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Metric properties of some special p-adic series expansions

by

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1. Introduction. Let $\mathbb{Q}$ be the field of rational numbers, p a prime number and $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ with respect to the p-adic valuation $|\ |_p$ defined on $\mathbb{Q}$ by

$$(1.1)\qquad |0|_p = 0 \quad \text{and} \quad |A|_p = p^{-a} \quad \text{if } A = p^a r/s, \quad \text{where } p \nmid r,\ p \nmid s.$$

Then $\mathbb{Q}_p$ is the field of p-adic numbers with p-adic valuation $|\ |_p$, the extension of the original valuation on $\mathbb{Q}$ (cf. Koblitz [12] or Schikhof [19]).

It is well known that every $A \in \mathbb{Q}_p$ has a unique series representation $A = \sum_{n=v(A)}^{\infty} c_n p^n$, $c_n \in \{0, 1, 2, \ldots, p-1\}$. In the discussion below we call the finite series $\langle A \rangle = \sum_{v(A) \le n \le 0} c_n p^n$ the *fractional part* of A. Then $\langle A \rangle \in S_p$, where we define $S_p = \{\langle A \rangle : A \in \mathbb{Q}_p\} \subset \mathbb{Q}$.

This set S_p is not multiplicatively or additively closed. The function $\langle A \rangle$ and set S_p have been used in the study of certain types of p-adic *continued fractions* by Mahler [14], Ruban [17] and Laohakosol [13] in particular.

Recently the fractional part $\langle A \rangle$ was used by the present authors [7], [8] to derive some new unique series expansions for any element $A \in \mathbb{Q}_p$, including in particular analogues of certain "Sylvester", "Engel" and "Lüroth" expansions of arbitrary *real* numbers into series with rational terms (cf. [16], Chap. IV). It turns out that p-adic and real Lüroth series may be regarded in some sense as algorithmic relatives of continued fractions, and there is some interest in studying possible parallels between the algorithms or digits inducing them. In the direction of metric and asymptotic results concerning digits, various analogies of this kind were previously established, especially by Jager and de Vroedt [5] and Salát [18] for real Lüroth series, and by Ruban [17] for p-adic continued fractions, in comparison with classical theorems of Khinchin (see e.g. [2], [6]) for real continued fractions. (In these developments, Haar measure for p-adic numbers replaces Lebesgue measure for real numbers.)

The main aim of the present paper is to state or derive some similar metric and asymptotic results for the p-adic Lüroth-type expansions referred to above. Here, as in other areas such as e.g. transcendence and diophantine approximation theory (cf. Sprindžuk [20]), there are also parallels with results in the partly analogous but different context of Laurent series over finite fields; see Paysant-Le Roux and Dubois [15] and the present authors [9], [10]. Consequently, many of the steps below are given only in *outline*, together with references to fuller parallel arguments where appropriate. (We thank an anonymous referee for some helpful comments.)

2. Lüroth-type algorithm and ergodic properties. The *Lüroth-type expansion* (see (2.1) below) of a p-adic number $A \in \mathbb{Q}_p$ was derived in [7] from the following algorithm for the "digits" $a_n = a_n(A) \in S_p$:

Define $a_0 = \langle A \rangle$ and $A_1 = A - a_0$ and observe that

$$a_0 = c \in S_p \Leftrightarrow v(A - c) \geq 1 \Leftrightarrow A - c \in X_p,$$

where $X_p = p\mathbb{Z}_p$ is the maximal ideal in the ring $\mathbb{Z}_p$ of all *p-adic integers*, i.e. p-adic numbers of order ≥ 0. If $A_n \neq 0$ $(n \geq 1)$ has already been defined, inductively define

$$a_n = \langle 1/A_n \rangle \quad \text{and} \quad A_{n+1} = (a_n - 1)(a_n A_n - 1),$$

so that $v(a_n) \leq -1$ for $n \geq 1$. If any $A_m = 0$, or $a_m = 0$, stop the algorithm.

This algorithm leads (cf. [7]) to a finite or convergent (relative to $|\ |_p$) expansion

$$A = a_0 + \frac{1}{a_1} + \sum_{n \geq 2} \frac{1}{a_1(a_1 - 1) \ldots a_{n-1}(a_{n-1} - 1)a_n}, \tag{2.1}$$

which is *unique* for A subject to the stated condition on the digits a_n. Another way of looking at it is in terms of operators a and T (where $a : X_p \setminus \{0\} \to S_p$, $T : X_p \to X_p$) such that $a(x) = \langle 1/x \rangle$, $T(0) = 0$ and otherwise $T(x) = (a(x) - 1)(xa(x) - 1)$. Then for $x = A_1 \in X_p$ we have $a_1 = a_1(x) = a(x)$ and more generally $a_n = a_n(x) = a_1(T^{n-1}x)$ if $0 \neq T^{n-1}x \in X_p$.

