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ESTIMATION OF REDUCED PALM DISTRIBUTIONS BY RANDOM METHODS FOR COX PROCESSES WITH UNKNOWN PROBABILITY LAW

Abstract. Let N_i, $i \geq 1$, be i.i.d. observable Cox processes on $[a, b]$ directed by random measures M_i. Assume that the probability law of the M_i is completely unknown. Random techniques are developed (we use data from the processes $N_1, \ldots, N_n$ to construct a partition of $[a, b]$ whose extremities are random) to estimate

$$L(\mu, g) = E(\exp(-(N(g) - \mu(g))) \mid N - \mu \geq 0).$$

1. Introduction. Let $[a, b]$ be a compact interval of $\mathbb{R}$ and N a Cox process on $[a, b]$ directed by a random measure M on $[a, b]$ (see [3]–[5] for detailed definition).

In [4], A. F. Karr gives state estimators $E(e^{-M(f)} \mid F_A^N)$, where

$$F_A^N = \sigma(N(g1_A) : g \in \mathcal{C}_+)$$

and $\mathcal{C}_+$ denotes the set of nonnegative continuous functions on $[a, b]$.

In the case of a Cox process, he proves, by means of Proposition 2.2 recalled in Section 2, that it is sufficient to estimate the Laplace functionals $L(\mu, g)$ of the reduced Palm process of N (see [4] and [5] for detailed definitions). A. F. Karr constructs an estimator $\widehat{L}_n(\mu, g)$ of $L(\mu, g)$ by means of fixed partitions. He shows that, under some conditions, for each compact subset K of $\mathcal{C}_+$ and each compact subset K' of $\mathcal{M}_p$,

$$\sup_{g \in K} \sup_{\mu \in K'} |\widehat{L}_n(\mu, g) - L(\mu, g)| \to 0 \quad \text{almost surely},$$

where $\mathcal{M}_p$ denotes the set of finite, integer-valued measures on $[a, b]$.

We construct in Section 3 an estimator $\widehat{L}_n(\mu, g)$ of the same Laplace functional $L(\mu, g)$ using random partitions, and we study its behaviour in

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Section 4. The interest of this partition is that it takes into account the number of points of the copies to construct locally the estimator.

2. Notations and results. Let N be a simple point process on $[a,b]$ and let Q'_N be the measure on $\mathcal{M}_p$ defined by

$$Q'_N(\Gamma)=\sum_{k=0}^{\infty}\frac{1}{k!}E\Big(\int_{[a,b]}\mathbf{1}_{\Gamma}\Big(\sum_{i=1}^{k}\varepsilon_{x_i}\Big)N^{(k)}(dx)\Big),$$

where ε_x is the point mass at x and $N^{(k)}$ is the factorial moment measure

$$\begin{aligned}N^{(k)}(dx)&=N^{(k)}(dx_1,\dots,dx_k)\\&=N(dx_1)(N-\varepsilon_{x_1})(dx_2)\dots\Big(N-\sum_{i=1}^{k-1}\varepsilon_{x_i}\Big)(dx_k).\end{aligned}$$

We define similarly a measure Q'_M with

$$M^k(dx)=M(dx_1)\dots M(dx_k).$$

The *compound Campbell measures* of N and M are respectively the measures C'_N on $\mathcal{M}_p\times\mathcal{M}_p$ and C'_M on $\mathcal{M}_p\times\mathcal{M}$ ($\mathcal{M}$ is the set of finite, not necessarily integer-valued measures on $[a,b]$) given by

$$\begin{aligned}&\int_{[a,b]}e^{-\mu(f)}e^{-\nu(g)}\,C'_N(d\mu,d\nu)\\&\quad=\sum_{k=0}^{\infty}\frac{1}{k!}E\Big[e^{-N(g)}\int_{[a,b]}e^{-\Sigma_{i=1}^k f(x_i)}e^{-\Sigma_{i=1}^k g(x_i)}\,N^{(k)}(dx)\Big],\\&\int_{[a,b]}e^{-\mu(f)}e^{-\nu(g)}\,C'_M(d\mu,d\nu)\\&\quad=\sum_{k=0}^{\infty}\frac{1}{k!}E\Big[e^{-N(g)}\int_{[a,b]}e^{-\Sigma_{i=1}^k f(x_i)}e^{-\Sigma_{i=1}^k g(x_i)}\,M^{k}(dx)\Big].\end{aligned}$$

Assume that for each k, the mean measure of $N^{(k)}$ is finite. Then there exists a disintegration of C'_N with respect to Q'_N, that is, a transition probability Q_N from $\mathcal{M}_p$ into itself such that

$$C'_N(d\mu,d\nu)=Q'_N(d\mu)Q_N(\mu,d\nu).$$

The probability distributions $\{Q_N(\mu,\cdot):\mu\in\mathcal{M}_p\}$ are the *reduced Palm distributions* of N.

