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On Dirichlet–Schrödinger operators with strong potentials

by

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Abstract. We consider Schrödinger operators $H = -\Delta/2 + V$ ($V \geq 0$ and locally bounded) with Dirichlet boundary conditions, on any open and connected subdomain $D \subset \mathbb{R}^n$ which either is bounded or satisfies the condition $d(x, D^c) \to 0$ as $|x| \to \infty$. We prove exponential decay at the boundary of all the eigenfunctions of H whenever V diverges sufficiently fast at the boundary ∂D, in the sense that $d(x,D^c)^2V(x) \to \infty$ as $d(x,D^c) \to 0$. We also prove bounds from above and below for $\mathrm{Tr}(\exp[-tH])$, and in particular we give criterions for the finiteness of such trace. Applications to pointwise bounds for the integral kernel of $\exp[-tH]$ and to the computation of expected values of the Feynman–Kac functional with respect to Doob h-conditioned measures are given as well.

1. Introduction. Let D be an open and connected proper subset of $\mathbb{R}^n$. On D, one can consider the *Dirichlet Laplacian*, Δ_D, or the positive operator $H_0 = -\Delta_D/2$, which is the self-adjoint operator (in $\mathrm{L}^2(D)$) associated with the closure of the quadratic form

$$Q_0(f) = \frac{1}{2}\int_D |\nabla f(x)|^2\,dx, \qquad f \in C_0^\infty(D). \tag{1.1}$$

If $V : D \to [0,\infty)$ is a locally bounded measurable function (the *potential* function), then one can also consider the Schrödinger operator $H = -\Delta/2 + V = H_0 + V$, with Dirichlet boundary conditions, which is the self-adjoint operator associated, by the above procedure, with the quadratic form

$$Q(f) = Q_0(f) + \int_D V(x)|f(x)|^2\,dx, \qquad f \in C_0^\infty(D). \tag{1.2}$$

The reason for the factor 1/2 in (1.1), (1.2) is simply that this is the usual normalization for the generator of the semigroup associated with Brownian motion, and we prove some of the main results of this paper by probabilistic methods. It should also be noted here, once for all, that a negative part of

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the potential V, belonging to the Kato class K_n (cf. [S]), could be added to H without altering most of our results, and we do not follow this way mainly in order to avoid cumbersome notation. Also, H_0 could often be replaced by general uniformly elliptic second-order operators in divergence form.

The Dirichlet–Schrödinger operator H has been studied in previous works of F. Cipriani and the present author [CG1–3], in connection with pointwise lower bounds for positive eigenfunctions of H, and with some contractivity properties of the semigroup associated with H, which are known as *intrinsic ultracontractivity*, and which were introduced by E. B. Davies [D1, D3], and later on studied, among many others, in [DS], [Da], [B2] [BD2]. It should be noticed that the main novel feature of [CG1–3] is the possibility of studying Dirichlet–Schrödinger operators with potentials which are *not* small, in the quadratic form sense, with respect to $-\Delta_D$, and whose spectral properties are then not to be expected to be necessarily similar to those of $-\Delta_D$.

We focus our attention here on the opposite case, in which the potential V diverges sufficiently fast at the boundary. We identify a class of potentials for which many spectral properties of H are essentially independent of the geometry of the domain D, and involve only the rate of divergence of V; this class is "close" to being optimal, as shown in Example 2.6. The characterizing property of this class is the condition $d(x)^2V(x) \to \infty$ as $d(x) \to 0$, where $d(x) = d(x, D^c)$ (the *distance function from the boundary*). In this connection, it should be noticed that we restrict our attention to domains D which either are bounded, or satisfy $d(x) \to 0$ as $|x| \to \infty$, mainly because other situations can be discussed by methods which are essentially those of [CS]. One should also notice the similarity with the theory of intrinsic ultracontractivity of Schrödinger operators in $\mathbb{R}^n$; in fact, if $V(x) = |x|^\alpha$, then the associated Schrödinger operator in $\mathbb{R}^n$ is intrinsically ultracontractive if and only if $\alpha > 2$ (see [D3]).

In Section 2, we prove exponential decay at the boundary of all the eigenfunctions of H, by means of probabilistic estimates which use old ideas of Carmona [C]. We begin by observing that, under the above condition on V, H has compact resolvent, and hence a purely discrete spectrum. Results similar to those of Section 2 have been obtained in [CG3], but in the case of a continuous V. Moreover, the techniques there were much more involved, since they involved weighted L^2-estimates for eigenfunctions in terms of the *Agmon metric* of V ([A], [CS]), and subsolution estimates.

In Section 3, we discuss the tracial properties of the semigroup $\exp[-tH]$ associated with H. More precisely, we prove two-sided estimates for $\mathrm{Tr}(\exp[-tH])$, and in particular we give an integral criterion for the finiteness of $\mathrm{Tr}(\exp[-tH])$ for all $t > 0$ in terms of $V(x)$ and $d(x)$, which is valid for *weak* potentials, that is, those which satisfy $V(x)d(x)^2 \to 0$ as $d(x) \to 0$. This generalizes a well known condition of Davies [D2] for the finiteness of $\mathrm{Tr}(\exp[t\Delta_D])$. A similar criterion is stated for potentials which diverge sufficiently fast at the boundary and have mild oscillation. The results are then applied to prove pointwise bounds for the heat kernel of H (that is, for the integral kernel of $\exp[-tH]$), and finally to prove some bounds for the expectation of the Feynman–Kac functional associated with V, with respect to the measure on the path space induced by the so-called Doob h-conditioned processes [Do]. Results of this latter type are typically known only for V in the Kato class K_n ([Da], [B2], [BD1]).