Although the conclusions of the next theorem are sharpened in Section 3 below it seems at least worth sketching briefly how they can also be deduced from the Ergodic Theorem, after proving that $x \in X_p \Rightarrow T(x) \in X_p$ and that the resulting operator $T : X_p \to X_p$ is *ergodic* relative to Haar measure μ on X_p.

THEOREM 1. (i) *For any given $k \in S_p$ with $v(k) \leq -1$, and all $x \in X_p$ outside a set of Haar measure* 0, *the digit value k has asymptotic frequency*

$$\lim_{n \to \infty} \frac{1}{n} \#\{r \leq n : a_r(x) = k\} = |k|_p^{-2}.$$

(ii) *For all $x \in X_p$ outside a set of Haar measure* 0 *there exists a single "Khinchin-type" constant*

$$\lim_{n\to\infty} |a_1(x)\dots a_n(x)|_p^{1/n} = p^{p/(p-1)}.$$

(iii) *For all $x \in X_p$ outside a set of Haar measure* 0,

$$|x - w_n|_p = p^{(-2p/(p-1)+o(1))n} \qquad \textit{as } n \to \infty,$$

where

$$w_n = w_n(x) = \sum_{r=1}^{n} \frac{\lambda_{r-1}}{a_r}, \qquad \lambda_0 = 1,\ \lambda_r = \frac{1}{a_1(a_1-1)\dots a_r(a_r-1)}.$$

For this and later theorems, a convenient description of the Haar measure μ on $X_p = p\mathbb{Z}_p$ is given in Sprindžuk [20], pp. 67–70. In particular, $\mu(C) = p^{-r}$ for any "circle", "disc" or "ball"

$$C = C(x, p^{-r-1}) := \{y \in \mathbb{Q}_p : |y - x|_p \le p^{-r-1}\}$$

of radius p^{-r-1}. So $\mu(X_p) = 1$, since $X_p = C(0, p^{-1})$.

Now note that every "digit" $a(x)$ lies in the set $S_p^* := \{\langle A\rangle : v(A) \le -1\}$. For any given digits $k_1, \dots, k_n \in S_p^*$, let

$$I_n = I_n(k_1, \dots, k_n) := \{x \in X_p : a_1(x) = k_1, \dots, a_n(x) = k_n\}$$

and call I_n a *basic* (*Lüroth*) *cylinder of rank n*. Also let $I_0 = X_p$.

The Lüroth-type expansion (2.1) of any $x \in I_n$ then has the form

$$x = w_n + \lambda_n \sum_{r>n} \frac{1}{a_{n+1}(a_{n+1}-1)\dots a_{r-1}(a_{r-1}-1)a_r},$$

where

$$\lambda_0 = 1, \quad \lambda_r = \frac{1}{k_1(k_1-1)\dots k_r(k_r-1)} \quad \text{for } 1 \le r \le n,$$

and

$$w_n = \sum_{r=1}^{n} \frac{\lambda_{r-1}}{k_r}.$$

Thus $x = w_n + \lambda_n T^n(x) = \psi_n(T^n(x))$, if $\psi_n = \psi_n(k_1, \dots, k_n) : X_p \to I_n$ is defined by $\psi_n(y) = w_n + \lambda_n y$ ($y \in X_p$). The "linear-type" map ψ_n is then 1-1 onto, with inverse map $T^n : I_n \to X_p$. In particular, $I_n = \mathrm{Im}(\psi_n) = w_n + \lambda_n X_p$. Since $X_p = C(0, p^{-1})$, it then follows that $I_n = C(w_n, p^{-1}|\lambda_n|_p)$ and has Haar measure $\mu(I_n) = p^{-v(\lambda_n)} = |\lambda_n|_p$. Hence

$$\mu(I_n) = \frac{1}{|k_1(k_1-1)\dots k_n(k_n-1)|_p} = \frac{1}{|k_1 \dots k_n|_p^2}, \tag{2.2}$$

since $v(k) = v(k-1)$ for $v(k) \le -1$.

