ANNALES
POLONICI MATHEMATICI
LXIII.1 (1996)

On the estimate of the fourth-order homogeneous coefficient functional for univalent functions

by Larisa Gromova and Alexander Vasil'ev (Saratov)

Abstract. The functional $|c_4+pc_2c_3+qc_2^3|$ is considered in the class $\mathbb{S}$ of all univalent holomorphic functions $f(z)=z+\sum_{n=2}^{\infty}c_nz^n$ in the unit disk. For real values p and q in some regions of the (p,q)-plane the estimates of this functional are obtained by the area method for univalent functions. Some new regions are found where the Koebe function is extremal.

Introduction. Let $\mathbb{S}$ be the class of all holomorphic univalent functions

$$f(z)=z+\sum_{n=2}^{\infty}c_nz^n$$

in the unit disk. We consider the functional

$$D_4(f)=c_4+pc_2c_3+qc_2^3$$

for real values p and q which is fourth-order homogeneous in the sense of rotation:

$$e^{-3i\alpha}\,D_4(e^{-i\alpha}f(e^{i\alpha}z))=D_4(f).$$

Many papers are devoted to the estimation of $|D_4|$ for different values of p and q (see [1], [2], [4], [5]). Special interest in this functional is connected with estimating the seventh coefficient $c_7^{(2)}$ in the class $S^{(2)}$ of odd univalent functions

$$f(z)=z+\sum_{n=2}^{\infty}c_{2n-1}^{(2)}z^{2n-1}.$$

1991 *Mathematics Subject Classification*: Primary 30C55.
Key words and phrases: univalent function, area method.
This research is supported by Russian Foundation of Basic Research, grant N 95-01-00345A.

Namely, $\max_{f\in S^{(2)}} |c_7^{(2)}| = \max_{f\in\mathbb{S}} 2^{-1}|D_4(f)|$ for $p = -1/2, q = 1/8$. P. Lehto [4] showed that $|D_4| \leq 4+6p+8q$ when $q \geq p^2/4+p/4+7/12$, with the Koebe function being extremal. Moreover, he found that if $p=-2$ and $q = 13/12$, then the Koebe function is not unique.

Here we find some other regions in the (p,q)-plane where the Koebe function is extremal and find new regions where the estimates are different from $4+6p+8q$.

We use the area method in the form given by N. Lebedev [3] for the estimate of the fourth coefficient for univalent functions.

Let $F^{(2)}(\zeta)$ belong to the class $\Sigma^{(2)}$ of all odd univalent functions $F(\zeta) = \zeta + a_1/\zeta + a_3/\zeta^3 + \dots$ in the exterior of the unit disk $|\zeta| > 1$. Then

$$\operatorname{Ln} \frac{F^{(2)}(\zeta) - F^{(2)}(t)}{\zeta - t} = \sum_{n,m=1}^{\infty} \omega_{nm}\zeta^{-n}t^{-m}, \qquad |t| > 1,$$

where ω_{pq} are the Grunsky coefficients. It is known [3] that

$$\begin{cases} c_4 = 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}(\omega_{11})^3, \\ c_3 = 2\omega_{13} + 3(\omega_{11})^2, \\ c_2 = 2\omega_{11}. \end{cases} \tag{1}$$

By the Grunsky inequality for any $l \in \mathbb{C}$ we get

$$|\omega_{33} + 2\omega_{13}l + \omega_{11}l^2| \leq |l^2| + 1/3,$$

and from (1),

$$D_4 = c_4 + pc_2c_3 + qc_2^3 = 2\omega_{33} + 4(2+p)\omega_{11}\omega_{13} + 2(5/3+4q+3p)\omega_{11}^3$$

and

$$\begin{aligned} |D_4 - 4(2+p)\omega_{11}\omega_{13} - 2(5/3+4q+3p)\omega_{11}^3 \\ + 4\omega_{13}l + 2\omega_{11}l^2| \leq 2|l|^2 + 2/3. \end{aligned} \tag{2}$$

For convenience we assume $\omega_{13} = \omega_3$ and $\omega_{11} = \omega_1$. Since $D_4(f) = e^{-3i\alpha}D_4(e^{-i\alpha}f(e^{i\alpha}z))$, we assume $D_4 \geq 0$ without loss of generality. The modulus on the left-hand side of (2) can be replaced by the real part, so

$$\begin{aligned} \operatorname{Re} D_4 \leq 2/3 + 2|l|^2 + \operatorname{Re}\{4((2+p)\omega_1 - l)\omega_3 \\ + 2(5/3+4q+3p)\omega_1^3 - 2\omega_1 l^2\}. \end{aligned}$$

The area theorem for odd univalent functions [3] states that

$$\sum_{n=1}^{\infty}(2n-1)|\omega_{1,2n-1}|^2 \leq 1.$$

Therefore $|\omega_1|^2 + 3|\omega_3|^2 \leq 1$ or $|\omega_3| \leq (1/\sqrt{3})\sqrt{1-|\omega_1|^2}$. Thus

$$|D_4| \leq 2/3 + 2|l|^2 + |4((2+p)\omega_1 - l)|\frac{1}{\sqrt{3}}\sqrt{1-|\omega_1|^2}$$
$$+ 2\,\mathrm{Re}\{(5/3+4q+3p)\omega_1^3 - \omega_1 l^2\}.$$

