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Asymptotic expansion of solutions of Laplace–Beltrami type singular operators

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Abstract. The theory of Mellin analytic functionals with unbounded carrier is developed. The generalized Mellin transform for such functionals is defined and applied to solve the Laplace–Beltrami type singular equations on a hyperbolic space. Then the asymptotic expansion of solutions is found.

0. Introduction. This paper may be regarded as a sequel and correction of [1], and we use similar notations.

In Section 2 we define directly the space of Mellin analytic functionals with not necessarily bounded carrier, without using the notion of Fourier analytic functionals.

Section 3 contains the definition of the generalized Mellin transform of a Mellin analytic functional as some Fourier analytic functional. It is shown in Theorem 2 that if the carrier of a Mellin analytic functional is compact then its generalized Mellin transform is the boundary value (in some special sense) of its ordinary Mellin transform.

In the next section we prove two Paley–Wiener type theorems for the Mellin transform of Mellin analytic functionals. In the proof of Lemma 5 we use estimates similar to those used in the proof of Theorem 3 in [1] but in a corrected form.

In Section 6 we apply the theory of the Mellin transform of Mellin analytic functionals to solve the equation $Pu = f$, where P is a Laplace–Beltrami type operator. We find a solution in the space of Mellin analytic functionals, by a method similar to that used in Section 7 in [1]. The estimate of F_j in [1], p. 274, is incorrect, because the "constant" A is not constant (depends on $\operatorname{Re} z_2$).

Here we find a correct but slightly worse estimate; thus the conclusion on the Laplace–Beltrami operator in Section 8 of [1] is not true. The main result

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of this section (and of the paper) is the asymptotic expansion of solution presented in Corollary 2.

1. Notation. We use the following notation: if $\theta \in [0,\pi)$ and $t \in \mathbb{R}_+$, then $I(t,\theta) = \{w \in \mathbb{C} : 0 \le |w| \le t, \operatorname{Arg} w \le \theta\}$, and $R(\theta) = \{w \in \mathbb{C} : |\operatorname{Arg} w| \le \theta\}$. We denote by $\overline{I(t,\theta)}$ and $\overline{R(\theta)}$ the directional compactification of $I(t,\theta)$ at 0 and of $R(\theta)$ at 0 and at $\{\infty\}$. We use the uniform notation $I(t,\theta)$ for $t \in \mathbb{R}_+ \cup \{\infty\}$ so that $I(\infty,\theta) = R(\theta)$.

$\mathcal{S}^\theta$ stands for the family of sectorial neighbourhoods of $\overline{I(t,\theta)}$ defined in the following way: $U \in \mathcal{S}^\theta$ iff there exist $\alpha,\beta \in (\theta,\pi)$ and a complex (bounded in the case $t < \infty$) neighbourhood V (in $\mathbb{C}$) of $\overline{I(t,\theta)}$ such that

$$U = V \cap (\{-\alpha < \operatorname{Arg} w < \beta\} \cup \{0\}).$$

Now, for $\theta = (\theta_1,\ldots,\theta_n)$, $\theta_j \in [0,\pi)$ and $t = (t_1,\ldots,t_n)$, $t_j \in \mathbb{R}_+ \cup \{\infty\}$, we define $I(t,\theta) = I(t_1,\theta_1) \times \ldots \times I(t_n,\theta_n)$ and $U \in \mathcal{S}^\theta$ iff $U = U_1 \times \ldots \times U_n$ with $U_j \in \mathcal{S}^{\theta_j}$.

We define $\Gamma_\sigma = \{x \in \mathbb{R}^n : \sigma x > 0\}$ for $\sigma = (\sigma_1,\ldots,\sigma_n)$, $\sigma_j \in \{-,+\}$.

2. Mellin analytic functionals. Let $b \in \mathbb{R}^n$, $t \in (\mathbb{R}_+ \cup \{\infty\})^n$ and $V \in \mathcal{S}^\theta$ be a sectorial neighbourhood of $\overline{I(t,\theta)}$.

DEFINITION 1. $F \in \widetilde{\mathcal{M}}_{(b)}(V)$ iff $F \in \mathcal{O}(V\#\{0\})$ and for every $\varepsilon > 0$ and every compact $K \subset V$, there exists a constant $C(\varepsilon,K)$ such that

$$|w^{-b}F(w)| \le C(\varepsilon,K)\exp(\varepsilon|\ln|w||) \quad \text{for } w \in K\#\{0\}.$$

DEFINITION 2. $\phi \in \underset{\sim}{\mathcal{M}}_{(b)}(V,\delta)$ iff $\phi \in \mathcal{O}(V\#\{0\})$ and for every compact $K \subset V$ there exists a constant C_K such that

$$|w^{b+1}\phi(w)| \le C_K \exp(-\delta|\ln|w||) \quad \text{for } w \in K.$$

Note that

$$\exp(-\delta|\ln|w||) = \begin{cases} |w|^{-\delta} & \text{if } |w| > 1,\\ |w|^{\delta} & \text{if } |w| < 1.\end{cases}$$

Now we define the space of test functions for Mellin analytic functionals:

DEFINITION 3.

$$\underset{\sim}{\mathcal{M}}_{(b)}(\overline{I(t,\theta)}) = \lim\{\underset{\sim}{\mathcal{M}}_{(b)}(V,\delta) : V \in \mathcal{S}^\theta,\ \delta > 0\}.$$

It is easy to see that the space $\underset{\sim}{\mathcal{M}}_{(b)}(\overline{I(t,\theta)})$ is endowed with the topology defined by some increasing sequence of seminorms, so we can consider the space $\underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)})$, dual to $\underset{\sim}{\mathcal{M}}_{(b)}(\overline{I(t,\theta)})$.

DEFINITION 4. The functionals in $\underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)})$, are called *Mellin analytic functionals* with carrier $\overline{I(t,\theta)}$.

Under the notation $\mu(\zeta) = e^{-\zeta}$, the following lemma is obvious (see [1]):

LEMMA 1. (a) $F \in \widetilde{\mathcal{M}}_{(b)}(V) \Leftrightarrow e^{b\zeta}(F\circ\mu) \in \widetilde{\mathcal{O}}^\theta(\mu^{-1}(V))$.

(b) $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)}) \Leftrightarrow e^{b\zeta}(f\circ\mu) \in [\underset{\sim}{\mathcal{O}}^\theta(\overline{\mu^{-1}(I(t,\theta))})]'$.

The next theorem gives a characterization of Mellin analytic functionals.

THEOREM 1. $\underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)}) \approx \widetilde{\mathcal{M}}_{(b)}(V\#\overline{I(t,\theta)}) \,/\, \sum_{j=1}^n \widetilde{\mathcal{M}}_{(b)}(V\#_j\overline{I(t,\theta)})$ *for every* $V \in \mathcal{S}^\theta$.

Proof. The isomorphism is given as follows: if $F \in \widetilde{\mathcal{M}}_{(b)}(V\#\overline{I(t,\theta)})$ then for $\phi \in \underset{\sim}{\mathcal{M}}_{(b)}(\overline{I(t,\theta)})$ we define

$$f[\phi] = (2\pi i)^{-n}\int_\gamma F(w)\phi(w)\,dw, \tag{1}$$

where $\gamma = \gamma_1 \times \ldots \times \gamma_n$, γ_j is an arbitrary curve in the domain of ϕ, encircling $\overline{I(t_j,\theta_j)}$ in the counterclockwise direction. If $t_j = \infty$, then $\gamma_j = \gamma_j^+ \cup \gamma_j^-$, and $\gamma_j^\pm \subset \{w_j \in \mathbb{C} : \theta_j < \pm\operatorname{Arg} w_j < \theta'_j\}$ for some $\theta'_j \in (\theta_j,\pi)$.

The inverse map will be defined below. First we shall prove the following two lemmas:

LEMMA 2. *Assume* $t = \infty$, $\theta \in [0,\pi)$, $z \in \mathbb{C} \setminus \overline{R(\theta)}$ *and* $f \in \underset{\sim}{\mathcal{M}}'_{(0)}(\overline{R(\theta)})$. *Define*

$$g_z(u) = (u(\ln z - \ln u))^{-1}\exp(-(\ln z - \ln u)^2), \tag{2}$$

$$F(z) = f[g_z]. \tag{3}$$

Then $g_z \in \underset{\sim}{\mathcal{M}}_{(0)}(\overline{R(\theta)})$ *and* $F \in \widetilde{\mathcal{M}}_{(0)}(V \setminus \overline{R(\theta)})$ *for every* $V \in \mathcal{S}^\theta$.

