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On local automorphisms and mappings that preserve idempotents

by

MATEJ BREŠAR and PETER ŠEMRL (Maribor)

Abstract. Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space H. Automorphisms and antiautomorphisms are the only bijective linear mappings θ of $B(H)$ with the property that $\theta(P)$ is an idempotent whenever $P \in B(H)$ is. In case H is separable and infinite-dimensional, every local automorphism of $B(H)$ is an automorphism.

Introduction and statements of the main results. A linear mapping θ of an algebra A into itself is called a *local automorphism* if for every $a \in A$, there exists an automorphism θ_a of A such that $\theta(a) = \theta_a(a)$. This notion was introduced by Larson and Sourour in [11]. They have proved that every surjective local automorphism of $B(X)$, the algebra of all bounded linear operators on an infinite-dimensional Banach space X, is an automorphism [11, Theorem 2.1] (for finite-dimensional spaces X, the result is somewhat different [11, Theorem 2.2]). The aim of this paper is to prove two theorems, which, for the case when X is a Hilbert space, generalize the result of Larson and Sourour.

Note that any local automorphism θ of an algebra A preserves idempotents, that is, for any idempotent $p \in A$, $\theta(p)$ is again an idempotent. The question arises whether this condition itself is sufficient for determining the structure of linear mappings. In our first theorem we give the answer for the case when $A = B(H)$ and θ is bijective.

THEOREM 1. *Let H be a Hilbert space and let $\theta : B(H) \to B(H)$ be a bijective linear mapping. Suppose that $\theta(P)$ is an idempotent whenever $P \in B(H)$ is. Then θ is either an automorphism or an antiautomorphism.*

Let us point out that we do not assume the continuity of θ. A recent paper [3] of the present authors also contains a result concerning mappings

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of $B(X)$ which preserve idempotents; however, the continuity in the weak operator topology was required.

We also remark that linear mappings preserving idempotents have already been treated on matrix algebras [4, 1, 3].

The second question that we pose here is: Can the assumption of the surjectivity in the result of Larson and Sourour be removed? We settle this question for separable Hilbert spaces.

THEOREM 2. *Let H be an infinite-dimensional separable Hilbert space. Then every local automorphism of $B(H)$ is an automorphism.*

It should be mentioned that there is also an analogous notion of local derivations (its definition should be self-explanatory), introduced independently by Kadison [10] and Larson and Sourour [11]. Local derivations and some related mappings were also considered in [2] and [3]. An inspection of these two papers shows that certain algebraical methods allow a unified approach to both local derivations and local automorphisms. This paper, however, is devoted to local automorphisms only.

Proofs. Throughout, H will be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. By $F(H)$ we denote the ideal of all operators in $B(H)$ of finite rank, and by $P(H)$ the set of all idempotent operators in $B(H)$ (that is, $P(H) = \{P \in B(H) \mid P^2 = P\}$). By a *projection* we mean a self-adjoint idempotent. Given $x, y \in H$, by $x \otimes y^*$ we denote a rank one operator defined by $(x \otimes y^*)u = \langle u, y\rangle x$. Operators $T, S \in B(H)$ are said to be *similar* if there exists an invertible operator $A \in B(H)$ such that $S = ATA^{-1}$. Since every automorphism of $B(H)$ is inner [6], a local automorphism θ of $B(H)$ can be characterized as a linear mapping such that the operators A and $\theta(A)$ are similar for every $A \in B(H)$.

The proofs of both Theorems 1 and 2 are based on the following simple lemma, which was also proved in [3].

