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On open maps of Borel sets

by

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Abstract. We answer in the affirmative [Th. 3 or Corollary 1] the question of L. V. Keldysh [5, p. 648]: can every Borel set X lying in the space of irrational numbers $\mathbb{P}$ not $G_\delta \cdot F_\sigma$ and of the second category in itself be mapped onto an arbitrary analytic set $Y \subset \mathbb{P}$ of the second category in itself by an open map? Note that under a space of the second category in itself Keldysh understood a Baire space. The answer to the question as stated is negative if X is Baire but Y is not Baire.

Introduction. In 1934 Hausdorff proved [3; 2, 4.5.14] that if $f : X \to Y$ is an open map from a completely metrizable space X onto a metrizable Y, then Y is also completely metrizable. Thus, open maps preserve the class G_δ of Borel sets. L. V. Keldysh proved [5, Th. 1] that this result is not true for Borel sets of higher class, namely, that there is a Borel set $X \subset \mathbb{P}$ of the first category for which there is an open map $f : X \to Y$ onto an arbitrary analytic set $Y \subset \mathbb{P}$ (see Theorem 1). In connection with this result a question was raised whether an analogous theorem holds for Baire spaces.

It is clear that if $f : X \to Y$ is an open map and $O \subset Y$ is an open (nonempty) set of the first category, so is $f^{-1}(O)$. Hence, open maps preserve the property of being a Baire space. Let $X_0 \subset \mathbb{P}$ be an analytic set such that $\mathbb{P} \setminus X_0$ does not contain a copy of the Cantor set $\mathbb{C}$. It is not hard to see that X_0 is a Baire space. Keldysh remarked that if Y satisfies the following condition:

(i) $Y \subset \mathbb{P}$ is an analytic set such that $M \setminus Y$ contains a copy of the Cantor set $\mathbb{C}$ for every G_δ-set $M \supset Y$,

then X_0 cannot be mapped onto Y by an open map [5]. Note that every Borel (non-G_δ) set $Y \subset \mathbb{P}$ (and analytic set $Y = X_0 \times \mathbb{P}$ in which every G_δ-subspace is Baire [12, Theorem 4]) satisfies the condition (i).

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All spaces in this paper are assumed to be metrizable, and all maps are continuous and onto. We denote by $\mathbb{P}$ and $\mathbb{Q}$ the spaces of irrational and rational numbers, respectively, and by $B(\tau)$ the Baire space of weight τ (= the Cartesian product of countably many discrete spaces of cardinality $\tau \geqslant \aleph_0$). It is known that every metrizable space X with $\operatorname{Ind} X = 0$ and $w(X) = \tau \geqslant \aleph_0$ can be embedded in $B(\tau)$ (for $\tau = \aleph_0$, $B(\tau) = \mathbb{P}$) [2, Theorem 7.3.15].

A set $Y \subset \mathbb{P}$ is called an *analytic set* (respectively, a *Borel set*) if there exists a map $f : \mathbb{P} \to Y$ (respectively, a one-to-one map $f : M \to Y$, where M is a G_δ-set in $\mathbb{P}$).

The notation $X \hookleftarrow Y$ means that X contains a relatively closed subset which is homeomorphic to Y, the symbol $\approx$ denotes a homeomorphism, and $[A]$ denotes the closure of A.

The space X is called *of the first category* (respectively, *of the second category*) if X can (respectively, cannot) be represented as a countable union of nowhere dense (n.d.) sets in X.

We say that X has a property L *everywhere* if every open subspace $U \subset X$ has property L. The space X is *Baire* iff X is everywhere of the second category. A subset of X is *clopen* if it is both closed and open in X.

We say that a pair of spaces X, Y is *exceptional* if either

(a) X is Baire and Y is not, or

(b) Y is of the first category and X is not.

It is clear that if there is an open map $f : X \to Y$ then X, Y is not an exceptional pair.

The following theorem gives a necessary and sufficient condition on Borel sets $X, Y \subset \mathbb{P}$ for the existence of an open map $g : X \to Y$; it shows that the answer to the Keldysh question [5] is affirmative.