Proof. For fixed $z \notin \overline{R(\theta)}$ let $U \in \mathcal{S}^\theta$ be such that $z \notin \overline{U}$. Then for $u \in U$ we have

$$|\ln z - \ln u| \ge |\operatorname{Arg} z - \operatorname{Arg} u| > 0,$$

and

$$\begin{aligned}|\exp(-(\ln z - \ln u)^2)| &= \exp(-\operatorname{Re}(\ln z - \ln u)^2)\\ &= \exp((\operatorname{Arg} z - \operatorname{Arg} u)^2 - (\ln|z| - \ln|u|)^2)\\ &\le C(z)\exp(2|\ln|z||\,|\ln|u|| - (\ln|u|)^2)\\ &\le C'(z)\exp(-\delta|\ln|u||) \quad \text{for every } \delta > 0.\end{aligned}$$

Hence, we have

$$|ug_z(u)| \le C'(z)\exp(-\delta|\ln|u||).$$

The holomorphy of F (in (3)) is obvious. For $z \in V \setminus \overline{R(\theta)}$ and $U \in \mathcal{S}^\theta$ such that $z \notin \overline{U}$, it follows from the continuity of f that there exists a constant C_U such that for every $\delta > 0$,

$$|F(z)| \le C_U \sup_{u\in U}(|\ln z - \ln u|)^{-1}|\exp(-(\ln z - \ln u)^2)|\exp(\delta|\ln|u||).$$

Set $a = |\operatorname{Arg} z|$. Then we have

$$\begin{aligned}|F(z)| &\le C_U|a-\theta|^{-1}\exp((\operatorname{Arg} z)^2 - (\ln z)^2)\\ &\quad\times \sup_{u\in U}\exp(-(\ln|u|)^2 + (2|\ln|z|| + \delta)|\ln|u||)\\ &\le C_U|a-\theta|^{-1}\exp(a^2)\exp(-(\ln|z|)^2 + (((2|\ln|z|| + \delta)/2)^2)\\ &= C'_U\exp(\delta|\ln|z||),\end{aligned}$$

putting $x = \ln|u|$, $y = (2|\ln|z|| + \delta)/2$ in the inequality $-x^2 + 2xy \le y^2$.

LEMMA 3. *If $t \in \mathbb{R}_+$, $\theta \in [0,\pi)$, $z \in \mathbb{C} \setminus \overline{I(t,\theta)}$, $f \in \underset{\sim}{\mathcal{M}}'_{(0)}(\overline{I(t,\theta)})$, and*

$$g_z(u) = (z-u)^{-1}, \tag{2'}$$

$$F(z) = f[g_z], \tag{3'}$$

then $g_z \in \underset{\sim}{\mathcal{M}}_{(0)}(\overline{I(t,\theta)})$ and $F \in \widetilde{\mathcal{M}}_{(0)}(V \setminus \overline{I(t,\theta)})$ for every $V \in \mathcal{S}^\theta$.

Proof. Similarly to the previous proof, if $U \in \mathcal{S}^\theta$ is such that $z \notin \overline{U}$ and if $u \in U$ then $|z-u| \ge A$ for some A (depending on z). Hence $|ug_z(u)| \le A^{-1}|u| \le C(z)\exp(-\delta|\ln|u||)$ for every δ with $0<\delta<1$, since U is bounded.

Now, if $V \in \mathcal{S}^\theta$, F is given by (3′) and $K \subset V$ is compact with $K \cap \overline{I(t,\theta)} \subset \{0\}$ then for some $U \in \mathcal{S}^\theta$ such that $U \cap K \subset \{0\}$ and for some constant $C_K = C_U$ we have (from the continuity of f) for every $\varepsilon > 0$ and every δ with $0<\delta<1$,

$$\begin{aligned}|F(z)| &\le C_K \sup_{u\in U}|u-z|^{-1}|u|^{1-\delta}\\ &\le C_K\sup_{u\in U}|u-z|^{-\varepsilon}|1-z/u|^{\varepsilon-1}|u|^{\varepsilon-\delta} \qquad \text{for } z \in K.\end{aligned}$$

If $|\operatorname{Arg} z - \operatorname{Arg} u| \ge \alpha$ for some $\alpha > 0$, then $|u-z| \ge |z|\sin\alpha$, and $|1-z/u| \ge \sin\alpha$, so

$$|F(z)| \le C_K|z|^{-\varepsilon}(\sin\alpha)^{-\varepsilon}(\sin\alpha)^{\varepsilon-1}\sup_{u\in U}|u|^{\varepsilon-\delta} = C_{K,\varepsilon}|z|^{-\varepsilon}.$$

If $|u| < |z|$ for every $u \in U$, then there exists a constant D_K such that $\sup_{u\in U}|u-z|^{-1} \le D_K$, hence

$$|F(z)| \le C_K D_K\sup_{u\in U}|u|^{1-\delta} = A_K \le B_K|z|^{-\varepsilon}.$$

Therefore if for $z = (z_1,\ldots,z_n)$, $u = (u_1,\ldots,u_n)$ we define

$$g_z(u) = g_{z_1}(u_1)\ldots g_{z_n}(u_n), \tag{2''}$$

where every g_{z_i} is given by (2) for $t_i = \infty$, or by (2′) for $t_i < \infty$, then it is easy to see that $g_z \in \underset{\sim}{\mathcal{M}}_{(0)}(\overline{I(t,\theta)})$ and the function F defined by

$$F(z) = f[g_z] \quad \text{for } f \in \underset{\sim}{\mathcal{M}}'_{(0)}(\overline{I(t,\theta)}) \tag{3''}$$

belongs to $\widetilde{\mathcal{M}}_{(0)}(V\#\overline{I(t,\theta)})$ for every $V \in \mathcal{S}^\theta$.

We can see that the map $f \to [F]$ given by (3″) is inverse to the map $[F] \to f$ given by (1). Indeed, if $\phi \in \underset{\sim}{\mathcal{M}}_{(0)}(\overline{I(t,\theta)})$ then

$$\begin{aligned}f[\phi] &= (2\pi i)^{-n}\int_\gamma F(z)\phi(z)\,dz\\ &= (2\pi i)^{-n}\int_\gamma f[g_z]\phi(z)\,dz = f\Big[(2\pi i)^{-n}\int_\gamma g_z(u)\phi(z)\,dz\Big].\end{aligned}$$

For calculating the last integral we use the substitution $z = e^{-\zeta}$ and $u = e^{-w}$ and we apply the Cauchy theorem. Suppose for simplicity that

$$g_z(u) = \prod_{j=1}^n (u_j(\ln z_j - \ln u_j))^{-1}\exp(-(\ln z_j - \ln u_j)^2).$$

Then we have

$$\begin{aligned}\int_\gamma g_z(u)\phi(z)\,dz &= \prod_{j=1}^n u_j^{-1}\int_\gamma\prod_{j=1}^n(\ln z_j - \ln u_j)^{-1}\exp(-(\ln z_j - \ln u_j)^2)\phi(z)\,dz\\ &= \int_{\mu^{-1}(\gamma)}\prod_{j=1}^n(w_j-\zeta_j)^{-1}\exp(-(w_j-\zeta_j)^2 + w_j - \zeta_j)\phi(e^{-\zeta})\,d\zeta\\ &= (2\pi i)^n\phi(e^{-w}) = (2\pi i)^n\phi(u).\end{aligned}$$

We call the function F defined for a Mellin a.f. $f \in \underset{\sim}{\mathcal{M}}'_{(0)}(\overline{I(t,\theta)})$ by (3), (3′) or (3″) the *standard defining function* of the functional f.

It is obvious that if $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)})$ then the function

$$F(z) = f[u^{-b}g_z(u)] \tag{4}$$

is a defining function for f, and we shall call it the *standard defining function* of the functional f.

Suppose $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)})$ for some $t \in (\mathbb{R}_+ \cup \{\infty\})^n$, and F is a defining function of f. If F can be extended to a function in $\widetilde{\mathcal{M}}_{(b)}(\overline{I(s,\theta)})$ for some $s < t$, then we say that carrier $f \subset \overline{I(s,\theta)}$.