LEMMA 1. *Let $\theta : B(H) \to B(H)$ be a linear mapping such that $\theta(P(H)) \subseteq P(H)$. Then the restriction of θ to $F(H)$ is a Jordan homomorphism of $F(H)$ into $B(H)$ (that is, $\theta(A^2) = \theta(A)^2$ for every $A \in F(H)$).*

Proof. Let $S \in F(H)$ be a self-adjoint operator. Then $S = \sum_{i=1}^n t_i P_i$, where the P_i are mutually orthogonal projections and t_i are real numbers. Since $P_i + P_j$, $i \neq j$, is again a projection, it follows that $\theta(P_i + P_j)^2 = \theta(P_i + P_j)$. Hence $\theta(P_i)\theta(P_j) + \theta(P_j)\theta(P_i) = 0$, which (by standard arguments) gives $\theta(P_i)\theta(P_j) = 0$, $i \neq j$. Note that this implies $\theta(S^2) = \theta(S)^2$. Replacing in this identity S by $S + T$, where S and T are both self-adjoint, it follows that $\theta(ST + TS) = \theta(S)\theta(T) + \theta(T)\theta(S)$. Since every operator $A \in F(H)$ can be written in the form $A = S + iT$ with $S, T \in F(H)$ self-adjoint, we get $\theta(A^2) = \theta(A)^2$.

From the proof of Lemma 1 it is evident that $\theta(P)\theta(Q) = \theta(Q)\theta(P) = 0$ whenever $P, Q \in P(H)$ satisfy $PQ = QP = 0$. Besides this simple observation and Lemma 1, the main tool in the proof of Theorem 1 is a result of Pearcy and Topping which implies that every operator in $B(H)$ is a linear combination of idempotents [12].

Proof of Theorem 1. Pick an idempotent P of rank one, and let us show that $Q = \theta(P)$ also has rank one. Set $X_1 = PB(H)P$, $X_2 = PB(H)(I-P)$, $X_3 = (I-P)B(H)P$, $X_4 = (I-P)B(H)(I-P)$; thus $B(H) = X_1 \oplus X_2 \oplus X_3 \oplus X_4$. Similarly, $B(H) = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$ where $Y_1 = QB(H)Q$, $Y_2 = QB(H)(I-Q)$, $Y_3 = (I-Q)B(H)Q$, $Y_4 = (I-Q)B(H)(I-Q)$. Take $A \in X_2$. For any $\alpha \in \mathbb{C}$ we then have $P + \alpha A \in P(H)$, and hence $Q + \alpha\theta(A) \in P(H)$. This clearly yields $\theta(A)Q + Q\theta(A) = \theta(A)$, whence $Q\theta(A)Q = 0$, and therefore we get $\theta(A) = Q\theta(A)(I-Q) + (I-Q)\theta(A)Q \in Y_2 \oplus Y_3$. Thus, $\theta(X_2) \subseteq Y_2 \oplus Y_3$, and similarly we see that $\theta(X_3) \subseteq Y_2 \oplus Y_3$.

Next we claim that $\theta(X_4) \subseteq Y_4$. As X_4 is isomorphic to $B(\operatorname{Ker} P)$, the result of Pearcy and Topping tells us that it suffices to show that $\theta(R) \in Y_4$ for any idempotent $R \in X_4$. Thus, we have to see that $\theta(R)Q = Q\theta(R) = 0$. Since P and R are idempotents such that $PR = RP = 0$, this is true indeed. Finally, as $X_1 = \mathbb{C}P$, we have $\theta(X_1) = \mathbb{C}Q$. Therefore, we conclude that $\theta(B(H)) \subseteq \mathbb{C}Q \oplus Y_2 \oplus Y_3 \oplus Y_4$. However, θ is onto, so it follows that $\mathbb{C}Q = Y_1 = QB(H)Q$. This means that Q has rank one.

By Lemma 1, $\theta|F(H)$ is a Jordan homomorphism. Since $F(H)$ is a locally matrix algebra, a result of Jacobson and Rickart [8, Theorem 8] tells us that $\theta|F(H) = \varphi + \psi$, where $\varphi : F(H) \to B(H)$ is a homomorphism and $\psi : F(H) \to B(H)$ is an antihomomorphism. Pick an idempotent P of rank one. Then $\theta(P)$ is the sum of the idempotents $\varphi(P)$ and $\psi(P)$; therefore, as $\theta(P)$ also has rank one, it follows that either $\varphi(P) = 0$ or $\psi(P) = 0$. Thus, at least one of φ and ψ has a nonzero kernel. Since the kernels of homomorphisms and antihomomorphisms are ideals, and since the only nonzero ideal of $F(H)$ is $F(H)$ itself, we have $\varphi = 0$ or $\psi = 0$. Thus, either $\theta|F(H) = \varphi$ or $\theta|F(H) = \psi$. There is no loss of generality in assuming that $\theta|F(H) = \varphi$ is a homomorphism—otherwise consider the mapping $A \to \theta(A)^t$ where B^t denotes the transpose of B relative to an arbitrary orthonormal basis fixed in advance.

