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On characterization of Dirichlet L-functions

by

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1. Introduction. Let $L(s,f)$ denote the Dirichlet series $\sum_{n=1}^{\infty} f(n)/n^s$. If f is purely recurring, then $L(s,f)$ is absolutely convergent for $\mathrm{Re}(s)>1$ and

$$L(s,f)=\frac{1}{N^s}\sum_{n=1}^{N} f(n)\zeta(s,n/N),$$

where N is a period of f and

$$\zeta(s,x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^s}$$

is the Hurwitz zeta function. We know that $L(s,f)$ can be extended analytically to the whole plane as a meromorphic function of order one and has only a simple pole with residue $(f(1)+\ldots+f(N))/N$ at $s=1$ unless $f(1)+\ldots+f(N)=0$, in which case there exists no pole in the whole plane and $L(s,f)$ is convergent for $\mathrm{Re}(s)>0$. We call f *even* (resp. *odd*) modulo N if, extending it periodically to all integers, $f(-x)=(-1)^d f(x)$ with $d=0$ (resp. $d=1$). Schnee [6] showed the functional equation

$$\left(\frac{N}{\pi}\right)^{s/2}\Gamma\left(\frac{s+d}{2}\right)L(s,f)=i^{-d}\left(\frac{N}{\pi}\right)^{(1-s)/2}\Gamma\left(\frac{1-s+d}{2}\right)L(1-s,T_N f),$$

where

$$T_N f(x)=\frac{1}{\sqrt{N}}\sum_{n=1}^{N} f(n)\exp\left(\frac{2\pi i n x}{N}\right).$$

We list some of the above properties of $L(s,f)$ as

(A) The Dirichlet series expansion of $L(s,f)$ is absolutely convergent for $\mathrm{Re}(s)>1$.

(B) $L(s,f)$ can be continued into the whole plane to a meromorphic function of finite order with a finite number of poles.

(C) For a non-negative integer d and a positive number N a functional equation holds in the form

$$\left(\frac{N}{\pi}\right)^{s/2}\Gamma\left(\frac{s+d}{2}\right)L(s,f)=\left(\frac{N}{\pi}\right)^{(1-s)/2}\Gamma\left(\frac{1-s+d}{2}\right)L(1-s,g),$$

where $L(s,g)$ is convergent in a half-plane.

In Section 2 we shall prove that (A), (B), and (C) characterize Dirichlet series with recurrent coefficients, following Chandrasekharan–Narashimhan [3] and modifying Siegel's proof [7] of Hamburger's theorem [4] on the Riemann zeta function. In Section 3 we characterize Dirichlet L-functions without using Euler products. We shall use Dirichlet L-functions in Section 4 to give a characterization of finite Dirichlet series in a way different from Toyoizumi's results in [8]. In Section 5 we shall extend the concept of equivalence and conductors of Dirichlet characters to general periodic functions.

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2. Characterization of recurring coefficients

LEMMA 2.1. *For functions* $f \neq 0$, *properties* (A), (B) *and* (C) *imply that* N *is a positive integer*, *the number* d *is* 0 *or* 1, f *is purely recurring*, *even or odd modulo* N *according as* $d = 0$ *or* 1, *and* $g = i^{-d}T_N f$.

Proof (for more details see Chandrasekharan–Narashimhan [3]). We put

$$\phi(s)=(2N)^s L(2s-d,f),\qquad \psi(s)=(2N)^s L(2s-d,g).$$

The given functional equation becomes

$$(2\pi)^{-s}\Gamma(s)\phi(s)=(2\pi)^{s-\delta}\Gamma(\delta-s)\psi(\delta-s),$$

where $\delta = d + 1/2$.

Let α, β be positive numbers such that

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^{2\alpha-d}},\qquad \sum_{n=1}^{\infty}\frac{g(n)}{n^{2\beta-d}}$$

converge absolutely. By (A) we may choose $\alpha < 1 + d$ (in fact, any $\alpha > (1 + d)/2$ would do).

We see from (B) and the functional equation that $\phi(s)$ has at most a finite number of poles r, all in the strip $\delta - \beta < \mathrm{Re}(r) < \alpha$.