THEOREM 0. *Let $X \subset \mathbb{C}$ be a Borel set*, *$Y \subset \mathbb{C}$ be an analytic set*, *and X be everywhere not $F_\sigma \cup G_\delta$. Then there exists an open map $f : X \to Y$ if and only if X, Y is not an exceptional pair.*

It is not hard to see that Theorem 0 is the sum of Theorems 1–4, and Saint Raymond's theorem [14, Theorem 5]: Let X be a Borel set in $\mathbb{C}$; then X is a union of F_σ and G_δ (in $\mathbb{C}$) iff $X \not\hookleftarrow \mathbb{P} \times \mathbb{Q}$. It can be seen that Theorem 4 is based on Theorems 1 and 2, and Theorem 3 uses Theorem 2, which uses Theorem 1. Lemma 2 and the first step of its proof strengthen the theorem of [10].

Remark. If $X, Y \subset \mathbb{C}$, X contains an open $F_\sigma \cup G_\delta$ (relative to $\mathbb{C}$) and $f : X \to Y$ is an open map then Y also contains an open $F_\sigma \cup G_\delta$ (relative to $\mathbb{C}$).

Indeed, suppose X contains an open $U = X_1 \cup X_2$, where X_1 is F_σ and X_2 is G_δ. It is clear that $f(X_1)$ is F_σ in $\mathbb{C}$, $T = U \setminus f^{-1}(f(X_1))$ is G_δ and $f{\restriction}T$ is an open map. Hence by Hausdorff's theorem $f(T)$ is G_δ in $\mathbb{C}$ and $f(U) = f(T) \cup f(X_1)$ is $F_\sigma \cup G_\delta$.

Note that $\mathbb{C}$ is embeddable in $\mathbb{P}$ and if $X \subset \mathbb{C} \subset \mathbb{P}$ is not $F_\sigma \cap G_\delta$ in $\mathbb{P}$, then X is not $F_\sigma \cap G_\delta$ in $\mathbb{C}$.

We close this section with an example of a Baire space $X \subset \mathbb{C}$ which is $F_\sigma \cup G_\delta$ and everywhere not $F_\sigma \cap G_\delta$. Thus, if Y is any Baire space which is everywhere not $F_\sigma \cup G_\delta$ then X, Y is not an exceptional pair, but no open $f : X \to Y$ exists, showing that the condition on X in Theorem 0 cannot be weakened.

Indeed, let $\mathbb{Q}' \approx \mathbb{Q}$ be a dense subset of $\mathbb{C}$ and $\mathbb{P}' = \mathbb{C} \setminus \mathbb{Q}'$. Let now $X_1 = \mathbb{Q}' \times \mathbb{Q}'$, $X_2 = \mathbb{P}' \times \mathbb{P}'$ and $X = X_1 \cup X_2$. Obviously X is a Baire space and $F_\sigma \cup G_\delta$ in $\mathbb{C}$, and X is everywhere not $F_\sigma \cup G_\delta$ since every $F_\sigma \cap G_\delta$ in $\mathbb{C}$ which is everywhere not F_σ and everywhere not G_δ is homeomorphic to $\mathbb{P} \times \mathbb{Q}$ (see [8], [13]) and we have a contradiction to the Baire Category Theorem. (Notice that X is homeomorphic to the space T, which has been characterized by van Douwen [1, Theorem 2.3].)

1. Main theorems.

1. Main theorems. The proofs of Theorems 1–4 use Lemmas 2–5 of Section 3, which use Proposition 0 of Section 2.

THEOREM 1. *Let $X, Y \subset \mathbb{P}$ be analytic sets, and X be a space of the first category and everywhere not a σ-compact space. Then there exists an open map $g : X \to Y$.*

R e m a r k. If $X \subset \mathbb{P}$ is an analytic set and X is not a σ-compact space, then $X \hookleftarrow \mathbb{P}$ and for every analytic set $Y \subset \mathbb{P}$ there exists a map $f : X \to Y$ [4; 6, §39; 10, Corollary 2].

P r o o f o f T h e o r e m 1. According to the above remark, if $U \subset X$ is an open set then $U \hookleftarrow \mathbb{P}$. Since $\mathbb{P} \times \mathbb{P} \approx \mathbb{P}$ we may suppose that $\mathbb{P}$ is n.d. in X. Let $X = \bigcup F_i'$, where F_i' is closed n.d. in X ($i \in \omega$). Obviously, for every F_i' there exists a sequence of closed n.d. (in X) subsets $P_{i,j} \approx \mathbb{P}$ such that

$$F_i' = \Big[\bigcup\{P_{i,j} : j \in \omega\}\Big]_X \setminus \bigcup\{P_{i,j} : j \in \omega\}, \tag{A}$$

and every set

$$F_i = F_i' \cup \bigcup\{P_{i,j} : j \in \omega\} \tag{B}$$

is closed n.d. in X. The reader can easily verify that every F_i is an analytic set and no open $U \subset F_i$ is σ-compact. By the above remark, it remains to apply Lemma 2. ∎