3. Generalized Mellin transform.

Let $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)})$ for $t = (t_1,\ldots,t_n)$, $t_j \in \mathbb{R} \cup \{\infty\}$. Let $F \in \widetilde{\mathcal{M}}_{(b)}(V\#\overline{I(t,\theta)})$ be a defining function of f. We denote by F_σ ($\sigma = (\sigma_1,\ldots,\sigma_n) \in \{-,+\}^n$) the restriction of F to $\Gamma^\theta_\sigma = (V\#\overline{I(t,\theta)}) \cap \Gamma_\sigma$.

DEFINITION 5. Let $F \in \widetilde{\mathcal{M}}_{(b)}(V\#\overline{I(t,\theta)})$. If $F = 0$ on Γ^θ_τ for all $\tau \neq \sigma$ then the Mellin analytic functional f assigned to F by the isomorphism (1) is called the *boundary value* of F and is denoted by $f = j_\sigma(F)$.

Suppose that $\{\chi_\sigma\}$ is an "exponential partition of unity", as in [1], Section 3. We can see that the functions $\chi_\sigma(-\ln w)w^{-z-1}$ are in $\underset{\sim}{\mathcal{M}}_{(b)}(\overline{I(t,\theta)})$ for z such that $-1 < \sigma_j(\operatorname{Re} z_j - b_j) < 0$ $(j = 1,\ldots,n)$ for every θ and $t \in (\mathbb{R}_+ \cup \{\infty\})^n$. Hence every Mellin a.f. f can be evaluated on these functions, so we can put for $z \in \{-1 < \sigma_j(\operatorname{Re} z_j - b_j) < 0,\ j = 1,\ldots,n\}$

$$G_\sigma(z) = \operatorname{sgn}\sigma f[\chi_\sigma(-\ln w)w^{-z-1}], \tag{5}$$

$$G^b_\sigma(u) = G_\sigma(b+iu). \tag{6}$$

LEMMA 4 (see [1]). *$G^b_\sigma \in \widetilde{\mathcal{O}}^\theta(U \cap (\mathbb{D}^n + i\Gamma_\sigma))$ for some tubular neighbourhood U of $\mathbb{D}^n$ in $\mathbb{D}^n + i\mathbb{R}^n$.*

Proof. We consider the case of $n = 1$; the proof generalizes to $n > 1$ without difficulty.

Fix $\varepsilon > 0$. By (1) we have

$$G_\sigma(z) = (2\pi i)^{-1}\operatorname{sgn}\sigma\Big[\int_{\gamma^-} F_-(w)\chi_\sigma(-\ln w)w^{-z-1}\,dw - \int_{\gamma^+} F_+(w)\chi_\sigma(-\ln w)w^{-z-1}\,dw\Big],$$

where $\gamma^\tau \subset \{\theta < \tau\operatorname{Arg} w < \theta+\varepsilon\}$, $\tau = +,-$. From Definition 1 it follows that

$$\Big|\int_{\gamma^\tau} F_\tau(w)\chi_\sigma(-\ln w)w^{-z-1}\,dw\Big|$$
$$\le C_\varepsilon \int_{\gamma^\tau} |w|^b \exp(\varepsilon|\ln|w||)|\chi_\sigma(-\ln w)||w|^{-\operatorname{Re} z-1}\exp(\operatorname{Im} z\operatorname{Arg} w)\,dw$$
$$\le C_\varepsilon \exp((\theta+\varepsilon)|\operatorname{Im} z|)\int_{\gamma^\tau} |w|^{b-\operatorname{Re} z-1}|\chi_\sigma(-\ln w)|\exp(\varepsilon|\ln|w||)\,dw$$
$$\le C_{\varepsilon,\varepsilon'}\exp((\theta+\varepsilon)|\operatorname{Im} z|) \quad \text{for } -1+\varepsilon' \le \sigma(\operatorname{Re} z - b) \le -\varepsilon'.$$

Since $\operatorname{Re} u = \operatorname{Im} z$ and $\operatorname{Im} u = -(\operatorname{Re} z - b)$, the lemma is proved.

DEFINITION 6. If f is a Mellin analytic functional, $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)})$, $t \in (\mathbb{R}_+\cup\{\infty\})^n$, the *generalized Mellin transform* of f is the Fourier analytic functional defined by

$$M_b f = \sum_\tau j_\tau(G^b_\tau) \in [\underset{\sim}{\mathcal{O}}^\theta(\mathbb{D}^n)]'. \tag{7}$$

PROPOSITION 1. *The map $M_b : \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)}) \to [\underset{\sim}{\mathcal{O}}^\theta(\mathbb{D}^n)]'$ given by (7) is an isomorphism with the inverse map*

$$M_b^{-1} g = w^b(\mathcal{F}g \circ \mu^{-1})$$

for $g \in [\underset{\sim}{\mathcal{O}}^\theta(\mathbb{D}^n)]'$, where $\mathcal{F}$ is the Fourier transformation in the space of Fourier hyperfunctions (see [1]).

Proof. This follows immediately from Definition 6 and from [1], Sections 2 and 3.

It is obvious that if $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{R(\theta)})$ and $\operatorname{carrier} f \subset \overline{I(t,\theta)}$ for some $t \in \mathbb{R}^n_+$, then $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)})$.

THEOREM 2. *If $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(t,\theta)})$ for some $t \in \mathbb{R}^n_+$, then*

$$M_b f = (2\pi i)^{-1} j_{+,\ldots,+}(\mathcal{M}f(b+i\cdot))$$

and

$$|\mathcal{M}f(z)| \le C_{\varepsilon,\varepsilon'}e^{(\theta+\varepsilon)|\operatorname{Im} z|}(te^\varepsilon)^{-\operatorname{Re} z} \quad \textit{for } \operatorname{Re} z \le b-\varepsilon', \tag{8}$$

where $\mathcal{M}f(z) = f[w^{-z-1}]$.

Proof. We show this for $n = 1$. We have (from (7) and (5))

$$M_b f = j_+(G^b_+) + j_-(G^b_-),$$
$$G_\pm(z) = \pm f[\chi_\pm(-\ln w)w^{-z-1}] \quad \text{for } -1 < \pm(\operatorname{Re} z - b) < 0.$$

If F is the defining function for f then F is holomorphic in $V \setminus \overline{I(t,\theta)}$ for some $V \in \mathcal{S}^\theta$, and for every $\varepsilon > 0$,

$$|F(w)| \le C_{\varepsilon,K}|w|^b e^{\varepsilon|\ln|w||} \quad \text{for } w \in K,$$

for every compact $K \subset V \setminus \overline{I(t,\theta)}$. Therefore, for $b-1 < \operatorname{Re} z < b$ we have

$$G_+(z) = f[\chi_+(-\ln w)w^{-z-1}] = (2\pi i)^{-1}\int_{\gamma^- - \gamma^+} F(w)\chi_+(-\ln w)w^{-z-1}\,dw$$
$$= (2\pi i)^{-1}\int_\gamma F(w)\chi_+(-\ln w)w^{-z-1}\,dw,$$

where γ is a bounded curve encircling $\overline{I(t,\theta)}$, as in Figure 1.