We start off from the integral

$$\frac{1}{2\pi i}\int_{(\alpha)}\Gamma(s)\phi(s)x^{-s}\,ds\qquad (x>0)$$

over the vertical line (α) with real point α. By the formula

$$\frac{1}{2\pi i}\int_{(\alpha)}\frac{\Gamma(s)}{y^s}\,ds = e^{-y}\qquad (y>0),$$

putting in the series representation of $L(s,f)$, it is, on the one hand,

$$\sum_{n=1}^{\infty} f(n)n^d e^{-n^2x/(2N)}.$$

The series representations and the functional equation together with the Phragmén–Lindelöf principle, $L(s,f)$ being of finite order, imply in a standard way that $|\phi(s)|$ can be estimated by a power of $|\mathrm{Im}(s)|$ in any given vertical strip. This enables one, on the other hand, to push the line of integration to $(\delta-\beta)$.

Using the functional equation,

$$\begin{aligned}\frac{1}{2\pi i}\int_{(\delta-\beta)}\Gamma(s)\phi(s)x^{-s}\,ds &= \frac{1}{2\pi i}\int_{(\delta-\beta)}\Gamma(\delta-s)\psi(\delta-s)(2\pi)^{2s-\delta}x^{-s}\,ds\\ &= \frac{1}{2\pi i}\int_{(\beta)}\Gamma(s)\psi(s)(2\pi)^{\delta-s}x^{s-\delta}\,ds\\ &= \left(\frac{2\pi}{x}\right)^{\delta}\sum_{n=1}^{\infty} g(n)n^d e^{-2\pi^2n^2/(Nx)}\end{aligned}$$

by a similar calculation in the last step as above.

It remains to collect the residues of $\Gamma(s)\phi(s)x^{-s}$. At any given pole r of order q the residue is of the form

$$x^{-r}P_r(\log x),$$

where P_r is a polynomial of degree $\le q$ with constant coefficients. Denoting by $P(x)$ their (finite) sum,

$$P(x)=\sum_r x^{-r}P_r(\log x),$$

we get

$$(*)\qquad \sum_{n=1}^{\infty} f(n)n^d e^{-n^2x/(2N)} = \left(\frac{2\pi}{x}\right)^{d+1/2}\sum_{n=1}^{\infty} g(n)n^d e^{-2\pi^2n^2/(Nx)} + P(x).$$

Following Siegel's idea, we multiply $(*)$ throughout by $x^d e^{-s^2x/(2N)}$ first with $s>0$, and integrate with respect to x over $(0,\infty)$. The left hand side becomes

$$F_1(s) = (2N)^{d+1}\Gamma(d+1)\sum_{n=1}^{\infty}\frac{f(n)n^d}{(s^2+n^2)^{d+1}},$$

and using the formula

$$\int_0^{\infty}\frac{e^{-(ax+b/x)}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{a}}e^{-2\sqrt{ab}} \qquad (a,b>0),$$

the first term on the right becomes

$$F_2(s) = (2\pi)^{d+1}\sqrt{N}\sum_{n=1}^{\infty}s^{-1}g(n)n^d e^{-2\pi ns/N},$$

both the resulting series being absolutely convergent.

Finally, the second term on the right becomes

$$F_3(s) = \int_0^{\infty}x^d P(x)e^{-s^2x/(2N)}\,dx.$$

The latter is a finite linear combination of integrals, absolutely convergent by $\mathrm{Re}(d-r) > d-\alpha > -1$, of the type

$$\int_0^{\infty}x^{d-r}(\log x)^m e^{-s^2x/(2N)}\,dx = \int_0^{\infty}\left(\frac{y}{s^2}\right)^{d-r}(\log y - 2\log s)^m e^{-y/(2N)}\,\frac{dy}{s^2}$$

with integers $m\ge 0$. This is $s^{2r-2d-2}$ multiplied by a polynomial in $\log s$ and we see that $F_3(s)$ can be extended to a single-valued regular function in the whole plane with the non-positive real axis deleted.

Our formula for $F_2(s)$ extends $sF_2(s)$ to a function regular and periodic with period iN for $\mathrm{Re}(s)>0$.

Finally, the series representation of $F_1(s)$ does, in fact, converge for all complex $s\ne \pm in$ ($n=1,2,\ldots$) representing a meromorphic function in the whole plane with poles of order $d+1$ at $\pm in$ only (unless $f(n)=0$).

From the periodicity of $sF_1(s) - sF_3(s) = sF_2(s)$ we see that N is a positive integer and

$$\lim_{s\to in}F_1(s)s(s-in)^{d+1} = (-i)^d N^{d+1}\Gamma(d+1)f(n)$$

is periodic in n with period N.