THEOREM 2. *Let $X, Y \subset \mathbb{P}$ be analytic, Baire spaces and everywhere $X \hookleftarrow \mathbb{P} \times \mathbb{Q}$. Then there exists an open map $g : X \to Y$.*

Proof. Every analytic set X can be represented as $X_1 \cup X_2$, where X_2 is a G_δ-set in $\mathbb{P}$ and X_1 is of the first category in X [6, §11]. Since X is everywhere of the second category, $[X_2] = X$ and if $U = X \setminus [X_1] \neq \emptyset$ then $U \subset X_2$ is a G_δ-set in $\mathbb{P}$ and $U \hookleftarrow \mathbb{P} \times \mathbb{Q}$. This implies that $U \hookleftarrow \mathbb{Q}$, which contradicts the Baire Category Theorem. Hence $[X_1] = X$.

Analogously $Y = Y_1 \cup Y_2$, where Y_2 is a G_δ-set in $\mathbb{P}$, Y_1 is of the first category, and $[Y_2] = Y$. We may suppose that $[Y_1] = Y$. Indeed, it is clear that there is an open map (projection) $\pi : \mathbb{P} \times Y \to Y$, hence we may consider $\mathbb{P} \times Y$ instead of Y. Taking a dense countable subset $\mathbb{Q}' \approx \mathbb{Q}$ in $\mathbb{P} \times Y_2$ we set

$$Y_1' = (\mathbb{P} \times Y_1) \cup \mathbb{Q}' \quad \text{and} \quad Y_2' = (\mathbb{P} \times Y_2) \setminus \mathbb{Q}'.$$

It is clear that $\mathbb{P} \times Y = Y_1' \cup Y_2'$, where Y_1' is a dense subset of the first category in $\mathbb{P} \times Y$, and Y_2' is a G_δ-set (in $\mathbb{P} \times \mathbb{P} \approx \mathbb{P}$) dense in $\mathbb{P} \times Y$.

Represent X_2 as

$$X_2 = \bigcap \{O_i : i \in \omega\}, \qquad \text{with each } O_i \text{ open in } \mathbb{P},$$

and let $F_i' = X \setminus O_i$. Similarly to the proof of Theorem 1 (see (A) and (B)) one defines closed n.d. sets

$$F_i = F_i' \cup \bigcup \{P_{i,j} : j \in \omega\}, \tag{B$'$}$$

where the $P_{i,j} \approx \mathbb{P} \times \mathbb{Q}$ are closed n.d. sets. Clearly, F_i is a subspace of the first category, and for a relatively open set $U \subset F_i$ we have $U \hookleftarrow \mathbb{P} \times \mathbb{Q}$. Hence U is not σ-compact and of the first category. By Theorem 1 there exists an open map of U onto every nonempty analytic set in Y. It is clear that X is everywhere not compact and the conditions (i)–(iv) of Lemma 3 hold, hence by Lemma 3 one obtains the assertion. ∎

LEMMA 1. *Let $X \subset \mathbb{P}$ be a space which is not a Baire space and not of the first category. Then $X = T \cup X_1$, where $X_1 = X \setminus T$ is an open space of the first category, $T = F \cup X_2$ is a closed Baire subspace, F is a closed n.d. set in T and in $F \cup X_1$, $X_2 = T \setminus F$ is an open (in X) Baire space.*

Proof. Define X_1 as the union of all open subsets of X of the first category. Since X is Lindelöf, X_1 is a maximal open subspace of the first category. Put $T = X \setminus X_1$ and $F = [X_1] \setminus X_1$. It is clear that F is n.d. in $[X_1]$, hence in X. Obviously, $X_2 = X \setminus [X_1] = T \setminus F \neq \emptyset$, otherwise X would be of the first category. The subspace T is everywhere of the second category, because there exists no open $V \subset X$ such that $V \cap T \neq \emptyset$ is of the first category in T (otherwise, as $V \cap X_1$ is of the first category, X_1 would not be maximal). Obviously, X_2 is dense in $T = F \cup X_2$, since $[X_1] = X_1 \cup F$ is of the first category in X and if there is an open $V \subset X$ with $\emptyset \neq V \cap T \subset F$, then $V \subset [X_1]$ is of the first category in X, $V \not\subset X_1$ and again X_1 would not be maximal. ∎

THEOREM 3. *Let X, Y be analytic sets in $\mathbb{P}$, Y be a Baire space and everywhere $X \hookleftarrow \mathbb{P} \times \mathbb{Q}$. Then there exists an open map $g : X \to Y$.*

Proof. Let X be a space of the first category. Since everywhere $X \hookleftarrow \mathbb{P} \times \mathbb{Q}$, X is everywhere not σ-compact by the Baire Category Theorem. Now we apply Theorem 1.