Take $x \in H$ with $\|x\| = 1$. Since $x \otimes x^*$ is an idempotent of rank one, there exist $u, v \in H$ such that $\theta(x \otimes x^*) = u \otimes v^*$ and $\langle u, v\rangle = 1$. Define a linear operator $T : H \to H$ by $Ty = \theta(y \otimes x^*)u$. Given $F \in F(H)$ and $z \in H$, we have

$$TFz = \theta(Fz \otimes x^*)u = \theta(F(z \otimes x^*))u = \theta(F)\theta(z \otimes x^*)u = \theta(F)Tz.$$

Thus, T satisfies

(1) $$TF = \theta(F)T \quad \text{for every } F \in F(H).$$

We claim that T is one-to-one. Indeed, if $Ty = 0$ for some $y \in H$, then we have $0 = \theta(x \otimes y^*)Ty = T(x \otimes y^*)y = \langle y, y\rangle Tx$; since $Tx = u \neq 0$, it follows that $y = 0$.

Our next goal is to show that

(2) $$TA = \theta(A)T \quad \text{for every } A \in B(H).$$

Of course, it is enough to show that $TP = \theta(P)T$ for every $P \in P(H)$. Set $S = TP - \theta(P)T$ and let us prove that $S = 0$. Note that it suffices to show that $SQ = 0$ for any idempotent Q of rank one satisfying either $PQ = QP = 0$ or $PQ = QP = Q$. Let us first consider the case when $PQ = QP = 0$. Then $\theta(P)\theta(Q) = 0$, and therefore, using (1), we get $SQ = TPQ - \theta(P)TQ = -\theta(P)\theta(Q)T = 0$. Now suppose that $PQ = QP = Q$. Then Q and $P - Q$ are idempotents such that $(P - Q)Q = Q(P - Q) = 0$, which yields $\theta(P - Q)\theta(Q) = 0$, that is, $\theta(P)\theta(Q) = \theta(Q)$. Applying (1) it follows that $SQ = 0$, and (2) is thereby proved.

As θ is onto, (2) shows that every operator in $B(H)$ leaves the range of T invariant. Hence T is bijective, and therefore, $\theta(A) = TAT^{-1}$ for every $A \in B(H)$. This means that θ is an automorphism and the proof is complete (we remark that using the closed graph theorem one can show that T is actually continuous).

Remark. Under the additional assumption that θ is norm-continuous, Theorem 1 is much easier to prove. Namely, using the fact that the set of real-linear combinations of mutually orthogonal projections in $B(H)$ is dense in the space of self-adjoint operators in $B(H)$, it can easily be shown (just adapt the argument given in Lemma 1) that θ is a Jordan automorphism. But then [7, Theorem 3.1] tells us that θ is either an automorphism or an antiautomorphism.

In order to prove Theorem 2, we establish two preliminary results.

LEMMA 2. *Let $T, S \in B(H)$ and let $A, B : H \to H$ be linear operators. Suppose that for each pair of vectors $x, y \in H$, the operators $T + x \otimes y^*$ and $S + (Ax) \otimes (By)^*$ are similar. Then*

$$\langle T^n x, y\rangle = \langle S^n Ax, By\rangle, \quad x, y \in H,\ n = 0, 1, 2, \ldots$$

Proof. Let λ be any complex number such that $|\lambda| > \max\{\|T\|, \|S\|\}$. Suppose that $\langle(\lambda - T)^{-1}x, y\rangle = 1$ for some $x, y \in H$. Then

$$(T + x \otimes y^*)(\lambda - T)^{-1}x = T(\lambda - T)^{-1}x + x = \lambda(\lambda - T)^{-1}x.$$