Hence, G_+ can be continued holomorphically to $\{\operatorname{Re} z < b\}$. Similarly, for $b < \operatorname{Re} z < b+1$,

$$G_-(z) = f[\chi_-(-\ln w)w^{-z-1}] = -(2\pi i)^{-1}\int_\gamma F(w)\chi_-(-\ln w)w^{-z-1}\,dw$$

and this function can be continued holomorphically to $\{\operatorname{Re} z > b\}$. Moreover,

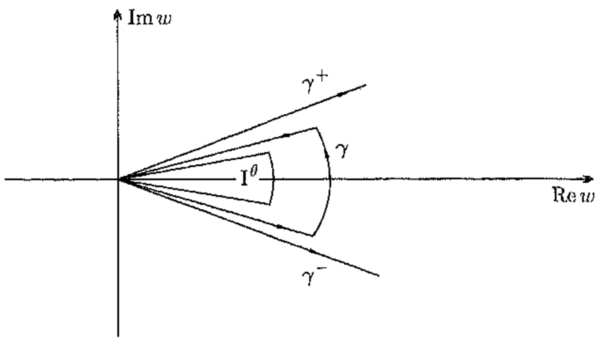

Fig. 1

$$\Big|\int_\gamma F(w)\chi_-(-\ln w)w^{-z-1}\,dw\Big|$$
$$\le C_\varepsilon \int_\gamma |w|^{b-\varepsilon}|w||w|^{-\operatorname{Re} z-1}\,dw = C_\varepsilon\int_\gamma |w|^{b-\varepsilon-\operatorname{Re} z}\,dw,$$

so that G_- can be continued holomorphically to $\{\operatorname{Re} z \le b\}$. Then for $z \in \{\operatorname{Re} z < b\}$ we have

$$\begin{aligned} &G_+(z) - G_-(z)\\ &= (2\pi i)^{-1}\Big[\int_\gamma F(w)\chi_+(-\ln w)w^{-z-1}\,dw + \int_\gamma F(w)\chi_-(-\ln w)w^{-z-1}\,dw\Big]\\ &= (2\pi i)^{-1}\int_\gamma F(w)w^{-z-1}\,dw = (2\pi i)^{-1}\mathcal{M}f(z). \end{aligned}$$

This means that

$$M_b f = j_+(G_+^b) - j_-(G_-^b) = j_+(G_+^b - G_-^b) = (2\pi i)^{-1} j_+(\mathcal{M}f(b+i\cdot)).$$

For $n > 1$, the idea of proof is the same. The condition (8) can be proved exactly as in [1], Theorem 4. It is obvious that for every $r \in \mathbb{N}_0^n$ the function $(\partial/\partial z)^r(\mathcal{M}f)$ also satisfies the condition (8).

Similarly, we prove

THEOREM 3. *If $f \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{R(\theta)})$ and* carrier $f \subset \overline{R(\theta)}\#I(t,\theta)$ *for some $t \in \mathbb{R}_+^n$ then*

$$\sum_\tau G_\tau(z) = (2\pi i)^{-1}\mathcal{M}f(z),$$
$$M_b f = j_-\Big(\sum_\tau G_\tau^b\Big) = j_-((2\pi i)^{-1}\mathcal{M}f(b+i\cdot)),$$

and

$$|\mathcal{M}f(z)| \le C_{\varepsilon,\varepsilon'}\exp((\theta+\varepsilon)|\operatorname{Im} z|)(te^{-\varepsilon})^{-\operatorname{Re} z}$$

for $\operatorname{Re} z \ge b + \varepsilon'$.

Now, we shall prove Paley–Wiener type theorems for Mellin a.f.

4. Characterization of Mellin analytic functionals

THEOREM 4. *Assume that F is a holomorphic function on the set $A = \{\alpha_- < \operatorname{Re} z < \alpha_+\}$, where $\alpha_+ = (\alpha_+^1,\ldots,\alpha_+^n)$, $\alpha_- = (\alpha_-^1,\ldots,\alpha_-^n)$, and $\alpha_\sigma = (\alpha_{\sigma_1}^1,\ldots,\alpha_{\sigma_n}^n)$ for $\sigma = (\sigma_1,\ldots,\sigma_n) \in \{-,+\}^n$, and for every $\varepsilon,\varepsilon' > 0$ there exists a constant $C_{\varepsilon,\varepsilon'}$ such that*

$$|F(z)| \le C_{\varepsilon,\varepsilon'}\exp((\theta+\varepsilon)|\operatorname{Im} z|) \tag{9}$$

on the set

$$A_{-\varepsilon'} = \{\alpha_- + \varepsilon' \le \operatorname{Re} z \le \alpha_+ - \varepsilon'\}.$$

Then, for every σ, there exists a unique Mellin a.f. $f^{\alpha_\sigma} \in \underset{\sim}{\mathcal{M}}'_{(\alpha_\sigma)}(\overline{R(\theta)})$ such that

$$M_{\alpha_\sigma} f^{\alpha_\sigma} = j_\sigma(F(\alpha_\sigma + i\cdot)).$$

P r o o f. Set $z = \alpha_\sigma + iu$. Then $\operatorname{Re} z = \alpha_\sigma - \operatorname{Im} u$ and $\operatorname{Im} z = \operatorname{Re} u$, so it is obvious that $F(\alpha_\sigma + i\cdot) \in \widetilde{\mathcal{O}}^\theta(\mathbb{D}^n + i(\Gamma_\sigma \cap \{0 < \sigma \operatorname{Im} u < \alpha_+ - \alpha_-\}))$. Taking $g^{\alpha_\sigma} = j_\sigma(F(\alpha_\sigma + i\cdot)) \in [\underset{\sim}{\mathcal{O}}^\theta(\mathbb{D}^n)]'$ and $f^{\alpha_\sigma} = M_{\alpha_\sigma}^{-1} g^{\alpha_\sigma}$ we see that $f^{\alpha_\sigma} \in \underset{\sim}{\mathcal{M}}'_{(\alpha_\sigma)}(\overline{R(\theta)})$.

THEOREM 5. *Suppose that F is a function holomorphic on the set $\{\sigma(\operatorname{Re} z - \omega) < 0\}$ for some $\sigma \in \{-,+\}^n$, and for every $\varepsilon,\varepsilon' > 0$ there exists a constant $C_{\varepsilon,\varepsilon'}$ such that*

$$|F(z)| \le C_{\varepsilon,\varepsilon'}\exp((\theta+\varepsilon)|\operatorname{Im} z|)(te^{\sigma\varepsilon})^{-\operatorname{Re} z} \quad \text{for } \sigma(\operatorname{Re} z - \omega) \le -\varepsilon'. \tag{10}$$

Then there exists a unique Mellin a.f. $f \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I_\sigma(t,\theta)})$, where $I_\sigma(t,\theta) = I_{\sigma_1}(t_1,\theta_1) \times \ldots \times I_{\sigma_n}(t_n,\theta_n)$, $I_+(t_j,\theta_j) = I(t_j,\theta_j)$ and $I_-(t_j,\theta_j) = R(\theta_j) \setminus I(t_j,\theta_j)$, such that

$$\mathcal{M}f(z) = F(z) \quad \text{for } z \in \{\sigma(\operatorname{Re} z - \omega) < 0\}.$$

First we shall prove the following lemma:

LEMMA 5. *Suppose that F is a function of one variable satisfying the assumptions of Theorem 5 for $\sigma = +$. For $\phi \in [0,\pi/2]$, $\mathring{\alpha} < \omega$, and $\tau = +,-$, define $R_\phi^\tau = \{z : z = \mathring{\alpha} + ir\tau e^{\tau i\phi},\ r \in \mathbb{R}_+\}$ and $R_\phi = R_\phi^- \cup R_\phi^+$. Then, for every $\phi \in (0,\pi/2)$ and $\tau,\tau' \in \{-,+\}$,*

$$\int_{R_0^\tau} F(z)\widetilde{\chi}_{\tau'}(z)w^z\,dz = \int_{R_\phi^\tau} F(z)\widetilde{\chi}_{\tau'}(z)w^z\,dz \quad \text{for } w \in U_\tau,$$

where $\widetilde{\chi}_\tau(z) = \chi^\theta_\tau(-i(z-\omega))$ *with* $\{\chi^\theta_\tau\}$ *an appropriate "exponential partition of unity" defined in* [1], *Section* 3, *and*

$$U_\tau = \{(\theta - \tau \operatorname{Arg} w) \operatorname{Re} e^{i\alpha} - (\ln|w| - \ln t) \operatorname{Im} e^{i\alpha} < 0,\ 0 < \alpha < \phi\}.$$