If X is Baire we apply Theorem 2.

In the third case, by Lemma 1, every open $U \subset X \setminus T$ satisfies the conditions of Theorem 1 and therefore can be mapped onto every open set $V \subset Y$, and T is a closed subspace satisfying the conditions of Theorem 2, hence there exists an open map $f : T \to Y$. By Lemma 4, there is an open extension $g : X \to Y$ of f. ∎

COROLLARY 1. *Let $X \subset \mathbb{C}$ be a Borel set everywhere not $F_\sigma \cup G_\delta$. Then for every analytic Baire space $Y \subset \mathbb{C}$ there exists an open map $g : X \to Y$.*

This follows from Theorem 3, since by the Saint Raymond's Theorem [14, Theorem 5; 7, Corollary 17] for every Borel not $F_\sigma \cup G_\delta$-set $U \subset X$ we have $U \hookleftarrow \mathbb{P} \times \mathbb{Q}$.

THEOREM 4. *Suppose that X, Y are analytic sets in $\mathbb{P}$ of the second category and everywhere $X \hookleftarrow \mathbb{P} \times \mathbb{Q}$. Suppose that X and Y contain open* (*nonempty*) *subsets of the first category. Then there exists an open map $g : X \to Y$.*

Proof. By Lemma 1, $X = X_1 \cup F^X \cup X_2$, where X_1 is an open subspace of X of the first category, F^X is n.d. in $X_1 \cup F^X$ and X_2 is an open (in X) Baire subspace such that F^X is a n.d. set in $F^X \cup X_2$ ($X_1 \cap X_2 = \emptyset$). Analogously we have $Y = Y_1 \cup F^Y \cup Y_2$ with the same properties. Similarly to the proofs of Theorems 1 and 2 (see (B′) and (B)) we take a closed n.d. set $F_0^X \supset F^X$ such that there exists an open map $f : F_0^X \to F^Y$. By Lemma 5 and Theorems 1 and 2, there exist open extensions $g_1 : F_0^X \cup X_1 \to F^Y \cup Y_1$ and $g_2 : F_0^X \cup X_2 \to F^Y \cup Y_2$ of f. Then it is easy to see that $g = g_1 \cup g_2 : X \to Y$ is an open extension of f. ∎

2. Terminology and basic facts

1.0. We denote by $A^{<\omega}$ the set of all finite sequences $u = \langle u(0), \ldots, u(k)\rangle$ of elements of A; $\emptyset$ denotes the empty sequence. The number $|u| = k+1$ is called the *length* of u; define $|\emptyset| = 0$.

If $u, v \in A^{<\omega}$, then $u^\frown v$ is the concatenation of the two sequences, i.e.

$$\langle u(0), \ldots, u(k)\rangle^\frown\langle v(0), \ldots, v(m)\rangle = \langle u(0), \ldots, u(k), v(0), \ldots, v(m)\rangle.$$

Of course, $u^\frown\emptyset = u$. The notation $s \subset t$ means that t extends s, i.e. that s is an initial segment of t and $s \neq t$.

A *tree* T on A is a subset of $A^{<\omega}$ such that $s \in T$ and $t \subset s \to t \in T$. If $t \subset s$ and $|t| + 1 = |s|$, we write $s = t^+$.

1.1. Let $T \subset B(\tau)$. A $\gamma(T)$-*system* is a family of open (in $T = T_\emptyset$) subsets T_s, indexed by some tree S satisfying the conditions:

(a) $\bigcup\{T_{s^+} : s^+ \in S\} = T_s$;
(b) $\operatorname{diam} T_s \to 0$ as $|s| \to \infty$.

1.2. A $\gamma^*(T)$-*system* is a $\gamma(T)$-system with the additional condition:

(c) for every fixed $n = 1, 2, \ldots$ the sets T_s, $|s| = n$, are pairwise disjoint clopen sets.