Thus, λ is an eigenvalue of $T + x \otimes y^*$. By the assumption, λ must also be an eigenvalue of $S + (Ax) \otimes (By)^*$, i.e.,

$$(S + (Ax) \otimes (By)^*)u = \lambda u$$

for some $u \neq 0$. This yields $u = \langle u, By\rangle(\lambda - S)^{-1}Ax$, and therefore,

$$\langle u, By\rangle = \langle u, By\rangle\langle(\lambda - S)^{-1}Ax, By\rangle.$$

By the previous relation, $\langle u, By\rangle \neq 0$. Thus, $\langle(\lambda - T)^{-1}x, y\rangle = 1$ implies $\langle(\lambda - S)^{-1}Ax, By\rangle = 1$. In a similar fashion one proves the reverse implication. Hence $\langle(\lambda - T)^{-1}x, y\rangle = 1$ if and only if $\langle(\lambda - S)^{-1}Ax, By\rangle = 1$. By linearity, it follows that

$$\langle(\lambda - T)^{-1}x, y\rangle = \langle(\lambda - S)^{-1}Ax, By\rangle$$

for all $x, y \in H$, $|\lambda| > \max\{\|T\|, \|S\|\}$. Using $(\lambda - T)^{-1} = \sum_{k=0}^{\infty} T^k/\lambda^{k+1}$ and $(\lambda - S)^{-1} = \sum_{k=0}^{\infty} S^k/\lambda^{k+1}$ we obtain the statement of the lemma.

We remark that in the proof of Lemma 2 we have used some ideas of Jafarian and Sourour [9].

LEMMA 3. *Let H be a separable Hilbert space and let θ be a local automorphism of $B(H)$. If the restriction of θ to $F(H)$ is a homomorphism, then θ is an automorphism.*

Proof. Of course, we may assume that H is infinite-dimensional. Let (e_k) be an orthonormal basis in H. Define $S \in B(H)$ by

$$S = \sum_{n=1}^{\infty} 2^{-n} e_n \otimes e_n^*.$$

Since $\theta(S) = A_S S A_S^{-1}$ for some invertible $A_S \in B(H)$, there is no loss of generality in assuming that $\theta(S) = S$ (otherwise replace θ by the mapping $T \mapsto A_S^{-1}\theta(T)A_S$).

Fix $u \in H$ such that $\|u\| = 1$. As $\theta(u \otimes u^*)$ is an idempotent of rank 1, we have

$$\theta(u \otimes u^*) = w \otimes v^*,$$

where $\langle w, v\rangle = 1$. Define $A, B : H \to H$ by

$$Ax = \theta(x \otimes u^*)w, \qquad Bx = \theta(u \otimes x^*)^*v.$$

Clearly, A and B are linear operators. Since $\theta|F(H)$ is a homomorphism, for all $x, y \in H$ we have

$$\begin{aligned}\theta(x \otimes y^*) &= \theta((x \otimes u^*)(u \otimes u^*)(u \otimes y^*)) \\ &= \theta(x \otimes u^*)(w \otimes v^*)\theta(u \otimes y^*) \\ &= (\theta(x \otimes u^*)w) \otimes (\theta(u \otimes y^*)^*v)^* = (Ax) \otimes (By)^*.\end{aligned}$$

Hence

$$\theta(S+x\otimes y^*)=\theta(S)+\theta(x\otimes y^*)=S+(Ax)\otimes(By)^*.$$

Thus, for each pair $x,y\in H$, the operators $S+x\otimes y^*$ and $S+(Ax)\otimes(By)^*$ are similar. By Lemma 2, it follows that

$$(3)\qquad \langle S^k x,y\rangle=\langle S^k Ax,By\rangle,\qquad x,y\in H,\ k=0,1,2,\ldots$$

