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On a direct sum decomposition of the Dem'yanenko matrix

by

Hirofumi Tsumura (Tokyo)

Introduction. In [S-Sch], Sands and Schwarz defined the generalized Dem'yanenko matrix associated with an arbitrary imaginary abelian field of odd prime power conductor. They investigated an interesting relation between this matrix and the relative class number of the field. In [D], Dohmae defined such a matrix for an arbitrary imaginary abelian field of odd conductor. Recently we succeeded in generalizing the above result as follows (see [T]). Let K be an imaginary abelian number field of arbitrary conductor. Let n be the conductor of K with $n \not\equiv 2 \pmod 4$ and let $[K:\mathbb{Q}] = 2d$. For $l \in \mathbb{Z}$ with $(l,n) = 1$ and $l > 1$, we defined the generalized Dem'yanenko matrix $\Delta(K,l) \in M(d,\mathbb{Q})$ (see [T], Definition 2.5). We proved a relation between $\det \Delta(K,l)$ and the relative class number h_K^-, which could be regarded as a generalization of the one in [S-Sch]. In fact, we verified that $\Delta(K,2)$ plays the same role as the matrices defined in [S-Sch] and [D]. Moreover, we verified that $\det \Delta(K,n+1)$ coincides with the generalized Maillet determinant defined by Girstmair in [G] (see [T], §§2 and 3).

In the present paper, we consider a direct sum decomposition of $\Delta(K,l)$ as follows. Let X_K^+ be the group of *even* characters of $\mathrm{Gal}(K/\mathbb{Q})$ and Y be a subgroup of X_K^+ of index $(X_K^+ : Y) = c$. We construct the matrices $\{\Delta_s(K,l,Y) \in M(d/c, \mathbb{Q}(\zeta_{2c})) \mid s = 1,\ldots,c\}$ such that the following theorem holds.

Theorem. *$\Delta(K,l)$ is similar to the matrix*

$$\begin{pmatrix} \Delta_1(K,l,Y) & & 0 \\ & \ddots & \\ 0 & & \Delta_c(K,l,Y) \end{pmatrix}.$$

In the case $Y = X_K^+$, we can see that $\Delta_1(K,l,X_K^+) = \Delta(K,l)$. In the case $Y = \{1\}$, $\Delta_s(K,l,\{1\})$ is essentially equal to the generalized Bernoulli number $B_{1,\chi}$ (see §3). In the previous paper ([T]), we treated these two cases. By the class number formula (cf. Proposition 4.9 of [W]), we got a relation between $\det \Delta(K,l)$ and h_K^- (see Theorem of [T]). As a generalization, the above theorem gives a kind of direct sum decomposition of the Dem'yanenko matrix.

In Section 1, we recall the definition of $\Delta(K,l)$. In Section 2, we define the matrices $\{\Delta_s(K,l,Y) \mid 1 \le s \le c\}$ and give the proof of above theorem. In Section 3, we give some remarks.

1. The generalized Dem'yanenko matrix. We make use of the same notations as in [T]. The generalized Dem'yanenko matrix $\Delta(K,l)$ was defined as follows. Let K be an imaginary abelian number field of degree $2d$ and let n be its conductor. For $x \in \mathbb{Z}$, let $R(x) = R_n(x)$ be the residue of x modulo n with $0 \le R(x) < n$, and x' be the integer with $xx' \equiv 1 \pmod n$ and $1 \le x' < n$. Let $G = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \{\sigma_a \mid \zeta_n \to \zeta_n^a,\ (a,n) = 1\}$. There is a canonical group isomorphism

$$(\mathbb{Z}/n\mathbb{Z})^\times \to G : a \pmod n \mapsto \sigma_a.$$

Let $H = \mathrm{Gal}(\mathbb{Q}(\zeta_n)/K)$ and $G_K = \mathrm{Gal}(K/\mathbb{Q}) \simeq G/H$. Note that we let σ_a denote both the element of G and its restriction to K, namely the element of G_K. Let T_K be the subset of $\{a \in \mathbb{Z} \mid 1 \le a < n,\ (a,n) = 1\}$ such that $H = \{\sigma_a \mid a \in T_K\}$. Let $J = \sigma_{-1}$ be complex conjugation. Since $J \notin H$, we can uniquely take a set $S_K \subset \{c \in \mathbb{Z} \mid 1 \le c < n/2\}$ such that $\{\sigma_c^{-1} \mid c \in S_K\} \cup \{\sigma_{-c}^{-1} \mid c \in S_K\}$ forms a set of representatives for $G/H \simeq G_K$.