Obviously, for every set $T \subset B(\tau)$ there exists a $\gamma^*(T)$-system.

2.1. Let now $T \subset X \subset B(\tau)$. A $\delta(T)$-*extension* of a $\gamma(T)$-system $\{T_s\}$ to X is a family of open sets X_s in X ($X_\emptyset = X$) such that

(d) $X_s \cap T = T_s$;
(e) $\operatorname{diam} X_s \to 0$ as $|s| \to \infty$;
(f) the sets

$$Z_s = X_s \setminus \bigcup\{X_{s^+} : s^+ \in S\}$$

are open in X;
(f$_1$) $X_s \setminus T = \bigcup\{Z_t : t \supseteq s, t \in S\}$;
(g) if T is a nowhere dense subset of X then the sets Z_s are nonempty.

2.2. A $\delta^*(T)$-*extension* is a $\delta(T)$-extension with the following additional condition:

(h) the sets X_s are pairwise disjoint for every fixed $|s| = n$ and the sets Z_s are pairwise disjoint and clopen in X.

Proposition 0. *Let T be a closed subset of $X \subset B(\tau)$. Then every $\gamma(T)$-system* (*respectively*, *$\gamma^*(T)$-system*) *has a $\delta(T)$-extension* (*respectively*, *$\delta^*(T)$-extension*).

Proof. Let S be the tree indexing the given $\gamma(T)$-system.

Fix $s \in S$, and suppose X_s has already been constructed (we take $X_\emptyset = X$). If T is a n.d. set, also take some $x_0 \in X_s \setminus T$. For each $s^+ \in S$ take an open set $O_{s^+} \subset X_s$ (with $x_0 \notin O_{s^+}$ if T is n.d.) such that $O_{s^+} \cap T = T_{s^+}$. For every $x \in X_s \setminus T$ take a neighbourhood $O(x) \subset X_s$ such that $O(x) \cap T = \emptyset$. The cover

$$\{O_{s^+}, O(x) : x \in X_s \setminus T,\ s^+ \in S\}$$

of X_s has a refinement $\lambda = \{U_\alpha : \alpha \in A\}$, where the U_α are clopen in X and pairwise disjoint. Put

$$Z_s = \bigcup\{U_\alpha \in \lambda : U_\alpha \cap T_s = \emptyset\}.$$

It is easy to see that

$$(*)\qquad Z_s = X_s \setminus V_s,$$

where

$$(**)\qquad V_s = \bigcup\{U_\alpha \in \lambda : U_\alpha \cap T_s \neq \emptyset\} \subset \bigcup\{O_{s^+} : s^+ \in S\}.$$

Define

$$X_{s^+} = O_{s^+} \cap V_s.$$

Then by $(*)$ and $(**)$,

$$Z_s = X_s \setminus \bigcup\{O_{s^+} \cap V_s : s^+ \in S\} = X_s \setminus \bigcup\{X_{s^+} : s^+ \in S\}.$$

Obviously, we have (d) (for s^+). We can get condition (e) to be also satisfied, choosing the sets O_s in a proper way and taking into account (b).

Conditions (f), (f_1) hold by the construction, and (g) follows from the fact that $x_0 \in Z_s$.

So, we have proved the existence of the required $\delta(T)$-extension. In the case of a $\gamma^*(T)$-system we consider for every $x \in T_{s^+}$ a neighbourhood $O(x)$ such that $O(x) \cap T \subset T_{s^+}$ instead of the set O_{s^+}, and choose $O(x)$ for $x \in X_s \setminus T$ and a refinement λ as above. Then put

$$X_{s^+} = \bigcup\{U_\alpha \in \lambda : U_\alpha \cap T \subset T_{s^+}\}.$$

It is clear that we have (h). ∎

3. Principal lemmas. A map $f : X \to Y$ is called *open* at $x \in X$ if there is a base $\mathcal{B}$ for X at x such that $\{f(U) : U \in \mathcal{B}\}$ is a base for Y at $f(x)$.

LEMMA 2. *Suppose $X, Y \subset B(\tau)$ and $X = \bigcup_{i\in\omega} F_i$, where the F_i are closed nowhere dense sets such that for every nonempty clopen* (*relative to F_i*) *set $V \subset F_i$ $(i \in \omega)$ and every nonempty clopen set $U \subset Y$ there exists a map $f : V \to U$. Then there exists an open map $g : X \to Y$.*

Proof. The proof is by induction. We will define a tree H, and for each $h \in H$ closed sets $F_h \subset$ some F_i, clopen subsets $O_h \subset X$, $Y_h \subset Y$, maps $g_h : F_h \to Y_h$, and trees H_h such that $h^\frown\langle v\rangle \in H$ if and only if $v \in H_h$. Always, h_n will denote an element of H of length n.