We claim that

$$(4)\qquad \langle Ae_i,e_n\rangle\langle Be_i,e_n\rangle=0,\qquad i\neq n.$$

Since the operator $S+(Ae_i)\otimes(Be_i)^*$ is similar to $S+e_i\otimes e_i^*$, 2^{-n} is an eigenvalue of $S+(Ae_i)\otimes(Be_i)^*$ for every $n\neq i$. Thus, there exists $x\neq 0$ such that

$$(S+(Ae_i)\otimes(Be_i^*))x=2^{-n}x.$$

That is,

$$(S-2^{-n})x=-\langle x,Be_i\rangle Ae_i.$$

Using $(S-2^{-n})e_n=0$, it follows that

$$0=\langle (S-2^{-n})x,e_n\rangle=\langle x,Be_i\rangle\langle Ae_i,e_n\rangle.$$

Suppose $\langle Ae_i,e_n\rangle\neq 0$. Then $\langle x,Be_i\rangle=0$, which yields $Sx=2^{-n}x$. But then $x=\lambda e_n$ for some $\lambda\neq 0$. Consequently, $\langle e_n,Be_i\rangle=0$. This proves (4).

Applying (3) we get

$$1=\langle e_i,e_i\rangle=\langle Ae_i,Be_i\rangle=\sum_{n=1}^{\infty}\langle Ae_i,e_n\rangle\overline{\langle Be_i,e_n\rangle},$$

and so (4) implies

$$(5)\qquad \langle Ae_i,e_i\rangle\overline{\langle Be_i,e_i\rangle}=1.$$

Fix positive integers i and j, $i\neq j$. By (3) we have

$$\begin{aligned}0&=\langle S^k e_i,e_j\rangle=\langle S^k Ae_i,Be_j\rangle\\&=\sum_{n=1}^{\infty}2^{-nk}\langle Ae_i,e_n\rangle\overline{\langle Be_j,e_n\rangle},\qquad k=0,1,2,\ldots\end{aligned}$$

Let $\lambda_n=\langle Ae_i,e_n\rangle\overline{\langle Be_j,e_n\rangle}$. We intend to show that $\lambda_n=0$ for every n. We have proved that

$$\sum_{n=1}^{\infty}2^{-nk}\lambda_n=0,\qquad k=0,1,2,\ldots,$$

and we know that $\sum_{n=1}^{\infty}|\lambda_n|<\infty$. Suppose there exists a positive integer n_0 such that $\lambda_{n_0}\neq 0$ and $\lambda_k=0$ for $k<n_0$. Let

$$\mu=\sum_{n>n_0}|\lambda_n|$$

and pick a positive integer k_0 such that $|\lambda_{n_0}|>2^{-k_0}\mu$. Then we have

$$\begin{aligned}0&=\Big|\sum_{n=1}^{\infty}2^{-k_0n}\lambda_n\Big|\geq 2^{-k_0n_0}|\lambda_{n_0}|-\sum_{n>n_0}2^{-k_0n}|\lambda_n|\\&=2^{-k_0n_0}\Big(|\lambda_{n_0}|-\sum_{n>n_0}2^{-k_0(n-n_0)}|\lambda_n|\Big)\geq 2^{-k_0n_0}\Big(|\lambda_{n_0}|-2^{-k_0}\sum_{n>n_0}|\lambda_n|\Big)\\&=2^{-k_0n_0}(|\lambda_{n_0}|-2^{-k_0}\mu)>0.\end{aligned}$$

This contradiction proves that $\langle Ae_i,e_n\rangle\overline{\langle Be_j,e_n\rangle}=0$, $i\neq j$. In particular, if $i\neq n$, then we have $\langle Ae_i,e_n\rangle\overline{\langle Be_n,e_n\rangle}=0$. Since $\overline{\langle Be_n,e_n\rangle}\neq 0$ by (5), it follows that $\langle Ae_i,e_n\rangle=0$, $i\neq n$. Therefore, for every i we have

$$(6)\qquad Ae_i=\alpha_i e_i$$

for some complex number α_i; observe that (5) implies that $\alpha_i\neq 0$. Similarly we see that Be_i must be a multiple of e_i; in view of (5) we then have

$$(7)\qquad Be_i=(1/\overline{\alpha}_i)e_i.$$

By (3) we have

$$(8)\qquad \langle x,y\rangle=\langle Ax,By\rangle,\qquad x,y\in H.$$

Applying (6), (7), (8) and the closed graph theorem one can easily show that A and B are bounded operators. Therefore, (8) implies that $A^*B=I$. Thus A^* is surjective. By (6) we see that the range of A is dense in H. Hence A^* is bijective, which yields $B=(A^*)^{-1}$.