Proof. It is sufficient to observe that if

$$\begin{aligned} z &= \mathring{\alpha} + ir\tau e^{i\tau\alpha} = \mathring{\alpha} + r\tau \sin\tau\alpha + ir\tau\cos\tau\alpha \\ &= \mathring{\alpha} - r\sin\alpha + ir\tau\cos\alpha = \mathring{\alpha} - r\operatorname{Im} e^{i\alpha} + ir\tau \operatorname{Re} e^{i\alpha}, \end{aligned}$$

then $\operatorname{Re} z = \mathring{\alpha} - r\operatorname{Im} e^{i\alpha}$, $\operatorname{Im} z = r\tau \operatorname{Re} e^{i\alpha}$,

$$\begin{aligned} w^z &= \exp(z \ln w) = \exp(\ln w(\mathring{\alpha} + ir\tau e^{i\tau\alpha})) \\ &= \exp((\ln|w| + i\operatorname{Arg} w)(\mathring{\alpha} - r\operatorname{Im} e^{i\alpha} + ir\tau \operatorname{Re} e^{i\alpha})) \end{aligned}$$

and

$$|w^z| = \exp(\mathring{\alpha}\ln|w| - r\tau \operatorname{Arg} w \operatorname{Re} e^{i\alpha} - r\ln|w| \operatorname{Im} e^{i\alpha}).$$

From the Cauchy theorem, for every $r > 0$ we have

$$\begin{aligned} &\int_0^r F(\mathring{\alpha} + i\tau y)\widetilde{\chi}_{\tau'}(\mathring{\alpha} + i\tau y) w^{\mathring{\alpha} + i\tau y}\, dy \\ &\quad - \int_0^r F(\mathring{\alpha} + i\tau y e^{i\tau\phi})\widetilde{\chi}_{\tau'}(\mathring{\alpha} + i\tau y e^{i\tau\phi}) w^{\mathring{\alpha} + i\tau y \exp(i\tau\phi)} e^{i\tau\phi}\, dy \\ &= \int_0^\phi F(\mathring{\alpha} + ire^{i\tau\alpha})\widetilde{\chi}_{\tau'}(\mathring{\alpha} + ir\tau e^{i\tau\alpha}) w^{\mathring{\alpha} + ir\tau\exp(i\tau\alpha)} e^{i\tau\alpha}\, d\alpha = J(r, w). \end{aligned}$$

We want to show that the last integral tends to zero as $r \to 0$. From the estimate (10) and the properties of χ_τ it follows that

$$\begin{aligned} |J(r,w)| &\le C_{\varepsilon,\varepsilon'} \int_0^\phi \exp((\theta+\varepsilon) r \operatorname{Re} e^{i\alpha} + (-\mathring{\alpha} + r\operatorname{Im} e^{i\alpha})(\varepsilon + \ln t) \\ &\qquad + \ln|w|\mathring{\alpha} - r\ln|w| \operatorname{Im} e^{i\alpha} - r\tau \operatorname{Arg} w \operatorname{Re} e^{i\alpha} + \varepsilon r \operatorname{Re} e^{i\alpha}) r\, d\alpha \\ &= C_{\varepsilon,\varepsilon'} (te^{\varepsilon})^{-\mathring{\alpha}} |w|^{\mathring{\alpha}} r \int_0^\phi \exp(((\theta + 2\varepsilon - \tau\operatorname{Arg} w)\operatorname{Re} e^{i\alpha} \\ &\qquad - (\ln|w| - \ln t - \varepsilon)\operatorname{Im} e^{i\alpha}) r)\, d\alpha. \end{aligned}$$

Therefore, $J(r,w)$ converges locally uniformly to zero as $r \to 0$, on the set U_τ.

Proof of Theorem 5. It is enough to prove this theorem in the case when $\sigma_j = +$ for $j = 1, \ldots, n$. By Theorem 4 there exists a unique Mellin analytic functional $f^\omega \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{R(\theta)})$ such that

$$M_\omega f^\omega = j_+(F(\omega + i\cdot)).$$

Therefore, it is sufficient to show that carrier $f^\omega \subset \overline{I(t,\theta)}$. We can represent f^ω in the form

$$f^\omega = \sum_\tau j_\tau(H_\tau),$$

where

$$\operatorname{sgn}\tau H_\tau(w) = \int_{\operatorname{Re} z = \mathring{\alpha}} F(z)\chi^\theta_\tau(-i(z-\omega)) w^z\, dz$$

for appropriate χ^θ_τ as in [1], Section 3, for a fixed $\mathring{\alpha} < \omega$ and $w \in \widehat{\Gamma}_\tau = \{0 < \tau \operatorname{Arg} w - \theta < 1\}$ (Proposition 1).

For $\psi = (\psi_1, \ldots, \psi_n)$, $\psi_j \in [0, \pi/2)$, define $R_\psi = R_{\psi_1} \times \ldots \times R_{\psi_n}$ (with R_{ψ_j} as in Lemma 5). Suppose $\tau, \tau' \in \{-,+\}^n$ and $\tau_j = \tau'_j$ for $j \ne k$, k fixed. Then for $w \in U_\tau \cap U_{\tau'}$ and for ϕ sufficiently close to $\pi/2$ we have, from Lemma 5,

$$\begin{aligned} \operatorname{sgn}\tau(H_\tau(w) - H_{\tau'}(w)) &= \int_{R_0} F(z)\widetilde{\chi}_\tau(z) w^z\, dz + \int_{R_0} F(z)\widetilde{\chi}_{\tau'}(z) w^z\, dz \\ &= \int_{R_\psi} F(z)\widetilde{\chi}_\tau(z) w^z\, dz + \int_{R_\psi} F(z)\widetilde{\chi}_{\tau'}(z) w^z\, dz, \end{aligned}$$

where $\psi_j = 0$ for $j \ne k$ and $\psi_k = \phi$.

We can write the above sum as

$$\begin{aligned} &\int_{R_\psi} F(z)\widetilde{\chi}_{\tau_1}(z_1)\ldots\widetilde{\chi}_+(z_k)\ldots\widetilde{\chi}_{\tau_n}(z_n) w^z\, dz \\ &\quad + \int_{R_\psi} F(z)\widetilde{\chi}_{\tau_1}(z_1)\ldots\widetilde{\chi}_-(z_k)\ldots\widetilde{\chi}_{\tau_n}(z_n) w^z\, dz = \int_{R_\psi} F(z)\widetilde{\chi}_{\widehat{\tau}}(\widehat{z}) w^z\, dz, \end{aligned}$$

where $\widehat{\tau} = (\tau_1, \ldots, \tau_{k-1}, \tau_{k+1}, \ldots, \tau_n)$ and $\widehat{z} = (z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n)$. If $w \in U_\tau \cap U_{\tau'}$ and $\phi \to \pi/2$, then the last integral converges to zero. Hence $H_\tau(w) = H_{\tau'}(w)$ on $U_\tau \cap U_{\tau'}$. Since we can fix k arbitrarily, there exists a function H such that $H(w) = H_\tau(w)$ on U_τ and $H \in \widetilde{\mathcal{M}}_{(\omega)}(V \# \overline{I(t,\theta)}) = \widetilde{\mathcal{M}}_{(\omega)}(\bigcup_\tau U_\tau)$. The proof is complete.

5. Differential operators in the space of Mellin analytic functionals. Consider a differential operator R of the form

$$R = \sum_{|\alpha| \le m} a_\alpha \left(x \frac{\partial}{\partial x} \right)^\alpha,$$

with $a_\alpha \in \mathbb{C}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ being a multiindex. Let $f \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t,\theta)})$ be a Mellin a.f. It follows from the properties of the Mellin transformation that also $Rf \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t,\theta)})$, and $\mathcal{M}(Rf)(z) = R(z)\mathcal{M}f(z)$. The polynomial $R(z) = \sum_{|\alpha|\le m} a_\alpha z^\alpha$ will be called the *Mellin symbol* of the operator R. If $\{R(z)=0\}\cap\{\operatorname{Re} z < \omega\} = \emptyset$, then the function $F(z) = \mathcal{M}g(z)/R(z)$ satisfies (10) for every Mellin a.f. $g \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t,\theta)})$. It follows from Theorem 5 that there exists a unique Mellin a.f. $f \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t,\theta)})$ such that $\mathcal{M}f(z) = \mathcal{M}g(z)/R(z)$. Thus, we can define the operator R^{-1} in the following way:

DEFINITION 7. For $g \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t,\theta)})$, $R^{-1}g = f$ is the Mellin a.f. such that $f \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t,\theta)})$ and $\mathcal{M}f(z) = \mathcal{M}(R^{-1}g)(z) = \mathcal{M}g(z)/R(z)$ on $\{\operatorname{Re} z < \omega\}$.