Denote by f_{E} and f_{O} the even and the odd part of f modulo N, respectively. Using $L(s,f) = L(s,f_{\mathrm{E}}) + L(s,f_{\mathrm{O}})$ in (C), the functional equations for $L(s,f_{\mathrm{E}})$ and $L(s,f_{\mathrm{O}})$ and the formula

$$\frac{\Gamma(s/2)}{\Gamma((1-s)/2)} = \frac{2^{1-s}}{\sqrt{\pi}}\Gamma(s)\cos\frac{s\pi}{2},$$

we get

$$L(s,T_N f_{\mathrm{E}})\cos\frac{s\pi}{2}-iL(s,T_N f_{\mathrm{O}})\sin\frac{s\pi}{2}=G(s)L(s,g)\cos\frac{(s-d)\pi}{2},$$

where

$$G(s)=\begin{cases}\displaystyle\prod_{j=0}^{d-1}\frac{s+j}{s-d+1+2j} & \text{if } d>1,\\ 1 & \text{if } d=0 \text{ or } 1.\end{cases}$$

Putting $s=4r+1+d$ for any positive integer r large enough, we get $L(4r+1+d,h)=0$, where $h=-\sin(d\pi/2)T_N f_{\mathrm{E}}-i\cos(d\pi/2)T_N f_{\mathrm{O}}$, implying $h=0$. Therefore, $T_N f_{\mathrm{E}}=0$ or $T_N f_{\mathrm{O}}=0$ according as $d\equiv 1$ or $0 \bmod 2$ and

$$L(s,g-i^{-d}T_N f)=(1-G(s))L(s,g).$$

The rational function $1-G(s)$ is thus the quotient of two Dirichlet series. Such a quotient or its reciprocal tends to a finite limit with an exponential speed, $O(e^{-as})$ as $s\to+\infty$, a speed a non-constant rational function cannot produce. Our $G(s)$ is only constant, $G(s)\equiv 1$ if $d=0$ or 1, implying also $g-i^{-d}T_N f=0$. The proof of Lemma 2.1 is complete.

3. Characterization of Dirichlet L-functions. Apostol ([1], [2]) characterizes Dirichlet L-functions corresponding to primitive characters by functional equation and Euler product. We replace the latter by an algebraic condition.

PROPOSITION 3.1. *Let $f\neq 0$ satisfy* (A), (B) *and* (C), *the latter with $g=W\bar{f}$, where W is a constant. By Lemma* 2.1, *N is an integer and assume that $f(n)=0$ if $(n,N)>1$ and that the field Q_f generated by the values $f(n)$ is algebraic over the rationals and is linearly disjoint from the Nth cyclotomic field C_N. Then f is a constant multiple of a primitive character* mod N.

Proof. By Lemma 2.1 we also know that f is purely recurring with period N and $T_N f=i^d W\bar{f}$.

Our algebraic assumption means that for any m relatively prime to N there is an automorphism τ_m of the composite field $Q_f C_N$ such that τ_m leaves Q_f invariant and $\tau_m(e^{2\pi i/N})=e^{2\pi i m/N}$.

We get

$$\begin{aligned}\tau_m(\sqrt{N}(T_N f)(k))&=\tau_m\Big(\sum_{n=1}^{N} f(n)e^{2\pi i n k/N}\Big)\\ &=\sum_{n=1}^{N} f(n)e^{2\pi i m n k/N}=\sqrt{N}(T_N f)(mk)\end{aligned}$$

and by the identity $T_N f = i^d W \bar{f}$,

$$\sqrt{N} i^d W \overline{f(mk)} = \tau_m(\sqrt{N} i^d W \overline{f(k)}) = \tau_m(\sqrt{N} i^d W) \overline{f(k)}.$$

Putting $k = 1$ here we get

$$\sqrt{N} i^d W \overline{f(m)} = \tau_m(\sqrt{N} i^d W) \overline{f(1)}.$$

This shows that $f(1) \neq 0$, otherwise $f = 0$, a contradiction. We may assume $f(1) = 1$ and dividing the last two equations we have

$$f(mk) = f(m)f(k).$$

If $(m, N) > 1$ then this holds trivially, both sides vanishing. Hence f is a character $\bmod N$ satisfying $(T_N f)(1) = i^d W \overline{f(1)} = i^d W$, i.e. $(T_N f)(n) = (T_N f)(1) \cdot \overline{f(n)}$. Such a character is known to be primitive (see e.g. [1], Lemma 1 or [5]) and the proof is complete.