A point process $N^{(\mu)}$ with probability law $Q_N(\mu,\cdot)$ is called a *reduced Palm process* of N.

Under the assumption that each M^k admits a finite mean measure there exist Palm distributions $Q_M(\mu, d\nu)$ satisfying

$$C'_M(d\mu, d\nu) = Q'_M(d\mu)Q_N(\mu, d\nu).$$

A random measure $M^{(\mu)}$ with distribution $Q_M(\mu, \cdot)$ is termed a *Palm process* of M. For further details on Palm distributions see [5].

In the context of Cox processes a key result is the following (see [4]):

PROPOSITION 2.1. *Let M be a random measure on $[a, b]$ with finite mean measure and let N be a Cox process directed by M. Then almost everywhere on $\mathcal{M}_p$ with respect to Q'_M, the reduced Palm process $N^{(\mu)}$ is a Cox process directed by the Palm process $M^{(\mu)}$.*

Under the same notations, we have (see [4]) the following proposition which allows us to deal with state estimation.

PROPOSITION 2.2. *For each Borel subset A of $[a, b]$ and each $f \in \mathcal{C}_+$,*

$$E(e^{-M(f)} \mid F_A^N) = \frac{E(e^{-M^{(\mu)}(A)}e^{-M^{(\mu)}(f)})}{E(e^{-M^{(\mu)}(A)})}\bigg|_{\mu=N_A},$$

where N_A denotes the restriction of N to A, and $F_A^N = \sigma(N(g1_A) : g \in \mathcal{C}_+)$.

We define

$$L_N(\mu, f) = E(\exp(-N^{(\mu)}(f))), \qquad L_M(\mu, f) = L_N(\mu, -\ln(1 - f)).$$

Thus, we only need to estimate $L_N(\mu, g)$ to estimate $E(e^{-M(f)} \mid F_A^N)$.

3. Definition of the estimator. Let $N_1, \ldots, N_n$ be i.i.d. copies of a Cox process N on $[a, b]$ assumed to satisfy $E(N^{(2)}([a, b])) < \infty$. N is directed by a random measure M. The problem is to construct an estimator $\widehat{L}_n(\mu, g)$ of the Laplace functional

$$L(\mu, g) = L_{N^{(\mu)}}(g) = E(\exp(-N^{(\mu)}(g))),$$

which can be interpreted as

$$L(\mu, g) = E(\exp(-(N(g) - \mu(g))) \mid N - \mu \geq 0).$$

We construct, for each realization r of the variable

$$R_n = \sum_{i=1}^{n} N_i([a, b]),$$

a random partition with fixed integers $k(r)$ growing to infinity with r and other fixed integers $\lambda_j(r)$ satisfying

$$\sum_{j=1}^{k(r)} \lambda_j(r) = r + 1.$$

Let $a = x_0 \le x_1 \le \ldots \le x_r \le x_{r+1} = b$ be the r ordered points of the n realizations of the process, and let the integers $\nu_j(r), j = 0, \ldots, k(r)$, be defined by

$$\nu_0 = 0, \quad \nu_j(r) = \nu_{j-1}(r) + \lambda_j(r), \quad j = 1, \ldots, k(r).$$

Then we have the random partition $\{A_j(r) : j = 1, \ldots, k(r)\}$, where

$$A_j(r) = [x_{\nu_{j-1}(r)}, x_{\nu_j(r)}[.$$

We study the estimator

$$\widehat{L}_n(\mu, g) = \frac{e^{\mu(g)} \sum_{i=1}^n (e^{-N_i(g)} \prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N_i(A_j(R_n)) \ge \mu(A_j(R_n))\}})}{\sum_{i=1}^n \prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N_i(A_j(R_n)) \ge \mu(A_j(R_n))\}}}.$$