Now, let R be a polynomial in one variable, for example z_1, and let $s_1,\ldots,s_q$ be the zeros of R, with multiplicities $k_1,\ldots,k_q$ respectively, such that $\operatorname{Re} s_1 \le \ldots \le \operatorname{Re} s_q < \omega_1$.

THEOREM 6. *If $f \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t,\theta)})$, then there exist a Mellin a.f. $g \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{R(\theta_1)} \times \overline{I(t',\theta')})$, constants C^j_{nr} and the Mellin a.f. $f_{jr} \in \underset{\sim}{\mathcal{M}}'_{(\omega')}(\overline{I(t',\theta')})$ ($j=1,\ldots,q$; $r=0,\ldots,k_j$; $n=r,\ldots,k_j$) such that*

$$\mathcal{M}f_{jr}(z') = \left(\frac{\partial}{\partial z_1}\right)^r \mathcal{M}f(s_j,z')$$

and

$$R^{-1}f = g + \sum_{j=1}^{q} w_1^{s_j} \sum_{r=0}^{k_j-1}\Big(\sum_{n=r}^{k_j-1} C^j_{nr}(\ln w_1)^{k_j-n-1}\Big) f_{jr}.$$

Proof. Write $R(z_1) = (z_1 - s_j)^{k_j} R_j(z_1)$. The function $\mathcal{M}f(z)/R(z_1)$ satisfies the condition (10) in the set $\{\operatorname{Re} z_1 < \operatorname{Re} s_1,\ \operatorname{Re} z' < \omega'\}$ and the condition (9) in the sets $\{\operatorname{Re} s_j < \operatorname{Re} z_1 < \operatorname{Re} s_{j+1},\ \operatorname{Re} z' < \omega'\}$ and $\{\operatorname{Re} s_q < \operatorname{Re} z_1 < \omega_1,\ \operatorname{Re} z' < \omega'\}$.

From Theorem 5 we can write

$$R^{-1}f = \sum_\sigma j_\sigma(H_\sigma),$$

where

$$\operatorname{sgn}\sigma H_\sigma(w) = \int_{\mathbb{R}^n} (\mathcal{M}f(\alpha+i\beta)/R(\alpha_1+i\beta_1))\widetilde{\chi}^\theta_\sigma(\alpha+i\beta) w^{\alpha+i\beta}\,d\beta$$

for some fixed α with $\alpha_1 < \operatorname{Re} s_1$ and $\alpha' < \omega'$ and for $w \in -\widehat{\Gamma}_\sigma$.

Now, fix $\widehat{\alpha}$ such that $\operatorname{Re} s_q < \widehat{\alpha}_1 < \omega_1$ and $\widehat{\alpha}' < \omega'$, and set $\widehat{z}_1 = \widehat{\alpha}_1 + i\beta_1$.

Consider the integral

$$\int_{\mathbb{R}} (\mathcal{M}f(z_1,z')/R(z_1))\widetilde{\chi}^{\theta_1}_{\sigma_1}(z_1) w_1^{z_1}\,d\beta_1$$

for fixed z'. It follows from the residue theorem that

$$\operatorname{Res}_{s_j}(\mathcal{M}f(z_1,z')/R(z_1))\widetilde{\chi}^{\theta_1}_{\sigma_1}(z_1) = 2\pi i(w_1^{s_j}/(k_j-1)!) \sum_{r=0}^{k_j-1}\binom{k_j-1}{r}(\ln w_1)^{k_j-r-1} G_j^{(r)}(s_j,z'),$$

where

$$G_j(z_1,z') = (\mathcal{M}f(z_1,z')/R_j(z_1))\widetilde{\chi}^{\theta_1}_{\sigma_1}(z_1).$$

Hence we have

$$\int_{\mathbb{R}} (\mathcal{M}f(z_1,z')/R(z_1))\widetilde{\chi}^{\theta_1}_{\sigma_1}(z_1) w_1^{z_1}\,d\beta_1 = \int_{\mathbb{R}} (\mathcal{M}f(\widehat{z}_1,z')/R(\widehat{z}_1))\widetilde{\chi}^{\theta_1}_{\sigma_1}(\widehat{z}_1) w_1^{\widehat{z}_1}\,d\beta_1 + 2\pi i\sum_{j=1}^{q} w_1^{s_j}\sum_{r=0}^{k_j-1}\Big(\sum_{n=r}^{k_j-1} C^j_{nr}(\ln w_1)^{k_j-n-1}\Big)\left(\frac{\partial}{\partial z_1}\right)^r \mathcal{M}f(s_j,z').$$

Multiplying this equality by $\widetilde{\chi}^{\theta'}_{\sigma'}(z')w'^{z'}$ and integrating the result over β' with fixed $\alpha' < \omega'$ we obtain

$$\begin{aligned}\operatorname{sgn}\sigma H_\sigma(w) = {} & \int_{\mathbb{R}^n} (\mathcal{M}f(\widehat{z}_1,z')/R(\widehat{z}_1))\widetilde{\chi}^{\theta}_{\sigma}(\widehat{z}_1,z') w_1^{\widehat{z}_1} w'^{z'}\,d\beta_1 d\beta' \\ & + 2\pi i\sum_{j=1}^{q} w_1^{s_j}\sum_{r=0}^{k_j-1}\Big(\sum_{n=r}^{k_j-1} C^j_{nr}(\ln w_1)^{k_j-n-1}\Big) \\ & \times \int_{\mathbb{R}^{n-1}} \bigg(\left(\frac{\partial}{\partial z_1}\right)^r \mathcal{M}f\bigg)(s_j,z')\widetilde{\chi}^{\theta'}_{\sigma'}(z') w'^{z'}\,d\beta'.\end{aligned}$$

If we denote the first term on the right by $\operatorname{sgn}\sigma\widehat{H}_\sigma(w)$, then we can put $g = \sum_\sigma j_\sigma(\widehat{H}_\sigma)$ and $f_{jr} = \mathcal{M}^{-1}(((\partial/\partial z_1)^r \mathcal{M}f)(s_j,\cdot))$.

6. The Laplace–Beltrami type operators. Let P be a differential operator of the form $P = R + Q$, where $R = R(x_1\partial/\partial x_1)$ is a polynomial in one variable, of degree at least 2, and $Q = x_1^2(\sum_{j=2}^n \tau_j \partial^2/\partial x_j^2)$, $\tau_j \in \mathbb{C}$.

Suppose $R(z_1) \neq 0$ on $\{\operatorname{Re} z_1 < d\}$. We define

$$\Omega_d = \{z \in \mathbb{C}^n : \operatorname{Re} z_1 < d,\ \operatorname{Re}(z_j + z_1) < d,\ j = 2, \ldots, n\},$$
$$\Omega'_d = \{z \in \mathbb{C}^n : \operatorname{Re} z_1 < d,\ \operatorname{Re} z_j < 0,\ j = 2, \ldots, n\},$$
$$\Omega''_d = \Omega'_d - (0, 2, \ldots, 2).$$

THEOREM 7. *Suppose $\theta_j \in [0, \pi/2)$, $j = 2, \ldots, n$, and $t = (t_1, \ldots, t_n)$ satisfies $|\tau_2|(t_1/t_2)^2 + \ldots + |\tau_n|(t_1/t_n)^2 < 1$. Let $f \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t,\theta)})$ for every $\omega \in \overline{\operatorname{Re} \Omega_d}$. Then there exists a unique Mellin analytic functional $u \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(\widehat{t}, \widehat{\theta})})$ for $\omega \in \overline{\operatorname{Re} \Omega''_d}$ such that $Pu = f$. Here $\widehat{\theta} = (\theta_1, \theta' + \pi/2)$ and $\widehat{t} = (t_1, t' e^{\pi/2})$.*

Proof. Applying the Mellin transformation to the equation $Pu = f$ and writing $F(z) = \mathcal{M}u(z)$ we obtain the functional equation

$$R(z_1)F(z) = -\sum_{j=2}^{n} \tau_j (z_j + 1)(z_j + 2) F(z_1 - 2, z' + 2e'_j) + \mathcal{M}f(z). \tag{11}$$