At the first step put $F_\emptyset = F_0$, $O_\emptyset = X$, $Y_\emptyset = Y$ and consider a map $g_\emptyset : F_\emptyset \to Y_\emptyset$. Suppose that we have already constructed $F_{h_n}, O_{h_n}, Y_{h_n}$, and g_{h_n}, and H_h for all $h \subset h_n$. Take a $\gamma^*(Y_{h_n})$-system. It is clear that there exist a tree H_h and a $\gamma^*(F_{h_n})$-system $\{T_{h_n{}^\frown\langle v\rangle} : v \in H_h\}$ such that for each $v \in H_h$ there is some $Y_{h_n{}^\frown\langle v\rangle} \in \gamma^*(Y_{h_n})$ with $T_{h_n{}^\frown\langle v\rangle} \subset g_{h_n}^{-1}(Y_{h_n{}^\frown\langle v\rangle})$. By Proposition 0, let $\{X_{h_n{}^\frown\langle v\rangle} : v \in H_h\}$ be a $\delta^*(F_{h_n})$-extension of $\gamma^*(F_{h_n})$ in $O_{h_n} = X_{h_n{}^\frown\langle\emptyset\rangle}$. Fix $v \in H_h$, and put $O_{h_n{}^\frown\langle v\rangle} = Z_{h_n{}^\frown\langle v\rangle}$, where $Z_{h_n{}^\frown\langle v\rangle}$ is as in 2.1(f). Let $F_{h_n{}^\frown\langle v\rangle}$ be the first nonempty intersection of $O_{h_n{}^\frown\langle v\rangle}$ with the sets F_i $(i \in \omega)$. Then by our condition there is a map $g_{h_n{}^\frown\langle v\rangle} : F_{h_n{}^\frown\langle v\rangle} \to Y_{h_n{}^\frown\langle v\rangle}$. We may put $h_n{}^\frown\langle v\rangle = h_{n+1} \in H$ and define the map $g : X \to Y$ as follows: $g\restriction F_h = g_h$ for all $h \in H$.

Fact. $g(X_{h_{n+1}}) = g(F_{h_{n+1}}) = Y_{h_{n+1}}$.

Indeed, by construction

$$X_{h_{n+1}} = T_{h_{n+1}} \cup \bigcup\{F_{p_{n+k+1}} : F_{p_{n+k+1}} \subset X_{h_{n+1}},\ p_{n+k+1} \in H,\ k \in \omega\}$$

and for every $F_{p_{n+k+1}} \subset X_{h_{n+1}}$ $(k > 0)$ there is $F_{p_{n+k}}$ $(h_n \subset p_{n+k} \subset p_{n+k+1})$ such that $g(F_{p_{n+k}}) \supset g(F_{p_{n+k+1}})$, hence

$$g(X_{h_{n+1}}) = g(T_{h_{n+1}}) \cup \bigcup\{g(F_{p_{n+1}}) : F_{p_{n+1}} \subset X_{h_{n+1}}\}.$$

It remains to remark that $g(T_{h_{n+1}}) \subset g(F_{h_{n+1}})$ and for every $F_{p_{n+1}} \subset X_{h_{n+1}}$ we have $g(F_{p_{n+1}}) \subset Y_{h_{n+1}} = g(F_{h_{n+1}})$.

Now, if $x \in X$, then $x \in F_{h_n}$ for some F_{h_n}. Let $\mathcal{B} = \{X_{h_{n+1}} : x \in X_{h_{n+1}}\}$. By our construction, $\mathcal{B}$ is a base at x and by the fact above g is an open map at x. ■

Lemma 3. *Let $X, Y \subset B(\tau)$, and let the following conditions for X and analogous conditions for Y hold*:

(i) *there exist open sets U_i^X $(i \in \omega)$ in $B(\tau)$ such that the set $G_\delta^X = \bigcap\{U_i^X : i \in \omega\}$ is dense in X (and $G_\delta^X \subset X$)*;

(ii) *the set $F_\sigma^X = X \setminus G_\delta^X$ is dense in X*;