4. Main result

PROPOSITION 4.1. *Assume that*:

(1) *There exists* $t > 0$ *such that* $E(e^{tM([a,b])}) < \infty$.
(2) *For each* $g \in \mathcal{C}_+$, $\mu \to L(\mu, g)$ *is continuous on* $\mathcal{M}_p$.
(3) *For each* k,

$$\sum_{n=1}^{\infty} \frac{(k(n))^k}{n^2} < \infty.$$

(4) $\lim_{r\to\infty} \inf_{j=1,\ldots,k(r)} \frac{\lambda_j(r)}{\ln(r)} = \infty.$

Then for each compact subset K *of* $\mathcal{C}_+$ *and each compact subset* K' *of* $\mathcal{M}_p$, *the estimator* $\widehat{L}_n(\mu, g)$ *satisfies*

$$\sup_{g\in K,\, \mu\in K'} |\widehat{L}_n(\mu, g) - L(\mu, g)| \to 0 \qquad \textit{almost completely.}$$

We mean that for all $\varepsilon > 0$,

$$P[\sup_{g\in K,\, \mu\in K'} |\widehat{L}_n(\mu, g) - L(\mu, g)| > \varepsilon]$$

is the general term of a convergent series.

Proof. Let K be a compact subset of $\mathcal{C}_+$ and K' a compact subset of $\mathcal{M}_p$. For each k, let $\mathcal{M}_p(k) = \{\mu \in \mathcal{M}_p : \mu([a, b]) = k\}$. We can assume that K' is a subset of $\mathcal{M}_p(k)$ for some fixed k. We form the decomposition

$$\widehat{L}_n(\mu, g) = \frac{e^{\mu(g)} E[e^{-N(g)} \prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N(A_j(R_n)) \ge \mu(A_j(R_n))\}}]}{E[\prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N(A_j(R_n)) \ge \mu(A_j(R_n))\}}]} \times \left(\frac{\frac{1}{n} \sum_{i=1}^n e^{-N_i(g)} \prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N_i(A_j(R_n)) \ge \mu(A_j(R_n))\}}}{E[e^{-N(g)} \prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N(A_j(R_n)) \ge \mu(A_j(R_n))\}}]} \right.$$

$$\times \frac{E[\prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N(A_j(R_n))\geq \mu(A_j(R_n))\}}]}{\frac{1}{n}\sum_{i=1}^{n}\prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N_i(A_j(R_n))\geq \mu(A_j(R_n))\}}}\Bigg)$$
$$= A_n \times (B_n/C_n)$$

and show that $A_n \to L(\mu, g)$, while $B_n \to 1$ almost completely and $C_n \to 1$ almost completely.

First, we need some lemmas.

LEMMA 4.2. *If l is the Lebesgue measure on $[a,b]$, then the random variable $\sup_{j=1,\ldots,k(R_n)} l(A_j(R_n))$ converges to 0 almost completely.*

Proof.

FIRST STEP. *Let $Z_1,\ldots,Z_r$ be r i.i.d. copies of the uniform law on $[0,1]$. Then the distribution of $\nu(A_j(r))/\nu([a,b])$ where $\nu = E(M)$ is the distribution of $Z_{\nu_j(r)} - Z_{\nu_{j-1}(r)}$.*

Proof. Conditionally on M, the distribution of the random variable (random partition) $M(A_j(r))/M([a,b])$ is the distribution of $Z_{\nu_j(r)} - Z_{\nu_{j-1}(r)}$ (see [1]). Then

$$\frac{\nu(A_j(r))}{\nu([a,b])} = \frac{\int_{\mathcal{M}_p} \big(\frac{M(A_j(r))}{M([a,b])}\big) M([a,b])\, P(dM)}{M([a,b])}$$

and hence the distribution of $\nu(A_j(r))/\nu([a,b])$ is the distribution of

$$(Z_{\nu_j(r)} - Z_{\nu_{j-1}(r)}) \frac{\int_{\mathcal{M}_p} M([a,b])\, P(dM)}{\nu([a,b])}.$$

The result is proved.

Recall that $R_n = \sum_{i=1}^{n} N_i([a,b])$.