We write $\omega_1 = xe^{i\varphi}, 0 \leq x \leq 1$, and put $l = (2+p)xe^{-i\varphi/2}\cos(3\varphi/2)$ and $y = |\sin(3\varphi/2)|$, $0 \leq y \leq 1$. Then $|(2+p)\omega_1 - l| = |2+p|xy$ and

$$|D_4| \leq 2/3 + 2b^2x^2(1-y^2) + 4|b|xy\frac{\sqrt{1-x^2}}{\sqrt{3}} + 2(a-b^2)x^3 + 2y^2(b^2-2a)x^3 = \varphi(x,y), \tag{3}$$

where $a = 5/3 + 4q + 3p$ and $b = 2 + p$.

1. The case $q \leq -3p/4 - 5/12$**.** Evidently, if $a = b = 0$, then $D_4 \leq 2/3$, so we omit this case. If $b = 0$ and $a < 0$, then the coefficient $z = ((b^2 - 2a)x - b^2)x^2$ of y^2 in (3) is positive for all $0 \leq x \leq 1$. Analogously, if $b \neq 0$ and $a = 0$, then $z \leq 0$ for all $0 \leq x \leq 1$. Let $x_0 = b^2/(b^2 - 2a)$; then $x_0 \in [0,1]$. If $a \neq 0$ and $b \neq 0$, then $0 < x_0 < 1$. Let $x_0 \leq x \leq 1$. Hence $z \geq 0$ and $\max_{0\leq y\leq 1}\varphi(x,y) = \varphi(x,1)$. If $0 < x < x_0$, then $z < 0$ and $\max_{0\leq y\leq 1}\varphi(x,y) = \varphi(x,y^*)$, where

$$y^* = \frac{|b|\sqrt{1-x^2}}{x\sqrt{3}[b^2(1-x)+2ax]}.$$

Elementary calculations show that the inequality $|p+2| < 2\sqrt{2}/\sqrt{3}$ implies that $y^* > 1$ is equivalent to $|p+2| < 2\sqrt{2}/\sqrt{3}$. Thus, if $|b| < 2\sqrt{2}/\sqrt{3}$ and $a < 0$, then

$$|D_4| \leq 2/3 - 2ax^3 + \frac{4|b|}{\sqrt{3}}x\sqrt{1-x^2} = 2\Phi_1(x) + 2/3,$$

and $\Phi_1(0) = 0$, $\Phi_1(1) > 0$. It is not difficult to show that $\Phi_1(x)$ has a unique maximum in (0,1) at the point x^*, where x^* is the unique root in (0,1) of the equation

$$3ax^2\sqrt{1-x^2} + \frac{4|b|}{\sqrt{3}}x^2 - \frac{2|b|}{\sqrt{3}} = 0.$$

Note that if $x{=}x_0$, then $\varphi(x,y)$ is a linear function of y and $\max_{0\leq y\leq 1}\varphi(x,y) = \varphi(x,1)$. If $x = 0$, then evidently $y^* > 1$.

Now, if $a = 0$ and $b \neq 0$, then $z \leq 0, x \in [0,1]$ and

$$y^{*2} = (1+x)/(3x^2b^2(1-x)).$$

Clearing up the inequality $y^* > 1$ we come to

$$\max_{0\leq y\leq 1}\varphi(x,y) = \varphi(x,1), \qquad \max_{0\leq x\leq 1}\Phi_1(x) = \Phi_1(1/\sqrt{2}). \tag{4}$$

If $a \neq 0$ and $b = 0$, then the maximum (4) holds again and $|D_4| \leq 8/3 + 8q + 6p$.

THEOREM 1. *If* $q \leq -3p/4 - 5/12$ *and* $|p+2| \leq 2\sqrt{2/3}$, *then*

$$|D_4| \leq 2/3 - 2(5/3 + 4q + 3p)x^{*3} + \frac{4|p+2|}{\sqrt{3}} x^* \sqrt{1 - x^{*2}}, \tag{5}$$

where x^* *is the unique root in* $(0,1)$ *of the equation*

$$\sqrt{3}(5 + 12q + 9p)x^2\sqrt{1-x^2} + |4 + 2p|(2x^2 - 1) = 0.$$

The inequality (5) *is sharp only for* $p = -2$ *and* $q = 13/12$.

