Q_2 respectively, satisfying the above conditions, then $\Phi(Q_1) \cap \Phi(Q_2) \neq \emptyset$ and $Q_1 \cap Q_2 \neq \emptyset$. So, the center of Q_2 belongs to $Q(x_1, 2r)$ and we can ensure that $\Psi(Q_2) \subset S_2(Q_1)$.

Thus, $E_\lambda \subset S_2(Q_1)$, and therefore

$$\beta(E_\lambda) \le \beta(S_2(Q_1)) \le C|Q_1|w(a,u) = \frac{C}{\lambda} w(a,u).$$

Assume now that the theorem is true for every finite family of J Dirac deltas, with $J \le N-1$. We now prove it for $J = N$.

In this case, if

$$E_\lambda = \Big\{(x,v) \in \mathbb{R}^n \times V : T\Big(\sum_{k=1}^{N} \delta_{(a_k,u_k)}\Big)(x,v) > \lambda\Big\},$$

where $\{(a_k,u_k)\}_{k=1}^N \subset A$, we can associate with each $(x,v) \in E_\lambda$ a cube $Q_{(x,v)}$ such that

$$\frac{1}{|Q_{(x,v)}|} \sum_{k=1}^{N} \chi_{\Phi(Q_{(x,v)})}(a_k,u_k)\chi_{\Psi(Q_{(x,v)})}(x,v) = \lambda.$$

Then the *size* of $Q_{(x,v)}$ will be $\frac{1}{\lambda}\#\{k : (a_k,u_k) \in \Phi(Q_{(x,v)})\}$.

Let Q_1 be one of biggest size, and assume that it has center $x_1 \in \mathbb{R}^n$ and side length r. For any $(x,v) \in E_\lambda$, if the associated cube $Q_{(x,v)}$ satisfies $\Phi(Q_{(x,v)}) \cap \Phi(Q_1) \neq \emptyset$, then $Q_{(x,v)} \cap Q_1 \neq \emptyset$; so, the center of $Q_{(x,v)}$ belongs to $Q_1^* = Q(x_1, 2r)$, and therefore $\Psi(Q_{(x,v)}) \subset S_2(Q_1)$. We have

$$E_\lambda \subset S_2(Q_1) \cup \Big\{(x,v) \in \mathbb{R}^n \times V : T\Big(\sum_{(a_k,u_k)\notin\Phi(Q_1)} \delta_{(a_k,u_k)}\Big)(x,v) > \lambda\Big\},$$

and thus

$$\begin{aligned}\beta(E_\lambda) &\le \beta(S_2(Q_1)) \\ &\quad + \beta\Big(\Big\{(x,v) \in \mathbb{R}^n \times V : T\Big(\sum_{(a_k,u_k)\notin\Phi(Q_1)} \delta_{(a_k,u_k)}\Big)(x,v) > \lambda\Big\}\Big).\end{aligned}$$

But

$$\beta(\Psi(S_2(Q_1)) \le C|Q_1|w(a_i,u_i) \le \frac{C}{\lambda} \sum_{(a_k,u_k)\in\Phi(Q_1)} w(a_k,u_k),$$

where $w(a_i,u_i) = \inf\{w(a_j,u_j) : (a_j,u_j) \in \Phi(Q_1)\}$.

We may now apply the induction hypothesis:

$$\beta(E_\lambda) \le \frac{C}{\lambda} \sum_{(a_k,u_k)\in\Phi(Q_1)} w(a_k,u_k) + \frac{C}{\lambda} \sum_{(a_k,u_k)\notin\Phi(Q_1)} w(a_k,u_k),$$

and finally,

$$\beta(E_\lambda) \le \frac{C}{\lambda} \sum_{k=1}^{N} w(a_k,u_k).$$

The last theorem can be extended to the fractional maximal operator defined by

$$M_\alpha f(x,v) = \sup_Q \frac{1}{|Q|^{1-\alpha/n}} \int_{\Phi(Q)} |f(y,u)|\,d\gamma(y,u)\chi_{\Psi(Q)}(x,v).$$

The action of M_α over one Dirac delta can be defined by

$$M_\alpha \delta_{(a,u)}(x,v) = \sup_Q \frac{1}{|Q|^{1-\alpha/n}} \chi_{\Phi(Q)}(a,u)\chi_{\Psi(Q)}(x,v).$$

With the same arguments as in Theorem 3, we can easily prove the following

THEOREM 5. *If M_α is the fractional maximal operator just defined*, $0 \le \alpha < n$, $1 \le q < \infty$, *and Φ and Ψ are set functions satisfying conditions* **I**, **II** *and* **III**, *then the following statements are equivalent*:

(i) *M_α is of weak type $(1,q)$ over finite sums of Dirac deltas.*

(ii) *There are m and C such that for every cube Q in $\mathbb{R}^n$*,

$$\frac{\beta(S_m(Q))^{1/q}}{|Q|^{1-\alpha/n}} \le Cw(x,u), \qquad \gamma\text{-a.e. } (x,u)\in\Phi(Q).$$

We may also obtain, as an immediate consequence of this theorem, the discrete version of the characterization obtained in [7] for M_α:

COROLLARY 6. *In the situation of Theorem* 5, *if condition* **II** *is replaced by* **II***, *then statement* (ii) *can be changed to*

(ii*) *There is $C > 0$ such that for every cube Q in $\mathbb{R}^n$*,

$$\frac{\beta(\Psi(Q))^{1/q}}{|Q|^{1-\alpha/n}} \le Cw(x,u), \qquad \gamma\text{-a.e. } (x,u)\in\Phi(Q).$$

Acknowledgments. The author is grateful to the referee for his useful suggestions, which have contributed to clarifying and improving Theorem 3.

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Reçu par la Rédaction le 22.10.1993;
en version modifiée le 5.1.1994 et 10.8.1994