SECOND STEP. *Let $0 < \delta < 1/2$ and $I_n = [n\nu([a,b])(1-n^{-\delta}), n\nu([a,b]) \times (1+n^{-\delta})]$. Then $P(R_n \notin I_n)$ is the general term of a convergent series.*

Proof. There exist random measures M_i associated with the processes N_i. Conditionally on $\{M_i : i = 1,\ldots,n\}$, R_n is a Poisson random variable with parameter $\sum_{i=1}^{n} M_i([a,b])$. We can write

$$P(R_n \notin I_n) = \int_{\mathcal{M}_p} \cdots \int_{\mathcal{M}_p} \sum_{r\notin I_n} e^{-\Sigma_{i=1}^{n} M_i([a,b])} \frac{(\sum_{i=1}^{n} M_i([a,b]))^r}{r!} P(dM_1)\ldots P(dM_n).$$

This expression is bounded from above by

$$\int\cdots\int_{\{(M_1,\ldots,M_n)\notin E_n\}} \sum_{r\notin I_n} e^{-\Sigma_{i=1}^{n} M_i([a,b])} \frac{(\sum_{i=1}^{n} M_i([a,b]))^r}{r!} P(dM_1)\ldots P(dM_n) + P((M_1,\ldots,M_n)\in E_n),$$

where E_n is the set

$$E_n = \Big\{(M_1, \ldots, M_n) : \Big|\sum_{i=1}^n M_i([a,b]) - n\nu([a,b])\Big| > \nu([a,b])n^{1-\delta}/2\Big\}.$$

The second term of the sum is bounded from above by

$$\sum_{r<n\nu([a,b])(1-n^{-\delta})} e^{-n\nu([a,b])(1-n^{-\delta}/2)} \frac{(n\nu([a,b])(1-n^{-\delta}/2))^r}{r!} + \sum_{r>n\nu([a,b])(1+n^{-\delta})} e^{-n\nu([a,b])(1+n^{-\delta}/2)} \frac{(n\nu([a,b])(1+n^{-\delta}/2))^r}{r!}.$$

Using the Stirling formula, we obtain the bound

$$\sum_{r<n\nu([a,b])(1-n^{-\delta})} e^{-n\nu([a,b])(1-n^{-\delta}/2)} \frac{(ne\nu([a,b])(1-n^{-\delta}/2))^r}{r^r} + \sum_{r>n\nu([a,b])(1+n^{-\delta})} e^{-n\nu([a,b])(1+n^{-\delta}/2)} \frac{(ne\nu([a,b])(1+n^{-\delta}/2))^r}{r^r}.$$

For large n, the first term is bounded from above by

$$n\nu([a,b])(1-n^{-\delta})e^{-n\nu([a,b])(1-n^{-\delta}/2)} \frac{(e(1-n^{-\delta}/2))^{n\nu([a,b])(1-n^{-\delta})}}{(1-n^{-\delta})^{n\nu([a,b])(1-n^{-\delta})}} \times e^{-n\nu([a,b])(1+n^{-\delta}/2)} \frac{(e(1+n^{-\delta}/2))^{n\nu([a,b])(1+n^{-\delta})-2}}{(1+n^{-\delta})^{n\nu([a,b])(1+n^{-\delta})-2}} \times \frac{(ne\nu([a,b])(1+n^{-\delta}/2))^2\pi^2}{6}.$$

Therefore the first term is the general term of a convergent series.

Now, to show the same for the second term, it is sufficient to see that the assumption (1) implies (using the Bernstein inequality) that

$$P\Big(\Big|\sum_{i=1}^n (M_i - \nu([a,b]))\Big| > \nu([a,b])n^{1-\delta}/2\Big) \le 2e^{-n(\nu([a,b])^2 n^{-2\delta}/4)/(4\,\mathrm{VAR}(M))}$$

if n is large enough since $\nu([a,b])n^{-\delta}/2 < \mathrm{VAR}(M)$. Thus the proof is complete since $0 < \delta < 1/2$.