Let X_K be the character group of G_K. Let $X_K^- = \{\chi \in X_K \mid \chi(-1) = -1\}$ and $X_K^+ = \{\chi \in X_K \mid \chi(-1) = 1\}$. For $\chi \in X_K$, let

$$\varepsilon_\chi = \frac{1}{[K:\mathbb{Q}]} \sum_{a \in S_K} \chi(a)(\sigma_a^{-1} + \chi(-1)\sigma_{-a}^{-1}). \tag{1.1}$$

Then $\{\varepsilon_\chi \mid \chi \in X_K\}$ are called the *orthogonal idempotents* of the group ring $\overline{\mathbb{Q}}[G_K]$, where $\overline{\mathbb{Q}}$ is an algebraic closure of $\mathbb{Q}$. Let $V = \overline{\mathbb{Q}}[G_K]$, and $V^- = ((1-J)/2)V = \{v \in V \mid Jv = -v\}$. Note that $\varepsilon_\chi \sigma_a^{-1} = \overline{\chi}(a)\varepsilon_\chi$. We can easily verify that $\{\varepsilon_\chi \mid \chi \in X_K\}$ forms a $\overline{\mathbb{Q}}$-basis for V, and $\{\varepsilon_\chi \mid \chi \in X_K^-\}$ forms a $\overline{\mathbb{Q}}$-basis for V^- (cf. [W], Chap. 6).

For $x \in \mathbb{Z}$ with $(x,n) = 1$, let $\xi(x) = (\sigma_x^{-1} - \sigma_{-x}^{-1})/2$. A short calculation shows that

$$\xi(xy) = \xi(x)\xi(y), \qquad \xi(-x) = -\xi(x) \tag{1.2}$$

and that $\{\xi(c) \mid c \in S_K\}$ forms a $\overline{\mathbb{Q}}$-basis for V^-.

We fix $l \in \mathbb{Z}$ with $l > 1$ and $(l, n) = 1$. For $b \in \mathbb{Z}$, let

$$A(b, l) = A_n(b, l) = \sum_{\substack{\zeta^l = 1 \\ \zeta \neq 1}} \frac{\zeta^{n-b}}{1 - \zeta^n} \in \mathbb{Q}.$$

Note that $A(b, 2) = (-1)^{b-1}/2$ in the case $l = 2$. We can easily verify that

$$A(R(n-a), l) = -A(R(a), l). \tag{1.3}$$

We consider the element of $\mathbb{Q}[G_K]$ defined by

$$\varrho = \varrho(K, l) = \sum_{\substack{a=1 \\ (a,n)=1}}^{n} A(R(a), l)\sigma_a^{-1}.$$

By (1.3), we have $\varrho \in V^-$. Since

$$G = \{\sigma_a^{-1}\sigma_c^{-1} \mid a \in T_K,\ c \in S_K\} \cup \{\sigma_a^{-1}\sigma_{-c}^{-1} \mid a \in T_K,\ c \in S_K\},$$

we have

$$\varrho = \sum_{b \in S_K} 2\Big(\sum_{a \in T_K} A(R(ab), l)\Big)\xi(b). \tag{1.4}$$

For $\alpha \in V^-$, let $L_\alpha : V^- \to V^-$ be defined by $L_\alpha(v) = \alpha v$. The following fact was proved in Proposition 2.4 of [T]. For each $c \in S_K$,

$$L_\varrho(\xi(c)) = \sum_{b \in S_K} 2\Big(\sum_{a \in T_K} A(R(abc'), l)\Big)\xi(b). \tag{1.5}$$

Definition 1.1 (The generalized Dem'yanenko matrix).

$$\Delta(K, l) = \Big(2\sum_{a \in T_K} A(R(abc'), l)\Big)_{b,c \in S_K} \in M(d, \mathbb{Q}).$$

By (1.5), we get the following.

Proposition 1.2. *The matrix of L_ϱ with respect to the basis $\{\xi(a) \mid a \in S_K\}$ is $\Delta(K, l)$.*

2. Definition of $\Delta_s(K, l, Y)$. Let Y be a subgroup of X_K^+ of index $(X_K^+ : Y) = c$. Then we can take representatives $\{\psi_1, \ldots, \psi_c\}$ of those classes in X_K/Y that consist of odd characters. Let

$$\lambda_s = \sum_{\chi \in Y} \varepsilon_{\psi_s \chi}, \tag{2.1}$$

and let $V_s = \lambda_s V$ for $s = 1, \ldots, c$. Since ψ_s is odd for any s, we have $V_s \subset V^-$. Let

$$\ker Y = \{\sigma_a \in G_K \mid \chi(a) = 1 \text{ for any } \chi \in Y\}.$$

We can verify that $|\ker Y| = (X : Y) = 2c$ (cf. [W], Chap. 3). So $(G_K : \ker Y) = d/c$. Since $J \in \ker Y$, we can take $\Gamma \subset S_K$ such that $\{\sigma_y^{-1} \mid y \in \Gamma\}$ forms a set of representatives of $G_K/\ker Y$.