More generally, for any $r_1 < \ldots < r_n$, we obtain

$$\mu\{x \in X_p : a_{r_1}(x) = k_1, \ldots, a_{r_n}(x) = k_n\} = |k_1 \ldots k_n|_p^{-2}. \tag{2.3}$$

In particular, $\mu\{x \in X_p : a_r(x) = k\} = |k|_p^{-2}$ for any $r \geq 1$ and $k \in S_p^*$. Thus the digit functions are *identically distributed* and *independent* random variables relative to μ.

Now, in a standard way quite similar to that followed by Jager and de Vroedt [5] for real Lüroth series, one can deduce that T is measure-preserving and ergodic. (In fact, the stronger *Bernoulli property* for T could be approached along lines analogous to some given in [1].)

Theorem 1 and some further conclusions then follow by making special choices for integrable functions f in the ergodic formula

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f(T^{r-1}x) = \int_{X_p} f\, d\mu \quad \text{a.e.} \tag{2.4}$$

For example, part (i) of Theorem 1 follows from consideration of the characteristic function f_k of a basic cylinder $I_1(k)$. Alternatively, use of the function $\widehat{f}(\cdot) = \log_p |a_1(\cdot)|_p$ leads to the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \log_p |a_r(x)|_p = \int_{X_p\setminus\{0\}} \widehat{f}\, d\mu = \frac{p}{p-1} \quad \text{a.e.,} \tag{2.5}$$

and this implies part (ii) of Theorem 1. The same function $\widehat{f}$ may be used in the deduction of part (iii), in combination with the following inequalities analogous to some appearing for Laurent series in [10]:

$$1 - \sum_{r=1}^{n+1} \log_p |a_r(x)|_p^2 \leq \log_p |x - w_n|_p \leq -1 - \sum_{r=1}^{n} \log_p |a_r(x)|_p^2. \tag{2.6}$$

The function $\widehat{f}$ may also be used to show that the operator T has *entropy*

$$h(T) = -\lim_{n\to\infty} \frac{1}{n} \log_e \mu(I_n) = \frac{2p\log_e p}{p-1}. \tag{2.7}$$

Lastly, it is interesting to note that, in contrast to (2.5), a truncation argument involving the function $\widetilde{f}(\cdot) = |a_1(\cdot)|_p$ leads to the conclusion

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} |a_r(x)|_p = \infty \quad \text{a.e.} \tag{2.8}$$

3. Sharper asymptotic estimates. The fact that the p-adic Lüroth-type digit functions $a_r(\cdot)$ define identically distributed and independent random variables on X_p paves the way for the introduction of methods and results of probability theory, which lead to sharper and deeper results than those considered earlier.

In the first place, the law of the iterated logarithm and the central limit theorem (cf. Theorems 3.16/17 in Galambos [4]) yield:

THEOREM 2. *Let* $A_{n,k}(x) = \#\{r \le n : a_r(x) = k\}$. *Then for almost all* $x \in X_p$,

$$\limsup_{n\to\infty} \frac{A_{n,k}(x) - n|k|_p^{-2}}{\sqrt{n \log\log n}} = \sqrt{2|k|_p^{-2}(1-|k|_p^{-2})}\,.$$

Further, *for any real* z,

$$\lim_{n\to\infty} \mu\Big\{x \in X_p : A_{n,k}(x) - n|k|_p^{-2} < \frac{z}{|k|_p}\sqrt{n(1-|k|_p^{-2})}\Big\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\,du.$$

Now define a sequence (t_n) of independent random variables t_n on X_p by

$$t_n(x) = \begin{cases} v(a_n(x)) & \text{if } |a_n(x)|_p \le n^2, \\ 0 & \text{otherwise.} \end{cases}$$

Then the expected value

$$E(t_n) = \sum_{p^r \le n^2} p^{-2r} r(p-1)p^r = E(v(a_n(\cdot))) + O\Big(\frac{\log n}{n^2}\Big),$$

and a similar calculation of $E(t_n^2)$ and the variance $\operatorname{var}(t_n)$ leads to the estimate

$$B_n := \sum_{r=1}^{n} \operatorname{var}(t_r) = \frac{pn}{(p-1)^2} + O(1) \quad \text{as } n \to \infty. \tag{3.1}$$

Next, since $t_n(x) \le 2\log_p n = o(\sqrt{B_n/\log\log B_n}\,)$, the law of the iterated logarithm implies

$$\limsup_{n\to\infty} \frac{\sum_{r=1}^{n} t_r - \sum_{r=1}^{n} E(t_r)}{\sqrt{2B \log\log B_n}} = 1 \quad \text{a.e.} \tag{3.2}$$