Now, for every $T\in B(H)$ and all $x,y\in H$ we have

$$\theta(T+x\otimes y^*)=\theta(T)+(Ax)\otimes((A^{-1})^*y)^*.$$

Thus, Lemma 2 tells us that

$$\langle Tx,y\rangle=\langle\theta(T)Ax,(A^{-1})^*y\rangle,\qquad x,y\in H,$$

which gives $A^{-1}\theta(T)A=T$, $T\in B(H)$. This proves the lemma.

We now have enough information to prove Theorem 2.

Proof of Theorem 2. Using Lemma 1 and the result of Jacobson and Rickart one shows in the same manner as in the proof of Theorem 1 that $\theta|F(H)$ is either a homomorphism or an antihomomorphism. In view of Lemma 3 it suffices to consider the situation when $\theta|F(H)=\psi$ is an antihomomorphism. But then, as θ maps $F(H)$ into itself, $\theta^2|F(H)=\psi^2$ is a homomorphism. Observe that θ^2 is also a local automorphism. Applying Lemma 3 we then find that θ^2 is an automorphism. In particular, θ^2 is onto, which implies that so is θ. Thus, θ satisfies the requirements of Theorem 1. Hence θ is either an automorphism or an antiautomorphism. But the latter cannot occur. Namely, as is known, in that case we would have $\theta(A)=$

VA^*V^{-1} for every $A \in B(H)$, where V is a bounded invertible conjugate-linear operator on H. On the other hand, $\theta(A)$ is always similar to A. In particular, it would follow that an operator A is one-to-one if and only if A^* is. But this is certainly not true (consider, for instance, the shift operator). Thus, θ is an automorphism. The proof of the theorem is complete.

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The upper bound of the number of eigenvalues for a class of perturbed Dirichlet forms

by

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Abstract. The theory of Markov processes and the analysis on Lie groups are used to study the eigenvalue asymptotics of Dirichlet forms perturbed by scalar potentials.

Introduction. Let $A(x,D) = \sum_{|\alpha|\le m} a_\alpha(x)(i^{-1}\partial/\partial x)^\alpha$ be a selfadjoint differential operator with the symbol $A(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha$. The Bohr–Sommerfeld quantization principle, according to which the volume $\sim h^d$ in the phase space should count for one eigenvalue of $A(x,D)$, leads us to the hypothesis that the number of eigenvalues of $A(x,D)$ which are less than λ should be approximately the volume of the set $A = \{(x,\xi) \mid A(x,\xi) < \lambda\}$. If $A(x,D)$ is elliptic and $\lambda \to \infty$, this hypothesis is asymptotically correct (cf. [10]). For the Schrödinger operator $-\Delta + V$, this "volume-counting" has been fully expressed in the form of the Cwikel–Lieb–Rosenblum inequality (cf. [13]). However, this inequality can also produce grossly inaccurate estimates for systems as simple as two uncoupled harmonic oscillators. Following Fefferman (cf. [5]), it is better to count the number of distorted unit cubes which can be packed disjointly inside the subset A instead of measuring the importance of A. This idea, called the SAK-principle, led to sharp estimates of eigenvalue asymptotics (cf. [5], [6]). Because counting the number of distorted unit cubes which fit inside A is not easy, this kind of estimate gives us only a qualitative description for the number of eigenvalues. (In [3], it is shown how we can count the number of proper boxes in the case of Schrödinger operators with polynomial potentials.)

The aim of this paper is to redefine the place of "volume-counting type" estimates and to give a quantitative description of the number of eigenvalues for operators defined as $\mathcal{D} + V$, where $\mathcal{D}$ is the infinitesimal generator of a (sub)markovian semigroup and V is a function. For $\mathcal{D}$ being a sum of

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