Proof of Lemma 4.2.

$$P(\sup_{j=1,\ldots,k(R_n)} l(A_j(R_n)) > \varepsilon) \le P\Big(\sup_{j=1,\ldots,k(R_n)} \nu(A_j(R_n)) > \frac{\varepsilon}{\sup_{x\in[a,b]} f(x)}\Big),$$

where f is the density of the measure ν. Therefore

$$P(\sup_{j=1,\dots,k(R_n)} l(A_j(R_n)) > \varepsilon)$$
$$\le \sum_{r\in N}\sum_{j=1}^{k(r)} P\Big(\frac{\nu(A_j(r))}{\nu([a,b])} > \frac{\varepsilon}{\sup_{x\in[a,b]} f(x)\nu([a,b])}\Big) P(R_n = r).$$

Hence, the result follows from the proofs above (see [2]).

LEMMA 4.3. *Under the assumptions of Proposition* 4.1, *for all* $\varepsilon > 0$,

$$P(\sup_{g\in K}\sup_{\mu\in K'} |A_n - L(\mu,g)| > \varepsilon)$$

is the general term of a convergent series.

Proof. Let us introduce

$$K'_{1,n} = \{\mu \in K' : \forall j = 1,\dots,k(R_n),\ \mu(A_j(R_n)) \le 1\},$$
$$K'_{2,n} = \{\mu \in K' : \forall j = 1,\dots,k(R_n),\ \mu(A_j(R_n)) \ge 1\}.$$

We have the inclusion

$$\{\sup_{g\in K}\sup_{\mu\in K'} |A_n - L(\mu,g)| > \varepsilon\}$$
$$\subseteq \{\sup_{g\in K}\sup_{\mu\in K'_{1,n}} |A_n - L(\mu,g)| > \varepsilon\} \cup \{\sup_{g\in K}\sup_{\mu\in K'_{2,n}} |A_n - L(\mu,g)| > \varepsilon\}.$$

Remember that K' is assumed to be a subset of $\mathcal{M}_p(k)$ for some fixed k.

If $\mu \in K'_{1,n}$ then

$$\prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N(A_j(R_n))\ge\mu(A_j(R_n))\}}$$
$$= \frac{1}{k!}\int\limits_{[a,b]^k} \prod_{j=1}^{k(R_n)} \mathbf{1}_{\{\Sigma_{j=1}^k \varepsilon_{x_j}(A_j(R_n))\ge\mu(A_j(R_n))\}} N^{(k)}(dx)$$

so that, with $\Gamma_n(\mu) = \{c \in \mathcal{M}_p : \prod_{j=1}^{k(R_n)} \mathbf{1}_{\{c(A_j(R_n))\ge\mu(A_j(R_n))\}} = 1\}$,

$$E\Big(\prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N(A_j(R_n))\ge\mu(A_j(R_n))\}}\Big) = \frac{1}{k!}E\Big(\int\limits_{[a,b]^k} \mathbf{1}_{\Gamma_n(\mu)}\Big(\sum_{j=1}^{k}\varepsilon_{x_j}\Big)N^{(k)}(dx)\Big).$$

Hence

$$E\Big(\prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N(A_j(R_n))\ge\mu(A_j(R_n))\}}\Big) = E(Q'_N(\Gamma_n(\mu)\cap\mathcal{M}_p(k))).$$

Similarly, if $\mu \in K'_{1,n}$ then

$$e^{\mu(g)} E\Big(e^{-N(g)} \prod_{j=1}^{k(R_n)} \mathbf{1}_{\{N(A_j(R_n)) \geq \mu(A_j(R_n))\}}\Big) = E\Big(\int_{\Gamma_n(\mu)\cap \mathcal{M}_p(k)} Q'_N(dc) L(c,g)\Big)$$

and therefore

$$\{\sup_{g\in K} \sup_{\mu \in K'_{1,n}} |A_n - L(\mu,g)| > \varepsilon\} \subseteq \left\{ \sup_{g\in K} \sup_{\mu\in K'_{1,n}} \left| \frac{E(\int_{\Gamma_n(\mu)\cap\mathcal{M}_p(k)} Q'_N(dc)\, L(c,g))}{E(Q'_N(\mathcal{M}_p(k)\cap \Gamma_n(\mu)))} - L(\mu,g)\right| > \varepsilon \right\}$$

and

$$\{\sup_{g\in K} \sup_{\mu \in K'_{1,n}} |A_n - L(\mu,g)| > \varepsilon\} \subseteq \left\{ \sup_{g\in K} \sup_{\mu\in K'_{1,n}} \frac{E(\int_{\Gamma_n(\mu)\cap\mathcal{M}_p(k)} Q'_N(dc)\, |L(c,g) - L(\mu,g)|)}{E(Q'_N(\mathcal{M}_p(k)\cap \Gamma_n(\mu)))} > \varepsilon \right\}.$$

