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Perfect powers in products of integers from a block of consecutive integers (II)

by

T. N. Shorey (Bombay) and Yu. V. Nesterenko (Moscow)

1. Introduction. For an integer $\nu > 1$, we define $P(\nu)$ to be the greatest prime factor of ν and we write $P(1) = 1$. Let $m \ge 0$ and $k \ge 2$ be integers. Let $d_1, \ldots, d_t$ with $t \ge 2$ be distinct integers in the interval $[1, k]$ and let $l > 2, y > 0$ and $b > 0$ be integers with $P(b) \le k$. We consider the equation

$$(m + d_1) \ldots (m + d_t) = by^l \tag{1}$$

in $m, t, d_1, \ldots, d_t, b, y$ and l. We always assume that the left hand side of equation (1) is divisible by a prime exceeding k. Consequently, there is an i with $1 \le i \le t$ such that $m + d_i$ is divisible by an lth power of a prime exceeding k. Thus $m + d_i \ge (k + 1)^l$ implying that $m > k^l$.

Equation (1) with $t = k$ and $b = 1$ is solved completely by Erdős and Selfridge [5] in 1975; a product of two or more consecutive positive integers is never a power. In fact, Erdős [4] proved in 1955 that for $\varepsilon > 0$, equation (1) with $b = 1$ and

$$t \ge k - (1 - \varepsilon)k \frac{\log \log k}{\log k}$$

implies that k is bounded by an effectively computable number depending only on ε. This was sharpened considerably by Shorey [7], [8] in 1986–87. Shorey [8] showed that equation (1) with

$$t \ge \frac{1}{2}\left(1 + \frac{4l^2 - 8l + 7}{2(l - 1)(2l^2 - 5l + 4)}\right)k \tag{2}$$

implies that k is bounded by an effectively computable absolute constant. Further, the assumption (2) has been relaxed for sufficiently large l. More precisely, Shorey [7] showed in 1986 that equation (1) with

$$t \ge kl^{-1/11} + \pi(k) + 2 \tag{3}$$

implies that $\min(k, l)$ is bounded by an effectively computable absolute constant.

The proofs of these results depend on the method of Roth and Halberstam on difference between consecutive ν-free integers, the results of Baker [1] on the approximations of algebraic numbers of the form $(A/B)^{m/n}$ with $A > B$ by rationals and the theory of linear forms in logarithms. The precise dependence on "A" in the irrationality measures of Baker [1] plays a crucial role in the proofs. Further, Baker's sharpening [3] on linear forms in logarithms is essential. Linear forms in logarithms with α_i's very close to 1 appear in the proofs and the best possible estimates of Shorey [7, Lemma 2], namely replacing $\log A$ in place of $\log A_1 \ldots \log A_n$ with $A = \max_{1\le i\le n} A_i$, for these linear forms in logarithms are required.

In this paper, we improve the results mentioned above on equation (1) whenever $l \geq 7$. For this, it is important to relax the assumption (2) of Baker [1] even though this makes the exponent of irrationality measure less precise. This is possible by appealing to a subsequent paper of Baker [2] in this direction. See Lemma 1. We shall also use an improved version, due to Loxton, Mignotte, van der Poorten and Waldschmidt [6], of Shorey [7, Lemma 2] cited above on linear forms in logarithms to relax the assumption (3). For stating the results of this paper, we define for $l \geq 7$,

$$\nu_l = \begin{cases} \dfrac{112l^2 - 160l + 29}{28l^3 - 76l + 29} & \text{if } l \equiv 1 \pmod 2, \\ \dfrac{112l^2 - 160l + 17}{28l^3 - 188l + 129} & \text{if } l \equiv 0 \pmod 2. \end{cases}$$

For $l \geq 7$, we observe that $\nu_l \geq 3/l$,

$$\nu_l \leq \begin{cases} \dfrac{4}{l}\left(1 - \dfrac{1}{(.875)l}\right) & \text{if } l \equiv 1 \pmod 2, \\ \dfrac{4}{l}\left(1 - \dfrac{1}{(1.412)l}\right) & \text{if } l \equiv 0 \pmod 2 \end{cases}$$

and

$$\nu_7 \leq .4832, \quad \nu_8 \leq .4556, \quad \nu_9 \leq .3878, \quad \nu_{10} \leq .3664,$$
$$\nu_{11} \leq .3243, \quad \nu_{12} \leq .3076, \quad \nu_{13} \leq .2787, \quad \nu_{14} \leq .2655.$$

We prove the following result.