Here $e'_j = (0, \ldots, 1, \ldots, 0)$ with the unit as the jth coordinate. We solve this equation by successive approximations, putting $F_0(z) = \mathcal{M}f(z)/R(z_1)$ and

$$F_k(z) = -[R(z_1)]^{-1} \sum_{j=2}^{n} \tau_j (z_j + 1)(z_j + 2) F_{k-1}(z_1 - 2, z' + 2e'_j) + F_0(z)$$

for $k = 1, 2, \ldots$ It follows that

$$\begin{aligned} F_k(z) = [R(z_1)]^{-1} \Bigg[& \sum_{j=1}^{k} (-1)^j [R(z_1 - 2) \ldots R(z_1 - 2j)]^{-1} \\ & \times \sum_{j_2=0}^{j} \sum_{j_3=0}^{j_2} \ldots \sum_{j_n=0}^{j_{n-1}} \binom{j}{j_2} \binom{j_2}{j_3} \ldots \binom{j_{n-1}}{j_n} \tau_2^{j_n} \tau_3^{j_{n-1} - j_n} \ldots \tau_n^{j - j_2} \\ & \times (z_2 + 1) \ldots (z_2 + 2j_n) \ldots (z_n + 1) \ldots (z_n + 2(j - j_2)) \\ & \times \mathcal{M}f(z_1 - 2j, z_2 + 2j_n, \ldots, z_n + 2(j - j_2)) + \mathcal{M}f(z) \Bigg]. \end{aligned}$$

From the assumptions on f and R we get

$$\begin{aligned} & |\mathcal{M}f(z_1 - 2j, z_2 + 2j_n, \ldots, z_n + 2(j - j_2))| \\ & \quad \leq C_{\varepsilon,\varepsilon'} \exp((\theta + \varepsilon)|\operatorname{Im} z|)(te^{\varepsilon})^{-\operatorname{Re} z} (t_1 e^{\varepsilon})^{2j} (t_2 e^{\varepsilon})^{-2j_n} \ldots (t_n e^{\varepsilon})^{-2(j - j_2)} \\ & \quad = C_{\varepsilon,\varepsilon'} \exp((\theta + \varepsilon)|\operatorname{Im} z|)(te^{\varepsilon})^{-\operatorname{Re} z} (t_1/t_2)^{2j_n} \ldots (t_1/t_n)^{2(j - j_2)} \end{aligned}$$

for $z \in (\Omega_d)_{-\varepsilon'}$, and $|R(z_1)| \geq B_{\varepsilon'}$, $|R(z_1 - 2)| \ldots |R(z_1 - 2j)| > (2^j (j!))^N$

for some $N \in \mathbb{N}$, $N \geq 2$. Therefore we have

$$\begin{aligned} & \frac{|z_2 + 1| \ldots |z_2 + 2j_n| \ldots |z_n + 1| \ldots |z_n + 2(j - j_2)|}{|R(z_1 - 2) \ldots R(z_1 - 2j)|} \\ & \leq 2^{-2j} (j!)^{-2} |z_2 + 1| \ldots |z_2 + 2j_n| \ldots |z_n + 1| \ldots |z_n + 2(j - j_2)| \\ & = (2^{-2j} [(2j)!]^{-1} |z_2 + 1| \ldots |z_2 + 2j_n| \ldots |z_n + 1| \ldots |z_n + 2(j - j_2)| \binom{2j}{j} \\ & \leq \prod_{l_2=1}^{2j_n} (|z_2 + l_2|/l_2) \ldots \prod_{l_n=1}^{2(j-j_2)} (|z_n + l_n|/l_n) \\ & = \prod_{l_2=1}^{2j_n} (|(\alpha_2 + l_2)^2 + \beta_2^2|^{1/2}/l_2) \ldots \prod_{l_n=1}^{2(j-j_2)} (|(\alpha_n + l_n)^2 + \beta_n^2|^{1/2}/l_n) \\ & = \prod_{l_2=1}^{2j_n} (1 + 2\alpha_2/l_2 + (\alpha_2^2 + \beta_2^2)/l_2^2)^{1/2} \ldots \\ & \qquad \ldots \prod_{l_n=1}^{2(j-j_2)} (1 + 2\alpha_n/l_n + (\alpha_n^2 + \beta_n^2)/l_n^2)^{1/2} \\ & = J(z_2, \ldots, z_n). \end{aligned}$$

We have the following estimates on Ω'_d:

$$\begin{aligned} J(z_2, \ldots, z_n) & < \prod_{l_2=1}^{2j_n} (1 + |z_2|^2/l_2^2)^{1/2} \ldots \prod_{l_n=1}^{2(j-j_2)} (1 + |z_n|^2/l_n^2)^{1/2} \\ & < K \exp((\pi/2)(|z_2| + \ldots + |z_n|)) \end{aligned}$$

for some constant K.

In view of all the statements above we obtain the following estimate for F_k:

$$\begin{aligned} |F_k(z)| \leq C'_{\varepsilon,\varepsilon'} & \exp((\theta + \varepsilon)|\operatorname{Im} z|)(te^{\varepsilon})^{-\operatorname{Re} z} \exp((\pi/2)|z'|) \\ & \times \Bigg[\sum_{j=1}^{k} \sum_{j_2=0}^{j} \ldots \sum_{j_n=0}^{j_{n-1}} \binom{j}{j_2} \ldots \binom{j_{n-1}}{j_n} \\ & \times (|\tau_2|(t_1/t_2)^2)^{j_n} \ldots (|\tau_n|(t_1/t_n)^2)^{j - j_2} + 1 \Bigg] \\ < C'_{\varepsilon,\varepsilon'} & \exp((\theta_1 + \varepsilon)|\operatorname{Im} z_1|) \end{aligned}$$

$$\times \exp((\theta' + \pi/2 + \varepsilon)|\mathrm{Im}\, z'|)(t_1 e^{\varepsilon})^{\mathrm{Re}\, z_1}(t' e^{\pi/2+\varepsilon})^{-\mathrm{Re}\, z'}$$

$$\times \sum_{j=0}^{k} (|\tau_2|(t_1/t_2)^2 + \ldots + |\tau_n|(t_1/t_n)^2)^j \quad \text{for } z \in (\Omega'_d)_{-\varepsilon'}.$$

This means that the sequence $\{F_k\}$ converges locally uniformly to a function F, holomorphic on Ω'_d and satisfying the condition: for every $\varepsilon, \varepsilon' > 0$ there exists a constant $K_{\varepsilon,\varepsilon'}$ such that

$$|F(z)| < K_{\varepsilon,\varepsilon'} \exp((\widehat{\theta} + \varepsilon)|\mathrm{Im}\, z|)(\widehat{t} e^{\varepsilon})^{-\mathrm{Re}\, z} \quad \text{for } z \in (\Omega'_d)_{-\varepsilon'}.$$

From the definition of F_k it follows that F is a solution of (11) on Ω''_d and satisfies (10), so by Theorem 5 there exists a unique Mellin a.f. $u \in \underset{\sim}{\mathcal{M}}'_{(b)}(\overline{I(\widehat{t},\widehat{\theta})})$, for every $b \in \mathrm{Re}\, \Omega''_d$, such that $\mathcal{M}u(z) = F(z)$ on Ω''_d. Since $\mathcal{M}u$ is a solution of (11), u is a solution of $Pu = f$.

Moreover, the functional u can be written as

$$u = \sum_{\sigma} j_{\sigma}(U_{\sigma}),$$

where

$$\operatorname{sgn} \sigma U_{\sigma}(w) = \int_{\mathrm{Re}\, z = \mathring{a}} F(z) \widetilde{\chi}_{\sigma}^{\widehat{\theta}}(z) w^z \, dz$$

for $w \in \{0 < \sigma \operatorname{Arg} w - \widehat{\theta} < 1 \text{ or } |w| > \widehat{t}\}$, $\mathring{a}$ fixed in $\mathrm{Re}\, \Omega''_d$ and $\{\chi_{\sigma}^{\widehat{\theta}}\}$ an appropriate "exponential partition of unity".