Using the definition of $\Gamma_n(\mu)$, we obtain

$$\Gamma_n(\mu)\cap\mathcal{M}_p(k) \subseteq B(\mu, \sup_{j=1,\dots,k(R_n)} l(A_j(R_n))).$$

Now, by the assumption (2) and since for each measure $\mu\in\mathcal{M}_p$, $g \to L(\mu,g)$ is continuous on $\mathcal{C}_+$, it follows that for all $\varepsilon>0$, there exists $\eta>0$ satisfying

$$\forall \mu \in K' \text{ (compact)}, \forall g\in K \text{ (compact)},$$
$$c\in B(\mu,\eta),\ g'\in B(g,\eta) \ \Rightarrow\ |L(c,g') - L(\mu,g)| < \varepsilon.$$

Actually, for all $\varepsilon>0$, there exists $\eta>0$ satisfying

$$\forall \mu \in K' \text{ (compact)}, \forall g\in K \text{ (compact)},$$
$$c\in B(\mu,\eta) \Rightarrow |L(c,g) - L(\mu,g)| < \varepsilon.$$

Finally, we get the inclusion

$$\{\sup_{g\in K} \sup_{\mu \in K'_{1,n}} |A_n - L(\mu,g)| > \varepsilon\} \subseteq \{\varepsilon > \varepsilon\} \cup \{\sup_{j=1,\dots,k(R_n)} l(A_j(R_n)) > \eta\}.$$

By Lemma 4.2, for all $\varepsilon>0$,

$$P\{\sup_{g\in K} \sup_{\mu \in K'_{1,n}} |A_n - L(\mu,g)| > \varepsilon\}$$

is the general term of a convergent series.

We must now show that

$$P\{\sup_{g\in K}\sup_{\mu\in K'_{2,n}}|A_n - L(\mu,g)| > \varepsilon\}$$

is the general term of a convergent series. We will use the convention that $\sup_{x\in\emptyset}|a(x)| = 0$. Thus, it suffices to show that $P(K'_{2,n}\neq\emptyset)$ is the general term of a convergent series. Recall that

$$K'_{2,n} = \{\mu\in K' : \exists j = 1,\ldots,k(R_n),\ \mu(A_j(R_n))\geq 2\}.$$

Since $\mu\in\mathcal{M}_p(k)$, we can write $\mu = \sum_{p=1}^{k}\varepsilon_{x_p}$ where ε_{x_p} is the point mass at x_p and the x_p are ordered on $[a,b]$. We set $x_0 = a$ and $x_{k+1} = b$. We also define

$$\inf(\mu) = \inf_{p=1,\ldots,k+1}(x_p - x_{p-1}).$$

Since K' is a compact set and

$$K'\subseteq\bigcup_{\mu\in K'}B(\mu,\inf(\mu)/3)$$

there exists a finite set $\{\mu_1,\ldots,\mu_l\}$ of elements of K' for which

$$K'\subseteq\bigcup_{r=1}^{l}B(\mu,\inf(\mu_r)/3).$$

Hence

$$K'_{2,n}\subseteq\bigcup_{r=1}^{l}(B(\mu,\inf(\mu_r)/3)\cap K'_{2,n}).$$

We have

$$\{K'_{2,n}\neq\emptyset\}$$
$$=\bigcup_{r=1}^{l}\{\exists\mu\in B(\mu_r,\inf(\mu)/3)\text{ and } j\in\{1,\ldots,k(R_n)\} : \mu(A_j(R_n))\geq 2\}.$$

It is then straightforward to obtain

$$\{K'_{2,n}\neq\emptyset\}\subseteq\bigcup_{r=1}^{l}\{\sup_{j=1,\ldots,k(R_n)} l(A_j(R_n)) > \inf(\mu_r)/6\}.$$

Lemma 4.2 completes the proof.

Lemma 4.4. *Under the assumptions of Proposition* 4.1, *for all* $\varepsilon > 0$,

$$P\{\sup_{g\in K}\sup_{\mu\in K'}|C_n - 1| > \varepsilon\}$$

is the general term of a convergent series.