Theorem. (a) *Equation* (1) *with*

$$l \geq 7, \quad t \geq \nu_l k \tag{4}$$

implies that k is bounded by an effectively computable number depending only on l.

(b) *Let $\varepsilon > 0$. There exists an effectively computable number C depending only on ε such that equation* (1) *with*

$$t \geq kl^{-1/3+\varepsilon} + \pi(k) + 2$$

implies that $\min(k,l) \leq C$.

2. A relaxation in the assumption (2) of Baker's paper [1]. In this section, we appeal to Baker's paper [2] in order to derive the following result.

LEMMA 1. *Let A, B, K and n be positive integers such that $A > B$, $K < n$, $n \geq 3$ and $\omega = (B/A)^{1/n}$ is not a rational number. For $0 < \phi < 1$, put*

$$\delta = 1 + \frac{2-\phi}{K}, \qquad s = \frac{\delta}{1-\phi}, \tag{5}$$

$$u_1 = 40^{n(K+1)(s+1)/(Ks-1)}, \qquad u_2^{-1} = K2^{K+s+1}40^{n(K+1)}.$$

Assume that

$$A(A-B)^{-\delta}u_1^{-1} > 1. \tag{6}$$

Then

$$\left|\omega - \frac{p}{q}\right| > \frac{u_2}{Aq^{K(s+1)}}$$

for all integers p and q with $q > 0$.

Proof. We put

$$\lambda_1 = 40^{n(K+1)}A, \qquad \lambda_2 = 40^{n(K+1)}(A-B)^{K+1}A^{-K} \tag{7}$$

and

$$\Lambda = \frac{\log \lambda_1}{\log \lambda_2}.$$

By (6) and $0 < \phi < 1$, we observe that $0 < \lambda_2 < 1$. We follow Baker [2] with $m_j = j/n$ for $0 \leq j \leq K$ to conclude that for integers r, p and q with $r > 0$ and $q > 0$, there exists a polynomial $P_r(X) \in \mathbb{Z}[X]$ satisfying

$$\text{(i) } \deg P_r \leq K, \qquad \text{(ii) } H(P_r) \leq \lambda_1^r,$$
$$\text{(iii) } P_r(p/q) \neq 0, \qquad \text{(iv) } |P_r(w)| \leq \lambda_2^r.$$

Here $H(P_r)$ denotes the maximum of the absolute values of the coefficients of P_r. For $r \geq 54$, Baker [2] gave sharper estimates (ii) and (iv) with 40 replaced by 4 in the definitions (7) of λ_1 and λ_2. We may assume that $|\omega - p/q| < 1/2$ and we define r as the smallest integer such that

$$\lambda_2^r \leq \frac{1}{2q^K}.$$

Then

$$\lambda_2^r > \frac{\lambda_2}{2q^K}$$

and

$$\lambda_1^r = (\lambda_2^r)^\Lambda \leq \left(\frac{\lambda_2}{2q^K}\right)^\Lambda = \lambda_1 2^{-\Lambda}q^{-K\Lambda}.$$

Further, we observe that

$$\frac{1}{q^K} \le \left|P_r\left(\frac{p}{q}\right)\right| \le \left|P_r\left(\frac{p}{q}\right) - P_r(\omega)\right| + |P_r(\omega)| \le \left|P_r\left(\frac{p}{q}\right) - P_r(\omega)\right| + \frac{1}{2q^K}\,.$$

Thus

$$\left|P_r\left(\frac{p}{q}\right) - P_r(\omega)\right| \ge \frac{1}{2q^K}\,.$$

On the other hand, we have

$$\left|P_r\left(\frac{p}{q}\right) - P_r(\omega)\right| = \Big|\int\limits_{p/q}^{\omega} P_r'(X)\,dX\Big| \le K2^K\lambda_1^r\left|\omega - \frac{p}{q}\right|.$$