2. Pointwise decay of eigenfunctions at the boundary. In this section we prove, by probabilistic methods, a pointwise upper bound for any eigenfunction of a Dirichlet–Schrödinger operator on a proper subdomain D of $\mathbb{R}^n$. The main assumption which we shall use below is the following (the notation hereafter will be that of the introduction):

ASSUMPTION 2.1. (i) *The domain D is either bounded, or has the property that* $\lim_{|x|\to\infty} d(x) = 0$;

(ii) $V : D \to \mathbb{R}$ *is a.e. non-negative and belongs to* $\mathrm{L}^\infty_{\mathrm{loc}}(D)$. *Moreover*,

$$\lim_{d(x)\to 0} d(x)^2 V(x) = \infty.$$

Part (i) of the assumption is motivated by the requirement that the spectral properties of the Dirichlet–Schrödinger operators considered cannot be discussed by straightforward modifications of the known methods for Schrödinger operators in $\mathbb{R}^n$. If it does not hold, one can study the corresponding operator, under the further assumption that $\lim_{t\to\infty} V(\gamma(t)) = \infty$ along any curve $\gamma : [0, \infty) \to D$ such that $\inf_{t\ge 0} d(\gamma(t)) > 0$, by a combination of the methods of the present paper, and of those of [CS]. Part (ii) gives a precise meaning to the statement "Schrödinger operators with strong potentials" appearing in the title; we assume local boundedness of V since we are mainly interested in the singularities of V at the boundary. Here and in the sequel, by a bounded function we mean an *essentially* bounded function.

We recall that the spectrum of an operator H on a Hilbert space $\mathrm{L}^2(X, \mu)$ is said to be *purely discrete* if it consists of isolated eigenvalues, each of which is of finite multiplicity. The *essential spectrum* of H is that part of the spectrum which is not purely discrete. Hereafter, by the *spectrum* of H we mean its $\mathrm{L}^2(D)$-spectrum.

The first technical result will be basic in what follows, and uses a well known argument for proving discreteness of the spectrum.

LEMMA 2.2. *Suppose that Assumption* 2.1(i) *holds, and that* $V(x) \to \infty$ *as* $d(x) \to 0$ (*which is true in particular if Assumption* 2.1(ii) *is satisfied*).

Then the resolvent H^{-1} is compact as an operator on $L^2(D)$, so that the spectrum of H is purely discrete.

Proof. We prove that the essential spectrum of H is empty. To this end fix $c \geq 0$ and notice that, by Assumption 2.1, H can be written as

$$H = -\tfrac{1}{2}\Delta + V_1^{(c)} + V_2^{(c)},$$

where the potentials $V_{1,2}$ satisfy the following conditions:

(i) $V_1^{(c)} \geq c$ for all $x \in D$.

(ii) $V_2^{(c)}$ is bounded with compact support.

In fact, simply choose $V_1^{(c)}(x) = V(x) \vee c$, $V_2^{(c)}(x) = V(x) - V_1^{(c)}(x)$, and notice that $V_2^{(c)}$ is bounded with compact support by the assumptions. By [D3, Lemma 1.6.5], it follows that

$$[H+a]^{-1} - [\tfrac{1}{2}\Delta + V_1^{(c)} + a]^{-1}$$

is compact for $a > 0$ large enough, so that in particular the operators H and $H_1 = H_0 + V_1^{(c)}$ have the same essential spectrum. However, the spectrum of H_1 is contained in $[c, \infty)$, since H_0 is a positive operator and $V_1^{(c)} \geq c$ by construction. Since moreover $c \geq 0$ is arbitrary, it follows that the essential spectrum of H is empty. ■

Remark 2.3. By the min-max characterization of eigenvalues, it follows that the assertion of Lemma 2.2 remains true also when $H_0 = -\Delta/2$ is replaced by an elliptic operator given, in the weak sense, by

$$[Lf](x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial f}{\partial x_j}(x)\right),$$

where the matrix-valued function $(a_{i,j})$ is measurable, locally bounded, and satisfies the strict ellipticity condition $(a_{i,j}) \geq \lambda 1$ for some $\lambda > 0$.

COROLLARY 2.4. *Under the assumptions of Lemma 2.2, the operator $\exp[-tH]$ maps $L^1(D)\cap L^\infty(D)$ into itself, and can be extended, for $p \in [1,\infty]$, to a positive contraction semigroup on $L^p(D)$, with generator H_p. If $p < \infty$, the corresponding semigroup is strongly continuous. Moreover, for all $p \in (1,\infty)$, $\exp[-tH_p]$ is compact on $L^p(D)$, the spectrum of H_p is independent of p, and any $L^2(D)$-eigenfunction of $H = H_2$ belongs to $L^p(D)$.*

Proof. H is the generator of a symmetric Markov semigroup, by [D3, Th. 1.8.1], therefore the first part of the statement follows from the application of [D3, Th. 1.4.1]. To prove compactness, notice that the compactness of $\exp[-tH]$ on $L^2(D)$ follows from the compactness of H^{-1}, so that the assertion is just an application of [D3, Th. 1.6.3]. ■

The following is the main result of this section:

THEOREM 2.5. *Suppose that Assumption 2.1 holds true, and moreover that*

$$V(x) \geq \gamma/d(x)^{2m} \tag{2.1}$$

for some $\gamma > 0$ and $m > 1$. Let $p \in [1,\infty]$, and consider any L^p-eigenfunction ψ of H_p, with eigenvalue E (> 0). Then there exists $k = k_\delta > 0$ such that the following pointwise upper bound holds:

$$|\psi(x)| \leq k \exp[-\delta/d(x)^{m-1}] \tag{2.2}$$

for all $\delta < \widehat{\delta}$, where

$$\widehat{\delta} = \widehat{\delta}(m) = \begin{cases} \gamma^{1/2}/2^{(2m+1)/2} & \text{if } 1 < m \leq 2, \\ \gamma^{1/2}(m-1)^{m-1}/(\sqrt{2}\, m^m) & \text{if } m \geq 2. \end{cases} \tag{2.3}$$

Proof. By the spectral theorem, ψ is an eigenvector of $\exp[-tH]$ with eigenvalue $\exp[-tE]$. For any $x \in D$, one then has, by the Feynman–Kac formula,

$$e^{-tE}\psi(x) = (e^{-tH}\psi)(x) = \mathbf{E}_x\Big(e^{-\int_0^t V(X_u)\,du}\psi(X_t)\Big),$$

where X_s is a standard n-dimensional Brownian motion, and $\mathbf{E}_x$ denotes expectation with respect to Wiener measure with starting point x. Since ψ is necessarily bounded, it follows that

$$|\psi(x)| = e^{tE}\Big|\mathbf{E}_x\Big(e^{-\int_0^t V(X_u)\,du}\psi(X_t)\Big)\Big| \leq e^{tE}\|\psi(x)\|_\infty \mathbf{E}_x\Big(e^{-\int_0^t V(X_u)\,du}\Big).$$

Let us fix $s \in (0,1)$, and consider $B(x,s) := B(x, sd(x))$. If τ_s denotes the first hitting time of $B(x,s)^c$, one has

$$|\psi(x)| \leq e^{tE}\Big[\mathbf{E}_x\Big(e^{-\int_0^t V(X_u)\,du}; \tau_s \geq t\Big) + \mathbf{E}_x\Big(e^{-\int_0^t V(X_u)\,du}; \tau_s < t\Big)\Big].$$

Define

$$V_s(x) = \inf_{y\in B(x,s)} V(y). \tag{2.4}$$

It follows that, since $V \geq 0$ a.e.,

$$\begin{aligned} |\psi(x)| &\leq e^{tE}\|\psi\|_\infty(e^{-tV_s(x)} + P_x(\tau_s < t)) \\ &\leq e^{tE}\|\psi\|_\infty\bigg(e^{-tV_s(x)} + c\bigg[\bigg(\frac{sd(x)}{\sqrt{t}}\bigg)^{0\vee(n-2)} + 1\bigg]e^{-s^2 d(x)^2/(2t)}\bigg), \end{aligned}$$

where the last inequality follows from eq. (2.4) of [C], for a suitable $c > 0$.

By adapting the ideas of Carmona to the present context, we now put $t = t(x)$ in the above formula, so as to obtain a general pointwise upper bound on $|\psi|$. Under Assumption 2.1, we set in particular $t(x)=\gamma^{-1/2}Ad(x)^\alpha$, for positive constants α, A. Then, on using again (2.1), it follows that

$$|\psi(x)|\le\|\psi\|_\infty e^{EA\gamma^{-1/2}d(x)^\alpha}\bigg(e^{-A\gamma^{1/2}/[(1+s)^{2m}d(x)^{2m-\alpha}]}+c\bigg[\bigg(\frac{s\gamma^{1/2}}{A^{1/2}d(x)^{\alpha/2-1}}\bigg)^{0\vee(n-2)}+1\bigg]e^{-s^2\gamma^{1/2}/[2Ad(x)^{\alpha-2}]}\bigg).$$

Setting now $\alpha=1+m$ (this choice is easily seen to be optimal), one finds

$$|\psi(x)|\le\|\psi\|_\infty e^{EA\gamma^{1/2}d(x)^{1+m}}\bigg(e^{-A\gamma^{-1/2}/[(1+s)^{2m}d(x)^{m-1}]}+c\bigg[\bigg(\frac{s\gamma^{1/2}}{A^{1/2}d(x)^{(m-1)/2}}\bigg)^{0\vee(n-2)}+1\bigg]e^{-s^2\gamma^{1/2}/[2Ad(x)^{m-1}]}\bigg).$$

Next, consider the function $f:\Omega:=\mathbb{R}^+\times(0,1)\to\mathbb{R}$ defined by

$$f(A,s)=\frac{A}{(1+s)^{2m}}\wedge\frac{s^2}{2A}.$$

An elementary computation shows that if $1<m\le 2$ then $\sup_\Omega f=2^{-(2m+1)/2}$, while if $m\ge 2$ then $\sup_\Omega f=\max f=(m-1)^{m-1}/(\sqrt{2}\,m^m)$. This implies the assertion. ■

It is possible to show that the methods used in the proof of the above theorem are, in general, not applicable to potentials which do not diverge sufficiently fast at the boundary. A counterexample is in fact provided by the following