COROLLARY 1. *Suppose that* $f \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t,\theta)})$, *for every* $\omega \in \mathrm{Re}\, \Omega_b$ *for some* $b \in \mathbb{R}$, t *is as in Theorem 7*, $s_1, \ldots, s_q$ *are the roots of* R *with multiplicities* $k_1, \ldots, k_q$ *respectively and* $\mathrm{Re}\, s_j < b$ $(j = 1, \ldots, q)$. *Then the solution* u *of the equation* $Pu = f$ *can be represented in the form*

$$u = \sum_{j=1}^{q} \sum_{p_j=0}^{l_j} w_1^{s_j+2p_j} \sum_{r=0}^{k_j-1} \Big(\sum_{n=r}^{k_j-1} C^j_{nrp} (\ln w_1)^{k_j-n-1} \Big) f_{jp_jr} + g,$$

where l_j *is a nonnegative integer with* $\mathrm{Re}\, s_j + 2l_j < b$, $f_{jp_jr} \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{I(t',\theta')})$ *are Mellin a.f. for every* $\omega' < -2$, *and* $g \in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{R(\theta_1) \times I(t',\theta')})$ *for every* $\omega \in \mathrm{Re}\, \Omega''_b$.

Proof. Define $d = \min\{\mathrm{Re}\, s_j : j = 1, \ldots, q\}$. From Theorem 7 there exists a solution F of (11) in Ω''_d. We can write this function as the sum of a series:

$$(12) \quad F(z) = [R(z_1)]^{-1} \Big[\mathcal{M}f(z) + \sum_{j=1}^{\infty} (-1)^j [R(z_1 - 2) \ldots R(z_1 - 2j)]^{-1}$$
$$\times \sum_{j_2=0}^{j} \sum_{j_3=0}^{j_2} \ldots \sum_{j_n=0}^{j_{n-1}} \binom{j}{j_2} \ldots \binom{j_{n-1}}{j_n} \tau_2^{j_n} \tau_3^{j_{n-1}-j_n} \ldots \tau_n^{j-j_2}$$
$$\times (z_2 + 1) \ldots (z_2 + 2j_n) \ldots (z_n + 1) \ldots (z_n + 2(j - j_2))$$
$$\times \mathcal{M}f(z_1 - 2j, z_2 + 2j_n, \ldots, z_n + 2(j - j_2)) \Big].$$

We can see that the series (12) converges locally uniformly on $\Omega''_b \setminus \{s_j + 2p_j : j = 1, \ldots, q,\ p_j = 0, \ldots, l_j\}$ and that F can be continued to a meromorphic function on Ω''_b with poles at $\{z_1 = s_j + 2p_j\}$ of degree k_j, for $p_j = 0, \ldots, l_j$, $j = 1, \ldots, q$. This means that F can be written in the form

$$F(z) = \frac{F_{jp}(z)}{(z_1 - s_j - 2p_j)^{k_j}},$$

locally on some neighbourhood of $\{z_1 = s_j + 2p_j\}$.

Similarly to the proof of Theorem 6 we obtain the formula

$$\operatorname{sgn} \sigma U_{\sigma}(w) = 2\pi i \sum_{j=1}^{q} \sum_{p_j=0}^{l_j} w_1^{s_j+2p_j} \sum_{r=0}^{k_j-1} \Big(\sum_{n=r}^{k_j-1} C^j_{nrp_j} (\ln w_1)^{k_j-n-1} \Big)$$
$$\times \int_{\mathbb{R}^{n-1}} \Big(\Big(\frac{\partial}{\partial z_1} \Big)^r F_{jp_j} \Big)(s_j + 2p_j, z') \widetilde{\chi}_{\sigma'}^{\theta'+\pi/2}(z') w'^{z'} \, d\beta'$$
$$+ \int_{\mathrm{Re}\, z = \widetilde{\alpha}} F(z) \widetilde{\chi}_{\sigma}^{\widehat{\theta}}(z) w^z \, dz,$$

where in the last term $\widetilde{\alpha}$ is fixed such that $c < \widetilde{\alpha}_1 < b$ and $c \geq \max\{\mathrm{Re}\, s_j + 2p_j : j = 1, \ldots, q,\ p_j = 0, \ldots, l_j\}$, and $\widetilde{\alpha}'$ is such that the last integral exists.

It is sufficient to put

$$\operatorname{sgn} \sigma' G_{\sigma'}^{jp_jr}(w') = \int_{\mathbb{R}^{n-1}} \Big(\frac{\partial}{\partial z_1} \Big)^r F_{jp_j}(s_j + 2p_j, z') \widetilde{\chi}_{\sigma'}^{\theta'+\pi/2}(z') w'^{z'} \, d\beta'$$

for $w' \in \{0 < \sigma' \operatorname{Arg} w' - (\theta' + \pi/2) < 1 \text{ or } |w'| > t' e^{\pi/2}\}$, and

$$\operatorname{sgn} \sigma G_{\sigma}(w) = \int_{\mathrm{Re}\, z = \widetilde{\alpha}} F(z) \widetilde{\chi}_{\sigma}^{\widehat{\theta}}(z) w^z \, dz$$

for $w \in \{0 < \sigma \operatorname{Arg} w - \widehat{\theta} < 1 \text{ or } |w| > \widehat{t}\}$.

Immediately we obtain the following

COROLLARY 2. *Suppose* f *is a Mellin a.f. such that* $\mathcal{M}f$ *is holomorphic on* $\Omega = \{z : \mathrm{Re}(z_j + z_1) < d,\ j = 2, \ldots, n\}$ *for some* $d \in \mathbb{R}$. *Then the* *solution u of the equation $Pu = f$ has the asymptotic expansion*

$$u = \sum_{p=0}^{\infty} \sum_{j=1}^{q} w_1^{s_j+p} \sum_{r=0}^{k_j-1} \Big(\sum_{n=r}^{k_j-1} C^j_{nrp} (\ln w_1)^{k_j-n-1} \Big) f_{jpr},$$

where $f_{jpr} \in \underset{\sim}{\mathcal{M}}'_{(\omega')}(\overline{I(\widehat{t'}, \widehat{\theta'})})$ for $\omega' < -2$. This means that for every $N \in \mathbb{N}$,

$$u - \sum_{p=0}^{N} \sum_{j=1}^{q} w_1^{s_j+2p} \sum_{r=0}^{k_j-1} \Big(\sum_{n=r}^{k_j-1} C^j_{nrp} (\ln w_1)^{k_j-n-1} \Big) f_{jpr}$$
$$\in \underset{\sim}{\mathcal{M}}'_{(\omega)}(\overline{R(\theta_1) \times I(\widehat{t'}, \widehat{\theta'})}) \quad \text{for } \omega \in \overline{\operatorname{Re} \Omega}.$$

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Holomorphic functions and Banach-nuclear decompositions of Fréchet spaces

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Abstract. We introduce a decomposition of holomorphic functions on Fréchet spaces which reduces to the Taylor series expansion in the case of Banach spaces and to the monomial expansion in the case of Fréchet nuclear spaces with basis. We apply this decomposition to obtain examples of Fréchet spaces E for which the τ_ω and τ_δ topologies on $H(E)$ coincide. Our result includes, with simplified proofs, the main known results—Banach spaces with an unconditional basis and Fréchet nuclear spaces with DN [2, 4, 5, 6]—together with new examples, e.g. Banach spaces with an unconditional finite-dimensional Schauder decomposition and certain Fréchet–Schwartz spaces. This gives the first examples of Fréchet spaces, which are not nuclear, with $\tau_0 = \tau_\delta$ on $H(E)$.

In this article we introduce a new decomposition method for holomorphic functions on domains in Fréchet spaces which admit a Banach-nuclear decomposition (Proposition 1). This decomposition reduces to the Taylor series expansion for Banach spaces and to the monomial expansion in the case of Fréchet nuclear spaces with basis. This allows a unified treatment of topological problems on a variety of Fréchet spaces—including Banach spaces and Fréchet nuclear spaces. We apply this decomposition to obtain examples of Fréchet spaces E for which the τ_ω and τ_δ topologies on $H(E)$ coincide. Our result includes, with simplified proofs, the main known results—Banach spaces with an unconditional basis and Fréchet nuclear spaces with DN [2, 4, 5, 6]—together with new examples, e.g. Banach spaces with an unconditional finite-dimensional Schauder decomposition and certain Fréchet–Schwartz spaces (see the examples given below). Combined with results in [7] this gives the first examples of Fréchet spaces, which are not nuclear, with $\tau_0 = \tau_\delta$ on $H(E)$.

The proof is quite technical and we could not avoid some complicated notation. To keep the technicalities to a minimum we confined ourselves in Propositions 3 and 4 to entire functions and indicated afterwards the mod-

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