Consequently,

$$\left|\omega - \frac{p}{q}\right| > (K2^{K+1}\lambda_1)^{-1}2^{\Lambda}q^{-\chi},$$

where $\chi = K - K\Lambda$. By (6), we observe that $-\Lambda \le s$ and $\chi \le K(s+1)$. Hence

$$\left|\omega - \frac{p}{q}\right| > \frac{u_2}{Aq^{K(s+1)}}\,.$$

3. Proof of Theorem (a). Let $\varepsilon_1 = (10^6 l^3)^{-1}$. Suppose that equation (1) with (4) is satisfied. We may assume that k exceeds a sufficiently large effectively computable number depending only on l. We denote by u_3, u_4 and u_5 effectively computable positive numbers depending only on l. We put

$$\tau = \left(1 + \frac{\varepsilon_1 l}{4}\right)\nu_l^{-1} < \frac{l}{2}, \qquad \tau_1 = (\nu_l^{-1} - 1)/(l-1). \tag{8}$$

We see from equation (1) that

$$m + d_i = a_i x_i^l \quad \text{for } 1 \le i \le t,$$

where a_i and x_i are positive integers satisfying

$$P(a_i) \le k, \qquad \Big(x_i, \prod_{p\le k} p\Big) = 1.$$

We write $S = \{a_1, \ldots, a_t\}$. We argue as in [8] to conclude that there exists a subset S_2 of S with $|S_2| \ge u_3 k$ and

$$a_i \le k^{\tau} \quad \text{for } a_i \in S_2. \tag{9}$$

Further we apply the method of Halberstam and Roth as in [8] for deriving that there exists a subset S_3 of S_2 with $|S_3| \ge u_4 k^{1-\varepsilon_1}$ such that

$$x_i > k^{2-\tau_1-5\varepsilon_1} \quad \text{for } a_i \in S_3. \tag{10}$$

In fact, (9) is valid with τ replaced by $\tau' = (1+\varepsilon'/4)\nu_l^{-1}$ where $\varepsilon' = (10^6 l^5)^{-1}$, and we use this estimate for deriving (10). Put $s_3 = |S_3|$. By permuting the subscripts of $d_1, \ldots, d_t$, there is no loss of generality in assuming that $a_1, a_2, \ldots, a_{s_3}$ are elements of S_3 and $a_1 < a_2 < \ldots < a_{s_3}$. Then we find, as in [8], an integer μ with $1 \le \mu < s_3$ such that

$$\log\left(\frac{a_{\mu+1}}{a_\mu}\right) \le \frac{u_5 \log k}{k^{1-\varepsilon_1}} \tag{11}$$

and

$$0 \ne \left|\left(\frac{a_\mu}{a_{\mu+1}}\right)^{1/l} - \frac{x_{\mu+1}}{x_\mu}\right| < \frac{2k}{a_{\mu+1}x_\mu^l}\,. \tag{12}$$

Now, we turn to applying Lemma 1 with

$$K = \begin{cases} (l-3)/2 & \text{if } l \equiv 1 \pmod 2, \\ (l-4)/2 & \text{if } l \equiv 0 \pmod 2, \end{cases} \tag{13}$$

and $A = a_{\mu+1}, B = a_\mu, n = l$. We put $\psi = (2-\phi)/K$, where ϕ will be chosen later in some special way and we put $\delta = 1+\psi$ with $2/(l-3) < \psi < 1$. By (11), we observe that

$$\frac{a_{\mu+1}-a_\mu}{a_{\mu+1}} < \frac{a_{\mu+1}-a_\mu}{a_\mu} < \frac{2u_5 \log k}{k^{1-\varepsilon_1}}\,.$$

Therefore, by (9), the left hand side of inequality (6) exceeds

$$\left(\frac{k^{1-\varepsilon_1}}{2u_5 \log k}\right)^{1+\psi} (u_1 k^{\tau\psi})^{-1}.$$

Thus, the assumption (6) is satisfied if $1+\psi-\tau\psi \ge 5\varepsilon_1$, which, by (8), reads

$$\nu_l \ge \frac{\psi}{1+\psi} + \frac{\varepsilon_1 l}{4}\cdot\frac{\psi}{1+\psi} + \frac{5\varepsilon_1\nu_l\psi}{1+\psi}.$$