We remark that Dirichlet characters do not always satisfy the algebraic condition, but Proposition 3.1 enables us to characterize e.g. the Legendre symbol by assuming f to be rational-valued.

4. Characterization of finite series. If in

$$F(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

$c_n = 0$ for n large enough and $\chi(n)$ is any Dirichlet character, then

$$L(s, f) = F(s) L(s, \chi)$$

has the purely recurring coefficients

$$f(n) = c_n * \chi(n) = \sum_{d \mid n} c_d \chi(n/d).$$

Conversely, we have

Theorem 4.1. *If for each Dirichlet character χ there is an N such that $f(n + N) = f(n)$ for n large enough, then $F(s)$ is a finite series.*

Proof. Denoting by μ the Möbius function we see that

$$F(s) = \frac{L(s, f)}{L(s, \chi)} = L(s, f) L(s, \mu\chi)$$

is a Dirichlet series absolutely convergent for $\mathrm{Re}(s) > 1$, representing a meromorphic function of order ≤ 1 in the whole plane.

We first claim that for any given complex number s ($\neq 1$) there is a Dirichlet character χ such that $L(s, \chi) \neq 0$. Since $L(s, f)$ can only have a first order pole at $s = 1$ as its only singularity, it will follow that $F(s)$ is regular for $s \neq 1$. Using the zeta function, $\zeta(s) = L(s, \chi)$ with $\chi = 1$, having the same singularity at $s = 1$, we shall even find that $F(s)$ is an entire function.

To prove the claim we first note that $\zeta(s, x)$ $(0 < x < 1)$, also a regular function in s in the whole plane with the exception of $s = 1$, satisfies

$$\frac{\partial^m \zeta(s,x)}{\partial x^m} = (-1)^m s(s+1)\dots(s+m-1)\zeta(s+m,x)$$
$$= (-1)^m s(s+1)\dots(s+m-1)\sum_{n=0}^{\infty}\frac{1}{(x+n)^{s+m}}$$

for $\mathrm{Re}(s+m) > 1$, implying

$$\left|\frac{\partial^m \zeta(s,x)}{\partial x^m}\right| \sim \frac{|s(s+1)\dots(s+m-1)|}{(x+1)^{\mathrm{Re}(s+m)}} \to \infty$$

as $m \to \infty$, provided that $s \neq 1, 0, -1, -2, \dots$ Hence $\zeta(s, x)$ cannot vanish identically in x for such an s and there exists a rational number $x = p/q$, $0 < p < q$, $(p, q) = 1$, such that $\zeta(s, p/q) \neq 0$. Now,

$$\frac{1}{q^s}\zeta\left(s, \frac{p}{q}\right) = \sum_{n=0}^{\infty}\frac{1}{(nq+p)^s} = \sum_{\substack{k=1 \\ k\equiv p \,(\mathrm{mod}\, q)}}^{\infty}\frac{1}{k^s}$$

can be represented as a linear combination of Dirichlet L-functions $\mathrm{mod}\, q$, showing that at least one of them does not vanish.

As to the remaining cases $s = 0, -1, -2, \dots$, we have $\zeta(s) \neq 0$ $(s = 0, -1, -3, \dots)$ and $L(s, \chi) \neq 0$ $(s = -2, -4, \dots)$ for any odd character χ.

For the rest of the proof we fix our Dirichlet L-function e.g. as $\zeta(s)$ and use the single relation

$$F(s)\zeta(s) = L(s, f).$$

f, being ultimately recurring, can be written as $f_\infty + f_{\mathrm{E}} + f_{\mathrm{O}}$; here $f_\infty(n)$ vanishes for n large enough, f_{E} and f_{O} are purely recurring with period N, even and odd, respectively.

From the respective functional equations we have

$$\zeta(-k) = L(-k, f_{\mathrm{E}}) = 0$$

for even, positive integers k, implying

$$0 = L(-k, f) = L(-k, f_\infty) + L(-k, f_{\mathrm{O}}).$$

From the functional equation of f_{O} we see that

$$|L(-k, f_{\mathrm{O}})| > e^{\frac{1}{2}k\log k}$$

for even k large enough, unless $T_N f_{\mathrm{O}} = 0$, $f_{\mathrm{O}} = 0$. The finite series $L(s, f_\infty)$ also tends to infinity but at a smaller rate, only exponentially, as $s \to -\infty$, unless it is a constant.