Proof. There are $k(R_n)^k$ possibilities to set k points of a measure of $\mathcal{M}_p(k)$ in the $k(R_n)$ intervals $A_j(R_n)$. Thus, we can write

$$\mathcal{M}_p(k) = \bigcup_{l=1}^{k(R_n)^k} \Gamma_{n,l},$$

where the $\Gamma_{n,l}$ are sets of measures having the same number of points in each $A_j(R_n)$. We then have

$$P\{\sup_{g\in K}\sup_{\mu\in K'}|C_n-1|>\varepsilon\} \le P\Big\{\bigcup_{l=1}^{k(R_n)^k}\Big|\frac{n^{-1}\sum_{i=1}^n \mathbf{1}_{\{N_i\in\Gamma_{n,l}\}}}{P(N\in\Gamma_{n,l})}-1\Big|>\varepsilon\Big\}.$$

Consequently,

$$P\{\sup_{g\in K}\sup_{\mu\in K'}|C_n-1|>\varepsilon\} \le \sum_{r\in\mathbb{N}} k(r)^k\varepsilon^{-4}E\Big(\frac{n^{-1}\sum_{i=1}^n \mathbf{1}_{\{N_i\in\Gamma_{n,l}\}}}{P(N\in\Gamma_{n,l})}\Big)^4 P(R_n=r)$$

and

$$P\{\sup_{g\in K}\sup_{\mu\in K'}|C_n-1|>\varepsilon\} \le \sum_{r\in N} k(r)^k\frac{\text{const}}{n^2}P(R_n=r).$$

Therefore

$$\begin{aligned}P\{\sup_{g\in K}\sup_{\mu\in K'}|C_n-1|>\varepsilon\} \le &\sum_{r\in I_n} k(r)^k\frac{\text{const}}{n^2}P(R_n=r)\\ &+\sum_{r<n\nu([a,b])(1-n^{-\delta})} k(r)^k\frac{\text{const}}{n^2}P(R_n=r)\\ &+\sum_{r>n\nu([a,b])(1+n^{-\delta})} k(r)^k\frac{\text{const}}{n^2}P(R_n=r).\end{aligned}$$

Let us consider the first term of this sum. Since $k(r)$ grows to infinity (see the construction of the random partition), we can write

$$\sum_{r\in I_n} k(r)^k\frac{\text{const}}{n^2}P(R_n=r) \le \text{const}\,\frac{k([n\nu([a,b])(1+n^{-\delta})])}{n^2}.$$

By the assumption (3), this is the general term of a convergent series.

For the second term of the sum, we can write

$$\sum_{r<n\nu([a,b])(1-n^{-\delta})} k(r)^k\frac{\text{const}}{n^2}P(R_n=r) \le \text{const}\,\frac{k([n\nu([a,b])(1-n^{-\delta})])}{n^2}.$$

The assumption (3) shows that this is the general term of a convergent series.

For the third term of the sum, we have

$$\sum_{r>n\nu([a,b])(1+n^{-\delta})} k(r)^k \frac{\text{const}}{n^2} P(R_n = r) \leq \sum_{r>n\nu([a,b])(1+n^{-\delta})} k(r)^k \frac{\text{const}}{r^2} \cdot \frac{r^2}{n^2} P(R_n = r).$$

Since $k(r)^k/r^2$ decreases for large r, for $n \geq n_0$ we have

$$\sum_{r>n\nu([a,b])(1+n^{-\delta})} k(r)^k \frac{\text{const}}{n^2} P(R_n = r) \leq \frac{\text{const}(k([n\nu([a,b])(1+n^{-\delta})]))^k}{([n\nu([a,b])(1+n^{-\delta})])^2} \sum_{r\in\mathbb{N}} \frac{r^2}{n^2} P(R_n = r).$$

Using the fact that R_n is a Poisson variable with parameter $n\nu([a,b])$ we obtain, for n large,

$$\sum_{r>n\nu([a,b])(1+n^{-\delta})} k(r)^k \frac{\text{const}}{n^2} P(R_n = r) \leq \frac{\text{const}(k([n\nu([a,b])(1+n^{-\delta})]))^k}{([n\nu([a,b])(1+n^{-\delta})])^2} (2\nu([a,b]))^2.$$

By the assumption (3), this implies that the third term of the sum is the general term of a convergent series.