LEMMA 2.1. *$\{\lambda_s\sigma_y^{-1} \mid y \in \Gamma\}$ forms a $\overline{\mathbb{Q}}$-basis for V_s for $s = 1, \ldots, c$.*

Proof. For $\sigma_a \in \ker Y$,

$$\lambda_s\sigma_a = \sum_{\chi\in Y} \varepsilon_{\psi_s\chi}\sigma_a = \sum_{\chi\in Y} \psi_s\chi(a)\varepsilon_{\psi_s\chi} = \psi_s(a)\lambda_s. \tag{2.2}$$

Thus we have the assertion.

Now we let $\Delta_s(K, l, Y)$ be the matrix of $L_\varrho|_{V_s}$ with respect to the basis $\{\lambda_s\sigma_y^{-1} \mid y \in \Gamma\}$ for V_s, for $s = 1, \ldots, c$. We determine the entries of $\Delta_s(K, l, Y)$. Since $J \in \ker Y$, we can take a set $\Omega \subset \{x \in \mathbb{Z} \mid 1 \le x < n/2\}$ such that $\ker Y = \{\sigma_x^{-1} \mid x \in \Omega\} \cup \{\sigma_{-x}^{-1} \mid x \in \Omega\}$. Hence we have

$$\varrho = 2\sum_{y\in\Gamma}\sum_{x\in\Omega}\sum_{a\in T_K} A(R(axy), l)\xi(\sigma_{xy}).$$

PROPOSITION 2.2. *Let $z \in \Gamma$. Then*

$$L_\varrho(\lambda_s\sigma_z^{-1}) = 2\sum_{y\in\Gamma}\sum_{x\in\Omega}\sum_{a\in T_K} A(R(axyz'), l)\psi_s(x)\lambda_s\sigma_y^{-1}$$

for $s = 1, \ldots, c$.

In order to prove Proposition 2.2, we prepare some notations. For $x \in \mathbb{Z}$, we define the functions $g(x)$ and $f(x)$ as follows. If $0 \le R(x) < n/2$ then we let $g(x) = R(x)$ and $f(x) = 1$, and if $n/2 < R(x) < n$ then we let $g(x) = n - R(x)$ and $f(x) = -1$. We can verify that $0 \le g(x) < n/2$ and $g(x) \equiv f(x)x \pmod{n}$. We can prove the following lemmas in the same manner as Lemmas 2.2 and 2.3 of [T].

LEMMA 2.3. *Let $z \in \Omega$. Then $\{g(yz) \mid y \in \Omega\} = \Omega$.*

LEMMA 2.4. *Let $y, z, w \in \Omega$ with $g(yz) = w$. Then $y = g(wz')$ and $\xi(\sigma_{yz}) = f(wz')\xi(\sigma_w)$.*

Proof of Proposition 2.2. Since $\xi(\sigma_a)\sigma_b^{-1} = \xi(\sigma_{ab})$ for $a, b \in \mathbb{Z}$, we have

$$\begin{aligned} L_\varrho(\lambda_s\sigma_z^{-1}) &= 2\sum_{y\in\Gamma}\sum_{x\in\Omega}\sum_{a\in T_K} A(R(axy), l)\xi(\sigma_{xy})\lambda_s\sigma_z^{-1} \\ &= 2\sum_{y\in\Gamma}\sum_{x\in\Omega}\sum_{a\in T_K} A(R(axy), l)(\lambda_s\sigma_x^{-1})\xi(\sigma_{yz}). \end{aligned} \tag{2.3}$$

Let $g(yz) = w$. It follows from Lemmas 2.2 and 2.3 that (2.3) is equal to

$$2\sum_{w\in\Gamma}\sum_{x\in\Omega}\sum_{a\in T_K} A(R(axg(wz')), l)(\lambda_s\sigma_x^{-1})f(wz')\xi(\sigma_w).$$

By (1.3), we have

$$A(R(ag(wz')), l)f(wz') = A(R(axwz'), l).$$

By (2.2), we have $\lambda_s \sigma_x^{-1} = \overline{\psi}_s(x)\lambda_s$ for $x \in \Omega$. Finally, we can verify that $\lambda_s \xi(\sigma_w) = \lambda_s \sigma_w^{-1}$ for $w \in \Gamma$. Thus we have the assertion.