We conclude first that $f_{\mathrm{O}} = 0$ and then $f_\infty = 0$. Hence $f = f_{\mathrm{E}}$ and

$$F(s) = \frac{L(s, f_{\mathrm{E}})}{\zeta(s)}.$$

By the respective functional equations

$$F(s) = N^{1/2-s}\frac{L(1-s, T_N f_{\mathrm{E}})}{\zeta(1-s)},$$

implying for $\mathrm{Re}(s) \le -1$

$$|F(s)| \le cN^{1/2-\mathrm{Re}(s)}.$$

An entire function of finite order, representable by a Dirichlet series for $\mathrm{Re}(s) > 1$ and satisfying an estimate like this is a finite series. A proof of this standard fact runs e.g. as follows.

By the Phragmén–Lindelöf principle $F(s)$ is bounded in any fixed vertical strip. The coefficient formula,

$$c_n = \lim_{T\to\infty} \frac{1}{2T} \int_{\sigma-iT}^{\sigma+iT} F(s)n^s\,ds,$$

valid first for $\sigma > 1$, but by the above boundedness for any σ, implies

$$|c_n| \le cN^{1/2-\sigma}n^\sigma \qquad (\sigma \le -1),$$

and letting $\sigma \to -\infty$ gives $c_n = 0$ $(n > N)$. (This proof even allows for a finite number of singularities, compare with Toyoizumi [8].)

5. An equivalence relation. In the set of all convergent Dirichlet series, we define the equivalence $L(s,f) \sim L(s,g)$ if there exist two nonzero finite series $L(s,h_1)$ and $L(s,h_2)$ such that $L(s,h_0) = L(s,f)L(s,h_1) - L(s,g)L(s,h_2)$ is a finite series. If D_i is the least common multiple of integers d such that $h_i(d) \ne 0$, then this means

$$\sum_{d\mid(n,D_1)} f(n/d)h_1(d) - \sum_{d\mid(n,D_2)} g(n/d)h_2(d) = 0$$

for n large enough. The conductor of $L(s,f)$ can be defined as the minimum of the primitive period of g for which $L(s,g) \sim L(s,f)$ and g is purely recurring.

THEOREM 5.1. *Our conductor of a Dirichlet L-function coincides with the ordinary conductor of the associated character.*

Proof. Assume that $L(s,\chi) \sim L(s,f)$, that is, for n large enough

$$\sum_{d\mid(n,D_1)} \chi(n/d)h_1(d) = \sum_{d\mid(n,D_2)} f(n/d)h_2(d).$$

Let M denote the primitive period of f. By putting rn as n in the above identity, where $r \equiv 1 \pmod{M}$ and $(r, D_1D_2) = 1$, the right hand side is invariant and the left hand side is multiplied by $\chi(r)$. There exist infinitely many n such that the left hand side is not zero, otherwise $L(s,\chi)L(s,h_1)$ would be a finite series. Therefore $\chi(r) = 1$.

If χ belongs to the modulus q and $a_1 \equiv a_2 \pmod{(q, M)}$, $(a_1, q) = (a_2, q) = 1$, then we can find an r with the above properties such that in addition $ra_1 \equiv a_2 \pmod{q}$. This implies $\chi(a_1) = \chi(a_2)$, i.e. χ can be defined $\operatorname{mod}(q, M)$ and the conductor of χ is $\le (q, M) \le M$.

The rest is obvious.

Two characters are said to be *equivalent* if their corresponding primitive characters are the same.

COROLLARY 5.2. *Dirichlet L-functions are equivalent if and only if their associated characters are equivalent.*

Proof. Assume in the identity in Theorem 5.1 that f is also a character. Putting rn as n with $(r, D_1D_2) = 1$, the left and right hand sides are multiplied by $\chi(r)$ and $f(r)$, respectively. Since the two sides are not identically zero, we have $\chi(r) = f(r)$ for $(r, D_1D_2) = 1$, so that χ and f are equivalent.