EXAMPLE 2.6. Let $n\ge 4$, $D=B(0,1)\setminus\{0\}$. It is known that each point of $\mathbb{R}^n$ has capacity zero, so that the Dirichlet boundary condition on any isolated point is irrelevant. Therefore, $-\Delta_D=-\Delta_{D_1}$, where $D_1=B(0,1)$. Therefore $-\Delta_D$ has bounded resolvent and a purely discrete spectrum. Let ψ_0 be the *ground state eigenfunction* of $-\Delta_D$, that is, the unique eigenvector corresponding to the lowest eigenvalue of $-\Delta_D$. Consider any potential V on D which is of the form $V(x)=|x|^{-\alpha}$, with $\alpha<2$, for $|x|\le 1/2$, which vanishes for $3/4\le|x|<1$, and which is of class $C^\infty(D)$. Then it is elementary to check that V belongs to $\mathrm{L}^p(D)$ for a suitable $p>n/2$, so that in particular V belongs to the Kato class K_n ([S]). Therefore, by a local Harnack principle and the above observations, it follows that $\psi_0^V(0)>0$, where ψ_0^V is the ground state eigenfunction of the Dirichlet–Schrödinger operator H (with Dirichlet boundary conditions on D) corresponding to V. The same conclusion holds for a potential V which is of the form $V(x)=|x|^{-2}(\ln(|x|+2))^{-\alpha}$, with $\alpha>1$, in a neighbourhood of the origin. In fact, V is easily shown to belong to the Kato class as well.

Thus, for the above choices of V, there is an eigenfunction of H which does not vanish on ∂D, so that there is no hope to enlarge very much the class of potentials which can be treated as above, unless further assumptions on D are made. In this respect, we notice that regularity in the sense of Davies [D3] is not to be expected to be strictly relevant to the present problem, since in fact the quadratic form inequality $c/|x|^2\le-\Delta$ (for some $c>0$, see [RS1], [D4]) shows that D is *strongly* regular in the sense of Davies.

A further result can be obtained with a proof which is very similar to that of Theorem 2.5, and which is more useful in some situations, since it involves (a regularization of) the potential V itself. In fact, the next result reminds the classical W.K.B. estimates in $\mathbb{R}^n$ (cf. [RS2]). It should be noticed that the following estimate for $|\psi(x)|$ involves *only*, for any fixed $x\in D$, the local behaviour of V near x. This improves the (non-local) estimates of [CG3], which involve the Agmon metric of V.

THEOREM 2.7. *Under Assumption* 2.1 *one has, for any eigenfunction* ψ *of* H_p,

$$|\psi(x)|\le ke^{-\delta d(x)\sqrt{V_s(x)}} \tag{2.5}$$

for all $\delta<1/2$, *where* $V_s(x)$ *is defined in* (2.4), *so that in particular* $d(x)^2V_s(x)\to\infty$ *as* $d(x)\to 0$. *If in addition* V *has mild oscillation in the sense that* $\inf_{y\in B(x,sd(x))}V(y)\ge cV(x)$ *for some* $c>0$, *then* $V_s(x)$ *in* (2.5) *can be replaced by* $cV(x)$.

Proof. The proof is identical to that of Theorem 2.5, but here one sets $t(x)=d(x)/\sqrt{V_s(x)}$. ■

3. Trace properties, the heat kernel, and Doob h-conditioned processes. In this section we first prove upper and lower bounds for the trace of $\exp[-tH]$, the semigroup generated by the Dirichlet–Schrödinger operator H considered, and in particular we give a necessary and sufficient condition for the finiteness of $\mathrm{Tr}(\exp[-tH])$ for all $t>0$, in terms of V and of the geometry of D. To this end, we assume either that D is regular in the sense of Davies [D3] *and* V is a weak potential in the sense that $V(x)d(x)^2\to 0$ as $d(x)\to 0$ (cf. Cor. 3.2), or that V diverges sufficiently fast at the boundary and has mild oscillations (cf. Remark 3.3(ii)).

We first remark that the finiteness of $\exp[-tH]$ follows from *intrinsic ultracontractivity* (IUC) of H [DS]; in turn, IUC for the Schrödinger operators considered has been discussed in detail in [CG3], and it is known that it holds either when V is "small", and the domain D satisfies some extremely weak regularity condition (the property of L^p-averaging in the sense of Staples [Sta] is sufficient), or when V is a strong potential (in the present sense) with mild oscillation and D is a Hölder domain of order zero ([SS], [B2]). We refer to [CG3, Th. 4.6] for the relevant details.

We also notice that, in Theorem 3.1 below in the case of regular regions, an *effective potential* is of the form $V'(x) = V(x) + \text{const.}\, d(x)^{-2}$. In fact, one is familiar with such a quantity from the theory of boundary behaviour of eigenfunctions, as discussed in [CG3], and in particular one knows that the Agmon metric of V' often determines the rate of vanishing of the ground state eigenfunction of H.

The method of proof of our first result is a modification of ideas of Davies [D2]. For results on two-dimensional regions below the graph of a function we refer to [vdB].

In part of the next theorem, it is assumed that D is a Hölder domain of order zero (cf. e.g. [B2], [CG1-2] for a thorough discussion of this notion). We just remind the reader that they are those domains D for which there exist $x_0 \in D$, $A > 1$ and $B \in \mathbb{R}$ such that

$$K_D(x, x_0) \le A \ln(1/d(x)) + B \tag{3.1}$$

for all $x \in D$; here K_D is the *quasi-hyperbolic metric* of D,

$$K_D(x, y) = \inf_\gamma \int_\gamma \frac{ds}{d(x)}, \tag{3.2}$$

where the infimum is taken over those rectifiable paths in D which join x to y, and s is arclength. We refer to [GHM], [GO], [GP], [V] for more details and connections with the theory of conformal mappings.