(iii) *for all sets $U \subset F_i^X$ and $V \subset F_j^Y$ clopen relative to $F_i^X = X \setminus U_i^X$ and $F_j^Y = Y \setminus U_j^Y$, respectively, there exists an open map $f : U \to V$ $(i, j \in \omega)$*;

(iv) *for all clopen sets $O \subset X$, $W \subset Y$ and each refinement $\lambda(W)$ of W consisting of clopen (in Y) pairwise disjoint sets, there is a refinement $\lambda(O)$ of O consisting of clopen (in X) pairwise disjoint sets such that the cardinality of $\lambda(O)$ is greater than or equal to the cardinality of $\lambda(W)$.*

Then there exists an open map $g : X \to Y$.

Proof. The proof is by an inductive process similar to that used to establish Lemma 2.

At step 0 consider an open map $f_0 : F_0^X \to F_0^Y$ (where $F_0^X = X\setminus U_0^X$, $F_0^Y = Y\setminus U_0^Y$) and put $O_0^X = X$, $O_0^Y = Y$. Obviously, we may suppose that $\operatorname{diam} O_0^X < 1$ and $\operatorname{diam} O_0^Y < 1$.

Suppose we obtained at step n clopen sets $O^X_{t_n}$ and corresponding sets $O^Y_{t(t_n)}$ with the following properties:

(a) $\operatorname{diam} O^X_{t_n} < 1/2^n$ and $\operatorname{diam} O^Y_{t(t_n)} < 1/2^n$;

(b) $[O^X_{t_n}] \subset U^X_{n-1}$ and $[O^Y_{t(t_n)}] \subset U^Y_{n-1}$ ($U^X_{-1} = U^Y_{-1} = B(\tau)$).

We also obtained closed n.d. sets

$$F^X_{t_n} \subset O^X_{t_n}, \qquad F^Y_{t(t_n)} \subset O^Y_{t(t_n)}$$

and maps

$$f_{t_n} : F^X_{t_n} \to F^Y_{t(t_n)}.$$

Consider some $\gamma^*(F^X_{t_n})$-system $\{T^X_s : s \in S\}$ ($T_\emptyset = F^X_{t_n}$) and its extension $\delta^*(F^X_{t_n}) = \{X_s : s \in S\}$ ($X_\emptyset = O^X_{t_n}$). Using the open sets $f_{t_n}(T^X_{t_n})$ we construct the $\gamma(F^Y_{t(t_n)})$-system $\{T^Y_s = f_{t_n}(T^X_{t_n}) : s \in S\}$ and an extension $\delta(F^Y_{t(t_n)}) = \{Y_s : s \in S\}$ to Y.

It is well known (see the proof of Theorem 7.3.15 in [2]) that for a given $\varepsilon > 0$ every open cover of the open subset $Z^X_s \subset X$ has a refinement consisting of clopen (in X) pairwise disjoint sets of diameter less than ε. Then, by (iv) we may suppose that $Z^X_s = \bigcup \lambda^X_s$ and $Z^Y_s = \bigcup \lambda^Y_s$, where λ^X_s and λ^Y_s are families of clopen pairwise disjoint sets of diameter less than $1/2^{n+1}$ and there is a surjection $t : \lambda^X_s \to \lambda^Y_s$. Denote by $\tau^X_{n+1} = \{O^X_{t_{n+1}}\}$ and $\tau^Y_{n+1} = \{O^Y_{t_{n+1}}\}$ the families of elements of all the obtained families λ^X_s and λ^Y_s . Choosing the sets $O(x)$ in the proof of Proposition 0 such that $[O(x)] \subset U^X_n$ we may suppose that $O^X_{t_{n+1}} \subset U^X_n$. Let $F^X_{t_{n+1}}$ be the first nonempty intersection of $O^X_{t_{n+1}}$ with F_i. By analogy we construct the sets $F^Y_{t(t_{n+1})}$ in Y and obtain open maps

$$f_{t_{n+1}} : F^X_{t_{n+1}} \to F^Y_{t(t_{n+1})}.$$

Now we define $g_\sigma : F^X_\sigma \to F^Y_\sigma$ and $g_\delta : G^X_\delta \to G^Y_\delta$. By definition, $g_\sigma \restriction F^X_{t_n} = f_{t_n}$ ($t_0 = 0$) . It is clear that $g_\sigma : F^X_\sigma \to F^Y_\sigma$ is a surjective map. Note that by the construction we have

$$g_\sigma(O^X_{t_n} \cap F^X_\sigma) = O^Y_{t(t_n)} \cap F^Y_\sigma . \tag{1}$$

In order to define $g_\delta : G^X_\delta \to G^Y_\delta$ first note that by (a) and (b) for every sequence

$$O^X_{t_1} \supset \ldots \supset O^X_{t_n} \tag{2}$$

there is some

$$x \in \bigcap_{t_n} O^X_{t_n} = \bigcap_{t_n} [O^X_{t_n}] \subset G^X_\delta .$$