We observe that the second summand on the right hand side of the preceding inequality does not exceed $2\varepsilon_1$, since

$$\frac{\psi}{1+\psi} = \frac{2-\phi}{K+2-\phi} < \frac{2}{K+1} \le \frac{4}{l-2}\,,$$

and the third summand is at most $5\varepsilon_1$, since $\nu_l < 1$ and $0 < \psi < 1$. Hence, the assumption (6) is satisfied if

$$\nu_l \ge \frac{\psi}{1+\psi} + 7\varepsilon_1. \tag{14}$$

We shall later choose ϕ depending only on l so that (14) is satisfied. Then, the assumption of Lemma 1 is valid. Hence, we conclude from Lemma 1 that

$$\left|\left(\frac{a_\mu}{a_{\mu+1}}\right)^{1/l} - \frac{x_{\mu+1}}{x_\mu}\right| > \frac{u_2}{a_{\mu+1}x_\mu^{K(s+1)}}\,. \tag{15}$$

We put $\theta = l - K(s+1)$. The parameter ϕ will be chosen later in such a way that $\theta > 0$. We observe from (5) that

$$\theta = l - \frac{K+2-\phi}{1-\phi} - K = l - \left(2 + \frac{\phi}{1-\phi}\right)(K+1)$$

which, by (13), implies that

$$\theta = \theta' - \frac{\phi(K+1)}{1-\phi},$$

where

$$\theta' = \begin{cases} 1 & \text{if } l \equiv 1 \pmod 2, \\ 2 & \text{if } l \equiv 0 \pmod 2. \end{cases}$$

Further, we see from (8) and (14) that

$$\tau_1 \le \frac{1}{(l-1)\psi} - \varepsilon_1.$$

Finally, we combine (12), (15) and (10) in order to derive that

$$k^{(2-\tau_1-5\varepsilon_1)\theta} < 2u_2^{-1}k,$$

which, since k is sufficiently large, implies that $(2-\tau_1-5\varepsilon_1)\theta < 1+\varepsilon_1$. Consequently,

$$\theta' - \frac{\phi(K+1)}{1-\phi} < \left(2 - \frac{1}{(l-1)\psi}\right)^{-1} + 8\varepsilon_1.$$

Let $l \equiv 1 \pmod 2$. Then, by substituting $\theta = 1$, $l = 2K+3$ and $\psi = (2-\phi)/K$, we get

$$(1-(K+2)\phi)(7K+8-(4K+4)\phi) - (2K+2)(2-3\phi+\phi^2) < 128\varepsilon_1 K.$$

Thus

$$(4K^2+10K+6)\phi^2 - (7K^2+20K+14)\phi + 3K + 4 < 128\varepsilon_1 K.$$

Let

$$\phi = \frac{24K+28.84}{14(4K^2+10K+6)}.$$

Then

$$(45.68)K^2 - (26.88)K - 116.8944 < 3 \cdot 10^6 \varepsilon_1 K^3.$$

We observe that the left hand side of the preceding inequality exceeds 12 since $K \ge 2$. On the other hand, the right hand side is less than one. This is a contradiction.

Let $l \equiv 0 \pmod 2$. Then

$$(4K^2+16K+15)\phi^2-(7K^2+35K+39)\phi+10K+18<128\varepsilon_1 K$$

and we choose

$$\phi=\frac{80K+127.82}{14(4K^2+16K+15)}$$

to obtain

$$(145.64)K^2-(12.6)K-531.7676<3\cdot 10^6\varepsilon_1 K^3,$$

leading to a contradiction. Finally, we compute ψ in either of the cases $l \equiv 1 \pmod 2$ and $l \equiv 0 \pmod 2$ to observe that the assumption (14) is valid. This completes the proof of Theorem (a).

4. Proof of Theorem (b). We follow the notation of [7, Lemma 2] where, under certain assumptions, the lower bound

$$\exp(-(C_9\tau_2 n^3)^{3n+3}\tau_1\log A) \tag{16}$$

for the absolute value of linear forms in logarithms was proved. This has been improved to

$$\exp(-(C_9 n)^n\tau_2^{n+1}\log A) \tag{17}$$

in [6, Theorem 1]. If we replace (16) by (17) for the case $n=2$ in the proof of [7, Lemma 6], the assertion of Theorem (b) follows.

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School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road
Bombay 400 005, India
E-mail: shorey@math.tifr.res.in

Department of Mathematics
University of Moscow
Moscow 119899, Russia
E-mail: nest@nw.math.msu.su

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