In the sequel, we shall also need the concept of *Whitney decomposition* $\mathcal{F}$ of D (see [Ste]). In fact, under the assumption that $\operatorname{Inr}(D) := \sup_{x\in D} d(x) < \infty$, we have

$$\mathcal{F} = \{\overline{Q}_j^k : j = 1, \ldots, N_k,\ k = 1, 2, \ldots\}$$

where $\overline{Q}_j^k$, the jth closed cube of the kth generation, has edge length 2^{-k}. In addition, the distance of any point $x \in \overline{Q}_j^k$ to the boundary of D is comparable to the edge length of $\overline{Q}_j^k$ in the sense that

$$\sqrt{n}\, 2^{-k} \le d(x) \le 4\sqrt{n}\, 2^{-k} \qquad \forall x \in \overline{Q}_k^j \tag{3.3}$$

(cf. [Ste]). Then $D = \bigcup_{j,k} \overline{Q}_j^k$. We choose an order in the Whitney decomposition by declaring that $\overline{Q}_j^k$ precedes $\overline{Q}_l^m$ if $k < m$ or if $k = m$ and $j < l$.

We define V_+ as the piecewise constant function

$$V_+(x) = V_j^k := \sup_{y\in Q_j^k} V(y) \qquad \text{for } x \in \overline{Q}_j^k, \tag{3.4}$$

where $\overline{Q}_j^k$ is the first cube to which x belongs in the Whitney decomposition of D. Notice that $V_+ \in L^\infty_{\text{loc}}(D)$ as well.

THEOREM 3.1. *Fix $t > 0$, consider a potential $V \ge 0$ a.e., $V \in L^\infty_{\text{loc}}(D)$, and assume that $\int_D \exp[-V(x)t]\,dx < \infty$. Let also $V_+ : D \to [0, \infty)$ be defined by* (3.4). *Then*

(i)

$$\operatorname{Tr}(e^{-tH}) \le \frac{1}{(\pi t)^{n/2}} \int_D \exp[-V(x)t]\,dx. \tag{3.5}$$

If moreover D is regular in the sense of Davies [D3], *so that there exist $c > 0$ and $b \in \mathbb{R}$ such that $c/d(x)^2 \le -\Delta + b$ in the sense of quadratic forms, and also $\int_D \exp[-[V(x) + c/(4d(x)^2)]t]\,dx < \infty$, then*

$$\operatorname{Tr}(e^{-tH}) \le \frac{e^b}{(\pi t)^{n/2}} \int_D \exp[-[V(x) + c/(4d(x)^2)]t]\,dx. \tag{3.6}$$

(ii) *Assume that* $\operatorname{Inr}(D) < \infty$. *Then*

$$\operatorname{Tr}(e^{-tH}) \ge \frac{1}{(4\pi t)^{n/2}} \int_D \exp[-[V_+(x) + 16\pi^2 n^2/d(x)^2]t]\,dx. \tag{3.7}$$

Proof. (i) *Upper bound.* In the case where there is no assumption on D, we proceed as follows. First, recall that the Golden–Thompson inequality says that, for positive self-adjoint operators A, B, one has

$$\operatorname{Tr}(e^{-(A+B)}) \le \operatorname{Tr}(e^{-A/2}e^{-B}e^{-A/2}).$$

It follows that

$$\operatorname{Tr}(e^{-Ht}) \le \operatorname{Tr}(e^{-Vt/2}e^{\Delta t/2}e^{-Vt/2}) = \int_D e^{-V(x)t} K_D(t/2, x, x)\,dx,$$

where $K_D(t, x, y)$ is the heat kernel of $-\Delta_D/2$, that is, the integral kernel of $\exp[-tH_0]$. Next, by the monotonicity of the heat kernel with respect to regions, and denoting by K_n the heat kernel of $-\Delta/2$ on $\mathbb{R}^n$, one has

$$0 < K_D(t, x, y) \le K_n(t, x, y) = \frac{1}{(2\pi t)^{n/2}} e^{-|x-y|^2/(2t)},$$

and this proves (3.6). As concerns (3.7), one has

$$H = \frac{1}{2}H + \frac{1}{2}H \ge \frac{1}{2}H + \left(\frac{1}{2}V + \frac{c}{4d^2}\right) - \frac{b}{2}$$

in the sense of quadratic forms, by definition of regularity. By the same procedure as above, (3.7) follows.

(ii) *Lower bound.* Let $\mathcal{F}$ be a Whitney decomposition of D, as introduced just before the statement of the present theorem, and let V_+ be defined as in (3.4). By monotonicity and decoupling of the Dirichlet Laplacian [RS2], it follows that, if $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are disjoint open sets, $\Omega \subset \mathbb{R}^n$ is open,

$\overline{\Omega_1 \cup \Omega_2}^{\text{int}} = \Omega$ and $\Omega \setminus (\Omega_1 \cup \Omega_2)$ has measure zero, then, in the sense of quadratic forms,

$$-\Delta_\Omega \le \Delta_{\Omega_1 \cup \Omega_2} = -\Delta_{\Omega_1} \oplus -\Delta_{\Omega_2},$$

so that in particular $-\Delta_D \le \bigoplus_{j,k}(-\Delta_{Q_j^k})$. Clearly, a similar procedure can be applied to the Schrödinger operator considered, so that, if one sets $H_j^k = -\Delta_{Q_j^k} + V_j^k$ (acting in $L^2(Q_j^k)$), then $H \le \bigoplus_{j,k} H_j^k$.