DEFINITION 2.5. For $s \in \mathbb{Z}$ with $1 \le s \le c$,

$$\Delta_s(K, l, Y) = \Big(2 \sum_{x\in\Omega} \sum_{a \in T_K} A(R(axyz'), l)\psi_s(x)\Big)_{y,z\in\Gamma}.$$

Proof of Theorem. It follows from Propositions 1.2 and 2.2 that $\Delta(K, l)$ is similar to $\operatorname{diag}(\Delta_1(K, l, Y), \ldots, \Delta_c(K, l, Y))$. By the definition of $\Delta_s(K, l, Y)$, we can see that $\Delta_s(K, l, Y) \in M(d/c, \mathbb{Q}(\zeta_{2c}))$, since $\psi_s(x) \in \langle \zeta_{2c} \rangle$ for $x \in \Omega$. This completes the proof of Theorem.

3. Some remarks. We calculate $\det \Delta_s(K, l, Y)$ as follows.

PROPOSITION 3.1. *For $s \in \mathbb{Z}$ with $1 \le s \le c$,*

$$\det \Delta_s(K, l, Y) = \prod_{\chi\in Y} (l\psi_s\chi(l) - 1)B_{1,\psi_s\chi} \prod_{\substack{p \text{ prime} \\ p \mid n}} (1 - \psi_s\chi(p)),$$

where $B_{1,\chi}$ is the generalized Bernoulli number (cf. [W], Chap. 4).

Proof. By Proposition 1.3 of [T], we have

$$L_\varrho(\varepsilon_\chi) = (l\overline{\chi}(l) - 1)B_{1,\overline{\chi}} \prod_{p\mid n} (1 - \overline{\chi}(p)),$$

where $\overline{\chi} = \chi^{-1}$ for $\chi \in X_K^-$. Since $\lambda_s = \sum_{\chi\in Y} \varepsilon_{\psi_s\chi}$ for $s = 1, \ldots, c$, this completes the proof.

Remark. In the case $Y = \{1\}$, it follows from Proposition 3.1 that

$$\Delta_s(K, l, \{1\}) = (l\psi_s(l) - 1)B_{1,\psi_s} \prod_{p\mid n} (1 - \psi_s(p)),$$

for $s = 1, \ldots, d$.

In [T] (see §3, (3.3)), we proved that

$$\sum_{a\in T_K} A(R(ac), n+1) = \sum_{a\in T_K} nB_1\bigg(\frac{R(ac)}{n}\bigg),$$

for $c \in S_K$, where $B_1(x) = x - 1/2$. Hence, by Definition 2.5, we have

$$\Delta_s(K, n+1, Y) = \bigg(2n \sum_{x\in\Omega} \sum_{a\in T_K} B_1\bigg(\frac{R(axyz')}{n}\bigg)\psi_s(x)\bigg)_{y,z\in\Gamma}.$$

We recall that det $\Delta(K, n+1)$ coincides with the generalized Maillet determinant D^* defined by Girstmair in [G] (see [T], §3). Hence we can regard $\{\Delta_s(K, n+1, Y) \mid 1 \leq s \leq c\}$ as direct summands in a direct sum decomposition of Girstmair's matrix.

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References

[D] K. Dohmae, *Demjanenko matrix for imaginary abelian fields of odd conductors*, Proc. Japan Acad. Ser. A 70 (1994), 292–294.

[G] K. Girstmair, *The relative class numbers of imaginary cyclotomic fields of degrees* 4, 6, 8 *and* 10, Math. Comp. 61 (1993), 881–887.

[S-Sch] J. W. Sands and W. Schwarz, *A Demjanenko matrix for abelian fields of prime power conductor*, J. Number Theory 52 (1995), 85–97.

[T] H. Tsumura, *On Demjanenko's matrix and Maillet's determinant for imaginary abelian number fields*, ibid., to appear.

[W] L. C. Washington, *Introduction to Cyclotomic Fields*, Springer, New York, 1982.

Department of Mathematics
Aoyama Gakuin Kotobu
4-4-25 Shibuya, Shibuya-ku
Tokyo 150, Japan
E-mail: tsumura@cc.aoyama.ac.jp

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