Hence

$$\limsup_{n\to\infty} \frac{\sum_{r=1}^{n} t_r - \sum_{r=1}^{n} E(v(a_r(\cdot)))}{\sqrt{2\frac{p}{(p-1)^2} n \log\log n}} = 1 \quad \text{a.e.} \tag{3.3}$$

Now let $U_n = \{x \in X_p : t_n(x) \ne v(a_n(x))\}$. Then

$$\mu(U_n) = \sum_{|k|_p > n^2} |k|_p^{-2} < \frac{1}{n^2},$$

and the Borel–Cantelli lemma yields $\mu(\limsup_{n\to\infty} U_n) = 0$. Thus, for almost all $x \in X_p$, there exists $n_0(x)$ with $t_n(x) = v(a_n(x))$ for $n \geq n_0(x)$. Therefore (3.3) now implies:

Theorem 3. *For almost all* $x \in X_p$,

$$\limsup_{n\to\infty} \frac{\sum_{r=1}^{n} v(a_r(x)) - c_1 n}{\sqrt{n \log\log n}} = \sqrt{2c_2},$$

where $c_1 = p/(p-1)$, $c_2 = p/(p-1)^2$. *Hence as* $n \to \infty$,

$$|a_1(x)\ldots a_n(x)|_p^{1/n} = p^{p/(p-1)} + O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad a.e.$$

The next theorem sharpens the last part of Theorem 1 above:

Theorem 4. *If* $w_n = w_n(x)$ *is defined as in Theorem* 1(iii) *then*

$$\limsup_{n\to\infty} \frac{v(x-w_n) + 2pn/(p-1)}{\sqrt{n\log\log n}} = \frac{\sqrt{8p}}{p-1} \quad a.e.$$

Hence

$$\frac{1}{n} v(x-w_n) = -\frac{2p}{p-1} + O\left(\sqrt{\frac{\log\log n}{n}}\right) \quad a.e.$$

By symmetry as in Feller [3], p. 205, Theorem 3 leads to

$$\liminf_{n\to\infty} \frac{\sum_{r=1}^{n} v(a_r^2(x)) - 2pn/(p-1)}{\sqrt{n\log\log n}} = -2\sqrt{2c_2} \quad \text{a.e.} \tag{3.4}$$

In combination with (2.6) above, this implies Theorem 4.

4. Average and individual estimates for digits. By (2.8) above, the average

$$\frac{1}{n}\sum_{r=1}^{n} |a_r(x)|_p \to \infty \quad \text{a.e. on } X_p$$

as $n \to \infty$. Theorem 5 estimates this average in probability over X_p:

Theorem 5. *For any fixed* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left\{x \in X_p : \frac{1}{n\log_p n}\left|\sum_{r=1}^{n} |a_r(x)|_p - (p-1)\right| > \varepsilon\right\} = 0,$$

i.e.

$$\frac{1}{n\log_p n}\sum_{r=1}^{n} |a_r(x)|_p \to p-1 \quad \textit{in probability over } X_p.$$

P r o o f. Consider the truncation method of Feller [3], Chapter 10, §2, as applied to the random variables U_r, V_r $(r \le n)$ defined by

$$\begin{aligned} &U_r(x) = |a_r(x)|_p, \quad V_r(x) = 0 && \text{if } |a_r(x)|_p \le n \log_p n,\\ &U_r(x) = 0, \qquad\quad\;\; V_r(x) = |a_r(x)|_p && \text{if } |a_r(x)|_p > n \log_p n. \end{aligned}$$

In that case

$$\begin{aligned} \mu\Big\{x \in X_p : \frac{1}{n\log_p n}\Big|\sum_{r=1}^{n} |a_r(x)|_p - (p-1)\Big| > \varepsilon\Big\} \tag{4.1}\\ \le \mu\{x : |U_1 + \ldots + U_n - (p-1)n\log_p n| > \varepsilon n \log_p n\}\\ + \mu\{x : V_1 + \ldots + V_n \ne 0\}, \end{aligned}$$

and

$$\begin{aligned} \mu\{x : V_1 + \ldots + V_n \ne 0\} &\le n\mu\{x : |a_1(x)|_p > n\log_p n\} \tag{4.2}\\ &= n \sum_{|k|_p > n\log_p n} |k|_p^{-2} < 1/\log_p n = o(1). \end{aligned}$$