Conversely, for every $x \in G^X_\delta$ there is a sequence (2). For $x \in G^X_\delta$ defined by (2) put $g_\delta(x) = y = \bigcap_{t(t_n)} O^Y_{t(t_n)} \subset G^Y_\delta$. By (a) and (b) we obtain a surjection $g_\delta : G_\delta^X \to G_\delta^Y$, since, by our construction, for every $y \in G_\delta^Y$ there is a sequence $O_{t(t_1)}^Y \supset \ldots \supset O_{t(t_n)}^Y$ (containing y), and, hence, for the x defined by (2), we have $g_\delta(x) = y$ and

$$g_\delta(O_{t_n}^X \cap G_\delta^X) = O_{t(t_n)}^Y \cap G_\delta^Y. \tag{3}$$

It remains to define $g : X \to Y$ as follows:

$$g(x) = \begin{cases} g_\sigma(x) & \text{if } x \in F_\sigma^X, \\ g_\delta(x) & \text{if } x \in G_\delta^X. \end{cases}$$

Fact. *The surjection $g : X \to Y$ is an open continuous map from X onto Y.*

Indeed, by (1) and (3) we have

$$g(O_{t_n}^X) = O_{t(t_n)}^Y.$$

Let $x \in G_\delta^X$. Take for x the sequence (2) of sets $O_{t_n}^X \ni x$. Obviously, they constitute a base at x, and the $O_{t(t_n)}^Y$ are a base at $g(x)$, hence $g : X \to Y$ is an open continuous map.

Let $x \in F_\sigma^X$; hence, for some $F_{t_n}^X$, $x \in F_{t_n}^X$. Let $\mathcal{B} = \{X_s : x \in X_s\}$ be a base at x, where X_s is constructed as above. Since

$$X_s = F_{t_n}^X \cup \bigcup \{Z_p^X : Z_p^X \subset X_s,\ p \in S\}$$

and every Z_p^X is the union of some $O_{t_{n+1}}^X$ for which

$$g(O_{t_{n+1}}^X) = O_{t(t_{n+1})}^Y \subset Z_p^Y$$

we see that $g(X_s) = Y_s$, hence g is an open map at x. ■

Lemma 4. *Let X, Y be metric spaces, T be closed in X, $\dim X = 0$ and for all nonempty open sets $U \subset X \setminus T$, $V \subset Y$, there exists an open map $\varphi : U \to V$. Then every open map $f : T \to Y$ can be extended to an open map $g : X \to Y$.*

Proof. According to Section 2, consider a $\gamma^*(T)$-system and its $\delta^*(T)$-extension in X. If $Z_s \neq \emptyset$ then there exists an open map $\varphi_s : Z_s \to f(T_s)$. Let $g\restriction Z_s = \varphi_s$ and $g\restriction T = f$.

Obviously, we thus obtain a map $g : X \to Y$ which is open at every $x \in X \setminus T$. The sets X_s containing $x \in T$ constitute a base at x. By (e) and (f_1) of Section 2 the sets $g(X_s)$ constitute a base at $g(x)$, and g is continuous and open. ■

Lemma 5. *Let $T^X \subset X \subset B(\tau)$ and $T^Y \subset Y \subset B(\tau)$ be closed n.d. sets in X and Y, and $f : T^X \to T^Y$ be an open map. Suppose that for every (nonempty) open $V \subset X \setminus T^X$ and $U \subset Y \setminus T^Y$ there exists an open map $\varphi : V \to U$. Then f has an open extension $g : X \to Y$ over X.*

P r o o f. The proof is, to some extent, similar to the proof of Lemma 3 or Lemma 4. (Define $g : X \to Y$ as follows: $g{\restriction}T^X = f$, $g(Z_s^X) = Z_s^Y$, where $g{\restriction}Z_s^X$ are open maps of sets chosen as at the beginning of the proof of Lemma 3.)

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