PROPOSITION 5.3. *Any positive integer N except for 2 is the conductor of a Dirichlet series.*

Proof. According to Corollary 5.2 there exists a Dirichlet series with conductor N if $N \equiv 0, 1$ or $3 \bmod 4$. We show that the conductor of the Dirichlet series

$$L(s,f) = \sum_{\substack{k=1 \\ k\equiv 1 \pmod{N}}}^{\infty} \frac{1}{k^s}$$

is N if $N \equiv 2 \pmod{4}$. Assume that $L(s,f) \sim L(s,g)$, where g is purely recurring with primitive period $M < N$. We have

$$\sum_{d\mid(n,D_1)} f(n/d)h_1(d) = \sum_{d\mid(n,D_2)} g(n/d)h_2(d)$$

for n large enough, but both sides being purely recurring, in fact for all n. This means

$$L(s,g)/L(s,f) = L(s,h_1)/L(s,h_2).$$

The left hand side is an ordinary Dirichlet series $\sum_{n=1}^{\infty} a(n)/n^s$ because $f(1) \ne 0$ and we see from the right hand side that $a(n) = 0$ if $(n, D_1D_2) = 1$. Let d_1 be the least integer such that $a(d_1) \ne 0$.

In any case except $2M = N$ we can find an integer (even a prime) q satisfying $q \equiv 1 \pmod{M}$, $q \not\equiv 1 \pmod{N}$ and $(q, D_1D_2) = 1$. From the identity

$$g(n)=\sum_{d|n}a(d)f(n/d)$$

we get

$$g(d_1)=a(d_1)f(1)=a(d_1)\neq 0,$$
$$g(d_1q)=a(d_1)f(q)=a(d_1)\cdot 0=0,$$

contradicting the fact that by $d_1\equiv d_1q \pmod{M}$, $g(d_1)=g(d_1q)$.

In the exceptional case $2M=N$ we have M odd since $N\equiv 2 \pmod 4$ and we can find an integer q satisfying $2q\equiv 1 \pmod{M}$, $(q,D_1D_2)=1$. We get $g(2d_1q)=g(d_1)\neq 0$ as established above, contradicting the fact that

$$g(2d_1q)=a(d_1)f(2q)+a(2d_1)f(q)=0,$$

since $2q\not\equiv 1 \pmod{N}$, N being even and $q\equiv (M+1)/2\not\equiv 1 \pmod{N}$, provided that $M>1$.

The identity

$$a+\frac{b}{2^s}+\frac{a}{3^s}+\frac{b}{4^s}+\ldots=\left(a-\frac{a-b}{2^s}\right)\zeta(s)$$

shows that no series has conductor $N=2$.

Proposition 5.4. *Let f and g be purely recurring with period N, such that $f(n)=g(n)=0$ for $(n,N)>1$. If $L(s,f)\sim Ł(s,g)$ and $g\neq 0$, then $f=\vartheta g$ with a constant ϑ.*

Proof. Let χ run over the characters mod N. Under our assumption we have the representations

$$f=\sum_{\chi}c_\chi\chi,\qquad g=\sum_{\chi}d_\chi\chi$$

with constants c_χ, d_χ.

The relation

$$L(s,f)L(s,h_1)-L(s,g)L(s,h_2)=L(s,h_0)$$

can be rewritten as

$$\sum_{\chi}(c_\chi L(s,h_1)-d_\chi L(s,h_2))L(s,\chi)=:\sum_{\chi}L(s,h_\chi)L(s,\chi)=L(s,h_0)$$

($L(s,h_*)$ all denoting finite series) or, in terms of the coefficients,

$$\sum_{\chi}\sum_{d|n}h_\chi(d)\chi(n/d)=0$$

for n large enough.

Assuming that not all $h_\chi=0$, let q be the least value such that there is a χ with $h_\chi(q)\neq 0$. Applying the identity for $n=pq$ with a prime p large enough, we get

$$\sum_{\chi} h_{\chi}(q)\chi(p) = \sum_{\chi}\sum_{d|pq} h_{\chi}(d)\chi(pq/d) = 0.$$

Since large primes p represent all reduced residue classes mod N, it follows that $\sum_{\chi} h_{\chi}(q)\chi = 0$ and $h_{\chi}(q) = 0$ for all χ, a contradiction. We infer that $L(s, h_{\chi}) = L(s, h_0) = 0$ for all χ.

We get $c_{\chi}L(s, f) - d_{\chi}L(s, g) = 0$ for any χ and, since not all $d_{\chi} = 0$, the statement follows.

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