By the min-max principle, and the properties of the direct sum of operators, it follows that

$$\operatorname{Tr}(e^{-Ht}) \ge \operatorname{Tr}(e^{-t\bigoplus_{j,k}H_j^k}) = \sum_{j,k}\operatorname{Tr}(e^{-tH_j^k}) = \sum_{j,k} e^{-tV_j^k}\operatorname{Tr}(\exp[t\Delta_{Q_j^k}/2]).$$

By a result of Davies [D2], it is known that, if Q is the unit cube in $\mathbb{R}^n$, then

$$\operatorname{Tr}(e^{s\Delta_Q/2}) \ge \frac{e^{-\pi^2 ns}}{(4\pi s)^{n/2}}$$

and hence, if $l_k = 2^{-k}$ is the edge length of Q_j^k, then

$$\operatorname{Tr}(\exp[t\Delta_{Q_j^k}/2]) = \operatorname{Tr}(e^{t\Delta_Q/(2l_k^2)}) \ge \frac{l_k^n e^{-\pi^2 nt/l_k^2}}{(4\pi t)^{n/2}},$$

so that

$$\begin{aligned}
\operatorname{Tr}(e^{-Ht}) &\ge \sum_{j,k} e^{-tV_j^k}\frac{l_k^n e^{-\pi^2 nt/l_k^2}}{(4\pi t)^{n/2}} \\
&= \frac{1}{(4\pi t)^{n/2}}\sum_{j,k}\operatorname{vol}(Q_j^k)e^{-t(V_j^k+\pi^2 n/l_k^2)} \\
&\ge \frac{1}{(4\pi t)^{n/2}}\sum_{j,k}\int_{Q_j^k} e^{-t(V_+(x)+16\pi^2n^2/d(x)^2)}\,dx \\
&= \frac{1}{(4\pi t)^{n/2}}\int_D e^{-t(V_+(x)+16\pi^2n^2/d(x)^2)}\,dx. \ \blacksquare
\end{aligned}$$

Corollary 3.2. *Assume that either D is bounded, or $\operatorname{Inr}(D) < \infty$. If D is regular in the sense of Davies* [D3], *and moreover $\lim_{d(x)\to 0} V(x)d(x)^2 = 0$, then $\operatorname{Tr}(\exp[-tH]) < \infty$ for all $t > 0$ if and only if $\int_D \exp[-t/d(x)^2]\,dx < \infty$.*

Remark 3.3. (i) If V is a strong potential in the above sense, which in addition has very mild oscillations, in the sense that $V_+(x) \le kV(x)$ for some $k > 1$ and for all $x \in D$, it is immediate from the proof of the above theorem that V_+ in (3.7) can be replaced by kV. Thus, under this assumption it follows that $\operatorname{Tr}(\exp[-tH]) < \infty$ for all $t > 0$ if and only if $\int_D \exp[-tV(x)]\,dx < \infty$.

(ii) One could obtain a variant of the previous lower bound for the trace of $\exp[-tH]$ when V is a strong potential, by the following procedure: consider a refinement of the Whitney decomposition of D which is constructed by requiring that the edge length of each cube Q_j^k is comparable to $d(x)^c$ for any $x \in Q_j^k$ and for some $c > 1$. Then the term which is proportional to $d(x)^{-2}$ in (3.7) will change into a term proportional to $d(x)^{-2c}$. However, $V_+(x)$ will be defined by taking the supremum over a much smaller cube, whose edge length is proportional to $d(x)^c$. Next, require that V has mild oscillations, $V_+(x) \le kV(x)$, which will be accomplished for a class of potentials which is *larger* than the one considered in item (i) above. If one then takes those potentials which satisfy $\lim_{d(x)\to 0} V(x)d(x)^{2c} = 0$, it follows that $\operatorname{Tr}(\exp[-tH]) < \infty$ for all $t > 0$ if and only if $\int_D \exp[-tV(x)]\,dx < \infty$.

Example 3.4. Let $\Omega_\gamma = \{\mathbf{x} = (x,y) \in \mathbb{R}^2 : x > 1,\ |y| < x^{-\gamma}\}$ for $\gamma \in (0,1)$ (cf. Example 1.9.5 of [D3]). Then it is known that

$$\operatorname{Tr}(e^{\Delta_{\Omega_\gamma} t}) \sim t^{(1-1/\gamma)/2} \quad \text{as } t \to 0.$$

If one sets $V(\mathbf{x}) = \text{const.}\, d(\mathbf{x})^{-k}$ (and also if one sets $V(\mathbf{x}) = \text{const.}\, x^{k\gamma}$) for $k > 2$, then a straightforward calculation involving Theorem 3.1 shows that

$$\operatorname{Tr}(e^{-Ht}) \sim t^{(1-1/\gamma)/k} \quad \text{as } t \to 0.$$

The next result involves the heat kernel of H, that is, the integral kernel of $\exp[-tH]$ $(t > 0)$, which, by general arguments, is positive and jointly continuous. In fact, we have

Corollary 3.5. *Let $k(t,x,y)$ be the heat kernel of H, and assume that $V(x) \ge \gamma/d(x)^{2m}$ for some $\gamma > 0$, $m > 1$, and $\int_D \exp[-V(x)t]\,dx < \infty$. Then, for some $k > 0$ and for δ as in Theorem* 2.5,

$$k(t,x,y) \le \frac{k}{t^{n/2}}\exp\left[-\delta\left(\frac{1}{d(x)^{m-1}} + \frac{1}{d(y)^{m-1}}\right)\right]\int_D e^{-V(x)t}\,dx. \tag{3.8}$$