This proves Lemma 4.4.

LEMMA 4.5. *Under the assumptions of Proposition* 4.1, *for all* $\varepsilon > 0$,

$$P\{\sup_{g\in K} \sup_{\mu\in K'} |B_n - 1| > \varepsilon\}$$

is the general term of a convergent series.

Proof. Using the notations of Lemma 4.4 and the fact that K is a compact set and hence is covered with a finite number of $B(g,\alpha)$, we obtain

$$P\{\sup_{g\in K} \sup_{\mu\in K'} |B_n - 1| > \varepsilon\} = P\Big\{\bigcup_{r=1}^{s} \bigcup_{l=1}^{k(R_n)^k} \sup_{g\in B(g_r,\alpha)} \Big| \frac{n^{-1}\sum_{i=1}^n e^{-N_i(g)} \mathbf{1}_{\Gamma_{n,l}}(N_i)}{E(e^{-N(g)}\mathbf{1}_{\Gamma_{n,l}}(N))} - 1\Big| > \varepsilon\Big\}.$$

Thus

$$P\{\sup_{g\in K}\sup_{\mu\in K'}|B_n-1|>\varepsilon\}$$

$$\leq P\bigg\{\bigcup_{r=1}^{s}\bigcup_{l=1}^{k(R_n)^k}\bigg|\frac{n^{-1}\sum_{i=1}^n e^{-N_i(g_r)}\mathbf{1}_{\Gamma_{n,l}}(N_i)}{E(e^{-N(g_r)}\mathbf{1}_{\Gamma_{n,l}}(N))}-1\bigg|>\frac{\varepsilon}{2}\bigg\}$$

$$+P\bigg\{\bigcup_{r=1}^{s}\bigcup_{l=1}^{k(R_n)^k}\sup_{g\in B(g_r,\alpha)}\bigg|\frac{n^{-1}\sum_{i=1}^n e^{-N_i(g)}\mathbf{1}_{\Gamma_{n,l}}(N_i)}{E(e^{-N(g)}\mathbf{1}_{\Gamma_{n,l}}(N))}$$

$$-\frac{n^{-1}\sum_{i=1}^n e^{-N_i(g_r)}\mathbf{1}_{\Gamma_{n,l}}(N_i)}{E(e^{-N(g_r)}\mathbf{1}_{\Gamma_{n,l}}(N))}\bigg|>\frac{\varepsilon}{2}\bigg\}.$$

We show that the first term of this sum is the general term of a convergent series exactly as in Lemma 4.4. For the second term, choose α satisfying

$$1-e^{-2\alpha}<\varepsilon/4 \quad \text{and} \quad e^{2\alpha}-1<\varepsilon/4.$$

The second term is then bounded from above by

$$P\bigg\{\bigcup_{r=1}^{s}\bigcup_{l=1}^{k(R_n)^k}\bigg|\frac{n^{-1}\sum_{i=1}^n e^{-N_i(g_r)}\mathbf{1}_{\Gamma_{n,l}}(N_i)}{E(e^{-N(g_r)}\mathbf{1}_{\Gamma_{n,l}}(N))}\bigg|>2\bigg\}$$

and thus by

$$P\bigg\{\bigcup_{r=1}^{s}\bigcup_{l=1}^{k(R_n)^k}\bigg|\frac{n^{-1}\sum_{i=1}^n e^{-N_i(g_r)}\mathbf{1}_{\Gamma_{n,l}}(N_i)}{E(e^{-N(g_r)}\mathbf{1}_{\Gamma_{n,l}}(N))}-1\bigg|>1\bigg\}$$

We complete the proof of Lemma 4.5 with the same method as in Lemma 4.4.

With Lemmas 4.3–4.5, the proof of Proposition 4.1 is complete.

5. Conclusion. We thus have a new estimator of the Laplace functional $L(\mu,g)$ which converges almost completely. The estimator of Karr converges almost surely but the conditions are not the same. The condition

(b) $\max_{j\leq l_n}\operatorname{diam} A_{n_j}\to 0$ as $n\to\infty$

has been replaced by

(4) $\lim_{r\to\infty}\inf_{j=1,\dots,k(r)}\lambda_j(r)/\ln(r)=\infty$.

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