Next $E(U_1 + \ldots + U_n) = nE(U_1)$ and $\operatorname{var}(U_1 + \ldots + U_n) = n\operatorname{var}(U_1)$, where

$$\begin{aligned} E(U_1) &= \sum_{|k|_p \le n\log_p n} |k|_p^{-1} = (p-1)\log_p([n\log_p n]) \tag{4.3}\\ &\sim (p-1)\log_p n \quad \text{ as } n \to \infty \end{aligned}$$

and

$$\operatorname{var}(U_1) < E(U_1^2) = \sum_{|k|_p \le \log_p n} 1 < pn(\log_p n). \tag{4.4}$$

Theorem 5 then follows from an application of (4.3) and (4.4) to Chebyshev's inequality:

$$\mu\{x : |U_1 + \ldots + U_n - nE(U_1)| > \varepsilon n E(U_1)\} \le \frac{n\operatorname{var}(U_1)}{(\varepsilon n E(U_1))^2} = o(1). \tag{4.5}$$

Note that the conclusion of Theorem 5 is not valid with probability one, since Theorem 3.13 in Galambos [4] implies that either

$$\limsup_{n\to\infty} \frac{1}{n\log_p n} \sum_{r=1}^{n} |a_r(x)|_p = \infty \quad \text{a.e.}$$

or

$$\liminf_{n\to\infty} \frac{1}{n\log_p n} \sum |a_r(x)|_p = 0 \quad \text{a.e.}$$

Regarding estimates for individual digits, now consider:

THEOREM 6. *Given any positive increasing function* $\psi(n)$ *of* n,

$$|a_n(x)|_p = O(\psi(n)) \ a.e. \Leftrightarrow \sum_{n=1}^{\infty} 1/\psi(n) < \infty.$$

In fact, $|a_n(x)|_p = O(\psi(n))$ *is false a.e. if the series diverges.*

Proof. Let $V_n = \{x \in X_p : |a_n(x)|_p > \psi(n)\}$. Since $\mu\{x : a_n(x) = k\} = |k|_p^{-2}$ by (2.3), it follows that

$$\mu(V_n) = \sum_{|k|_p > \psi(n)} |k|_p^{-2} \le p/\psi(n).$$

If $\sum_{n=1}^{\infty} \psi(n)^{-1} < \infty$, then the Borel–Cantelli lemma (cf. [4], p. 36) yields $\mu(\limsup V_n) = 0$. Hence $|a_n(x)|_p > \psi(n)$ for at most finitely many n, for almost all $x \in X_p$. Thus $|a_n(x)|_p = O(\psi(n))$ a.e.

If $\sum_{n=1}^{\infty} \psi(n)^{-1}$ diverges, the Abel–Dini theorem (Knopp [11], p. 290) implies that there exists a positive increasing function $\theta(n)$ with $\theta(n) \to \infty$ as $n \to \infty$ such that $\sum_{n=1}^{\infty} \psi(n)^{-1}\theta(n)^{-1}$ also diverges. Then let $W_n = \{x \in X_p : |a_n(x)|_p > \psi(n)\theta(n)\}$. The independence of the random variables a_n implies the independence of the sets W_n. Also

$$\sum_{n=1}^{\infty} \mu(W_n) = \sum_{n=1}^{\infty} \sum_{|k|_p > \psi(n)\theta(n)} |k|_p^{-2} > \frac{1}{p} \sum_{n=1}^{\infty} \frac{1}{\psi(n)\theta(n)} = \infty.$$

Thus the Borel–Cantelli lemma yields $\mu(\limsup W_n) = 1$, and so $|a_n(x)|_p > \psi(n)\theta(n)$ holds with probability one, for infinitely many n. Thus $|a_n(x)|_p = O(\psi(n))$ is false a.e.

COROLLARY. *For almost all* $x \in X_p$,

$$\limsup_{n\to\infty} \frac{\log |a_n(x)|_p - \log n}{\log \log n} = 1.$$

Proof. Theorem 6 implies that $|a_n(x)|_p = O(n(\log n)^{\alpha})$ a.e. for any $\alpha > 1$, while $|a_n(x)|_p = O(n(\log n)^{\beta})$ is false a.e. for any $\beta \le 1$. The corollary then follows by choosing $\alpha = 1 + \varepsilon$, $\beta = 1 - \varepsilon$ ($\varepsilon > 0$).

(Note that the corresponding lower limit is *not* finite a.e., since (2.3) earlier shows that $|a_n(x)|_p$ can take any particular constant value p^N ($N \ge 1$) for all n, and all x in a set of positive measure.)

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