Assume, in addition to the above assumptions, that D is a Hölder domain in the sense that (3.1) *holds, and that V has mild oscillations in the sense that $V(x) \le a + b/d(x)^{2r}$ for some $r < 2m+1$. Then there exists $\varepsilon > 0$ ($\varepsilon = \sqrt{b}A^r/(r-1)$ will work) and $c_t > 0$ such that*

$$k(t,x,y) \ge c_t \exp\left[-\varepsilon\left(\frac{1}{d(x)^{r-1}} + \frac{1}{d(y)^{r-1}}\right)\right]. \tag{3.9}$$

Proof. By Mercer's theorem (cf. [RS1], [DS]), one has the following eigenfunction expansion of the heat kernel:

$$k(t,x,y)=\sum_{n=0}^{\infty}e^{-tE_n}\psi_n(x)\psi_n(y),$$

where ψ_n are the eigenvectors of H relative to the eigenvalues E_n counted with their multiplicity and arranged in non-decreasing order, and where the series is locally uniformly convergent in $D\times D$. Therefore,

$$\begin{aligned}k(t,x,y)&\le\sum_{n=0}^{\infty}|\psi_n(x)|\,|\psi_n(y)|e^{-tE_n}\\&\le k_\delta\exp\left[\delta\left(\frac{1}{d(x)^{m-1}}+\frac{1}{d(y)^{m-1}}\right)\right]\operatorname{Tr}(e^{-tH})\qquad(\text{by }(2.2))\\&\le k_\delta\exp\left[\delta\left(\frac{1}{d(x)^{m-1}}+\frac{1}{d(y)^{m-1}}\right)\right]\int_D e^{-V(x)t}\,dx\qquad(\text{by }(3.5)).\end{aligned}$$

As concerns the lower bound, it has been shown in [CG3] that, under the present assumptions, H is intrinsically ultracontractive in the sense of [D1], [DS]. By Theorem 3.2 of [DS], it follows that

$$k(t,x,y)\ge c_t\psi_0(x)\psi_0(y)$$

for a suitable $c_t>0$, since a corresponding upper bound holds. The assertion follows by recalling that, by (4.11) of [CG3], one has

$$\psi_0(x)\ge A\exp[-B/d(x)^{r-1}].\ \blacksquare$$

Remark 3.6. (i) In the proof of Corollary 3.5 we have used the fact that the Schrödinger operator considered is intrinsically ultracontractive. We briefly recall that this means what follows (cf. [DS] for details): let $U_{\psi_0}:\mathrm{L}^2(D,\psi_0^2dx)\to\mathrm{L}^2(D,dx)$ be the operator defined by $U_{\psi_0}(f)=f\psi_0$ for all $f\in\mathrm{L}^2(D,\psi_0^2\,dx)$, where ψ_0 is the ground state eigenfunction of H relative to the eigenvalue E_0. Next, define the operator $\widetilde{H}$, acting on $\mathrm{L}^2(D,\psi_0^2\,dx)$, by $\widetilde{H}=U_{\psi_0}^{-1}\circ(H-E_0)\circ U_{\psi_0}$. Then H is said to be *intrinsically ultracontractive* if $\exp[-t\widetilde{H}]$ is bounded, for all $t>0$, as an operator from $\mathrm{L}^2(D,\psi_0^2dx)$ to $\mathrm{L}^\infty(D,\psi_0^2dx)$. A survey on many of the consequences of intrinsic ultracontractivity can be found in [B2].

Therefore, a general argument of [DS] allows one to conclude that k is pointwise comparable with $\psi_0(x)\psi_0(y)$, so that

$$(3.10)\qquad a_t\psi_0(x)\psi_0(y)\le k(t,x,y)\le b_t\psi_0(x)\psi_0(y)\qquad(a_t,b_t>0).$$

In fact, the lower bound in (3.9) follows from (3.10). Inequality (3.8) does not follow in general from (3.10), since intrinsic ultracontractivity need not hold. One should also notice that it is not possible, in general, to give more detailed information about the constant c_t in (3.9); in fact, this is due to the very indirect method of proof, which passes through intrinsic ultracontractivity.

(ii) In a domain of finite area $|D|$, the integral in (3.8) is certainly finite for all t, and tends to $|D|$ as $t\to 0$, by dominated convergence.

(iii) The pointwise bounds for the heat kernel can be improved by a Gaussian factor by methods which are by now standard (cf. [D3, Sec. 3], [CKS], [B2, Cor. 2.7]). However, this does not give any further information on the decay of $K(t,x,y)$ as x or y approaches any fixed $z\in\partial D$.

(iv) Another pointwise upper bound for k which involves V itself can be obtained by making use of Theorem 2.7.

To state our last result, we need some more notation. Let h be a positive superharmonic function on D (no boundary condition is considered now in the definition of superharmonicity); we denote by $L_+(D)$ the set of positive superharmonic functions. Let also $P(t,x,y)$ be the transition density function of Brownian motion killed when it hits ∂D. Then it is well known that $P(t,x,y)$ is the integral kernel of $\exp[\Delta_Dt/2]$, that is, the heat kernel of $-\Delta_D/2$. Let us consider a new transition density function on D, defined by

$$(3.11)\qquad P_h(t,x,y)=P(t,x,y)h(y)/h(x).$$

The process on the path space determined by P_h is known as the *Doob h-conditioned diffusion* [Do]; we let $\mathbf{E}_h^{(x)}$ denote expectation with respect to the measure associated with the Doob h-process.

Several results on the lifetimes of such processes are known in the literature (cf. [CMC], [B1–2], [BD1] and references quoted therein). In [B2], a result concerning the expected value of the Feynman–Kac formula with respect to the measure associated with the Doob h-diffusion is given; however, it was necessary there to assume that V belongs to the Kato class K_n, thus being a small perturbation of the Dirichlet Laplacian. The next result therefore can be seen in part as complementary to that of Bañuelos, since it discusses the case in which V is a strong potential, and in part as a generalization of it, since it also allows potentials which do not belong to the Kato class, but nevertheless do not diverge too fast at the boundary.

THEOREM 3.7. *Assume that D is a Hölder domain, in the sense that (3.1) holds true; in particular, this is true for any John, BMO, NTA and Lipschitz domain. Assume also that either V is a strong potential with mild oscillations, so that*

$$a_1+b_1/d(x)^{2m}\le V(x)\le a_2+b_2/d(x)^{2r}$$

for some $m>1$, $r<2m+1$, $a_{1,2}\in\mathbb{R}$ and $b_{1,2}>0$, or that V is a weak *potential, so that*

$$V(x) \le a_3 + b_3/d(x)^{2c}$$

for some $c \le 1$. Then

$$\sup_{x\in D,\, h\in L_+(D)} \frac{\psi_0(x)}{h(x)} \int_D \psi_0(y)h(y)\,dy < \infty. \tag{3.12}$$

Moreover, let E_0 (>0) be the lowest eigenvalue of the Dirichlet–Schrödinger operator associated with V. Then

$$\lim_{t\to\infty} \mathbf{E}_x^{(h)}\Big(\exp\Big[-\int_0^t V(X_s)\,ds\Big]; \tau_D > t\Big) = \frac{\psi_0(x)}{h(x)} \int_D \psi_0(y)h(y)\,dy. \tag{3.13}$$

Proof. Let X_t be a standard n-dimensional Brownian motion, and let τ_D be the lifetime of Brownian motion killed when it hits ∂D. Since V is a.e. positive, by applying a Feynman–Kac formula and by the properties of Doob diffusions, one has

$$1 \ge \mathbf{E}_h^{(x)}\Big(\exp\Big[-\int_0^t V(X_s)\,ds\Big]; \tau_D > t\Big) = \frac{1}{h(x)} \int_D k(t,x,y)h(y)\,dy \ge c_t \frac{\psi_0(x)}{h(x)} \int_D \psi_0(y)h(y)\,dy,$$

whenever intrinsic ultracontractivity holds, by (3.10), for a suitable $c_t > 0$; therefore, (3.12) holds under this assumption. Moreover, it is well known that, once more under the assumption of intrinsic ultracontractivity,

$$\lim_{t\to\infty} \frac{e^{E_0 t}k(t,x,y)}{\psi_0(x)\psi_0(y)} = 1$$

uniformly in x, y (cf. [D3], [B2]). Therefore, (3.13) holds whenever one is able to prove intrinsic ultracontractivity. Therefore, the proof is completed by noting that, in a Hölder domain D, a Dirichlet–Schrödinger operator corresponding to a strong potential with mild oscillations is always intrinsically ultracontractive by [CG3, Th. 4.6(iii)]. The same holds for $0 \le V(x) \le a d(x)^{-c} + b$ for some $c \in [0,2]$, by part (i_a) of the same theorem. ■

Remark 3.8. Similar results could be obtained under other conditions on D and V, since the proof works whenever $H = -\Delta_D/2 + V$ is intrinsically ultracontractive on D. Other sufficient conditions for this to hold are given in Theorem 4.6 of [CG3], and we have stated in the theorem above only the most significant cases. It should also be noticed that, at least in the case of strong potentials with mild oscillations, one has pointwise bounds on ψ_0 (cf. eq. (2.7) and (4.11) of [CG3]) which may allow giving estimates on the right hand side of (3.13).

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The one-sided minimal operator and the one-sided reverse Hölder inequality

by

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Abstract. We introduce the one-sided minimal operator, m^+f, which is analogous to the one-sided maximal operator. We determine the weight classes which govern its two-weight, strong and weak-type norm inequalities, and show that these two classes are the same. Then in the one-weight case we use this class to introduce a new one-sided reverse Hölder inequality which has several applications to one-sided (A_p^+) weights.

1. Introduction. In our papers [1] and [2] we introduced a new operator, the minimal operator, so named since it is analogous to the Hardy–Littlewood maximal operator. Given a measurable function f, define the minimal function of f, $\mathrm{m}f$, by

$$\mathrm{m}f(x) = \inf_I \frac{1}{|I|} \int_I |f|\,dy,$$

where the infimum is taken over all cubes I with sides parallel to the coordinate axes which contain x. In [1] we used the minimal operator to study the structure of functions which satisfy the reverse Hölder inequality; in [2] we considered the weighted norm inequalities which hold for the minimal operator, and applied this to the problem of differentiability of the integral.

The maximal operator, as originally defined by Hardy and Littlewood, was a one-sided maximal operator on $\mathbb{R}$ (see [4]). The weighted norm inequalities for the one-sided maximal operator were first considered by Sawyer [10] and then by Martín-Reyes and others [5]–[8]. In light of this we define a one-sided minimal operator.

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