COROLLARY. *If* $f(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathbb{S}$, *then*

1) $$|c_4 - 2c_2c_3 + qc_2^3| \leq \begin{cases} 8(q-1) & \text{for } q \geq 13/12, \\ (4/3)(7-6q) & \text{for } q < 13/12, \end{cases}$$

2) $$|c_4 + pc_2c_3 - (3p/4 + 5/12)c_2^3| \leq 2/3 + 2|p+2|/\sqrt{3} \quad \text{if } |p+2| \leq 2\sqrt{2/3}.$$

Now we consider the case $|p+2| > 2\sqrt{2/3}$. We want y^* to be bigger than 1 again. This condition implies the inequality

$$\Psi(x) = 3x^2b^4 + 6x^3b^2(2a - b^2) + 3x^4(2a - b^2)^2 - b^2(1 - x^2) < 0.$$

To prove this, the sign of $\Psi'(x)$ can be determined or

$$u(x) = \frac{\Psi'(x)}{2x} = 6x^2(b^2 - 2a)^2 - 9b^2x(b^2 - 2a) + b^2(1 + 3b^2) > 0.$$

From $\Psi(x_0) = (4b^4a - 4a^2b^2)(b^2 - 2a)^{-2} < 0$ it will follow that $\Psi(x) < 0$. The equation $u(x) = 0$ has two real roots

$$x_{1,2} = \frac{9b^2 \pm b\sqrt{3(3b^2 - 8)}}{12(b^2 - 2a)}.$$

We put $b > 0, x_1 < x_2$. So $x_2 < x_0$. We want $\Psi(x_1)$ to be negative. To simplify the form of the corresponding curve in the (p,q)-plane we find that the inequality

$$q < -\frac{9}{128}p^4 - \frac{9}{16}p^3 - \frac{209}{128}p^2 - \frac{89}{32}p - \frac{127}{96} = A(p)$$

implies $\Psi(x_1) < 0$. Hence if $q < A(p)$, then $y^* > 1$, and the considerations of Theorem 1 remain true, so $\max_{0 \leq y \leq 1} \varphi(x,y) = \varphi(x,1)$. Obviously, if $q < A(p)$, then $q \leq -3p/4 - 5/12$.

THEOREM 2. *If* $|p+2| > 2\sqrt{2/3}$ *and*

$$q < -\frac{9}{128}p^4 - \frac{9}{16}p^3 - \frac{209}{128}p^2 - \frac{89}{32}p - \frac{127}{96},$$

then the estimate (5) *holds.*

2. The case $q > -3p/4 - 5/12$**.** The coefficient $z = x^2(-2ax - b^2(1-x))$ of y^2 in $\varphi(x, y)$ is negative, and $\varphi(x, y) \leq \varphi(x, y^*)$, where

$$y^* = \frac{|b|\sqrt{1-x^2}}{\sqrt{3}x(b^2 - (b^2 - 2a)x)}$$

and

$$\begin{aligned}|D_4| &\leq 2/3 + 2b^2x^2 + 2(a - b^2)x^3 - \frac{2b^2(1-x^2)}{3((b^2 - 2a)x - b^2)} \\ &= 2/3 + 2\Phi_2(x).\end{aligned}$$

Now we look for the region of the (p, q)-plane where the Koebe function is extremal. In this case we should have the inequality $\Phi_2(x) \leq \Phi_2(1)$. We assume

$$a \leq b^2 \leq 2a, \tag{6}$$

$$b^2 < a(9 - \sqrt{17})/4, \tag{7}$$

$$b^2 < 6a^2, \tag{8}$$

$$a \geq 1/3. \tag{9}$$

We reduce the inequality $\Phi_2(x) \leq \Phi_2(1)$ to a common denominator taking into account (6) and get it in the equivalent form:

$$\Psi(x) = 3(b^2 + (2a - b^2)x)\{b^2(1+x) + (a - b^2)(1 + x + x^2)\} - b^2(1+x) \geq 0.$$

The inequality (6) implies that $\Psi''(x)$ is decreasing and since by (7), $\Psi''(1)/6 = 8a^2 - 9ab^2 + 2b^4 > 0$ it follows that $\Psi''(x) > 0$ for $x \in [0, 1]$. Ψ' increases and by (8), $\Psi'(0) = 6a^2 - b^2 > 0$. Hence, $\Psi(x)$ increases and by (9), $\Psi(0) = b^2(3a - 1) \geq 0$ and therefore $\Psi(x) \geq 0$ for all $x \in [0, 1]$.

Now we consider the region containing the point $(0, 0)$ of the (p, q)-plane. In the case $a \geq 0$, $a \leq b^2/2$ we make $\Phi_2(x)$ bigger, so

$$\Phi_2(x) \leq b^2x^2 + (a - b^2)x^3 + b^2(1 - x^2)/(6a) = g(x).$$

If

$$q > p^2/24 - 7p/12 - 1/4 + \frac{|p+2|}{24}((p+2)^2 + 4)^{1/2},$$

then $b^2 < 9a^2/(1 + 3a)$, $g'(x) > 0$ for $x \in [0, 1]$ and

$$|D_4| \leq 2/3 + 2\Phi_2(x) \leq 2/3 + 2g(x) \leq 2/3 + 2g(1) = 2/3 + 2\Phi_2(1).$$

Hence, here the Koebe function is also extremal.

THEOREM 3. *If*

$$\frac{1}{12(9 - \sqrt{17})}(12p^2 + 5(3\sqrt{17} - 11)p + 5\sqrt{17} + 3) \leq q \leq p^2/4 + p/4 + 7/12$$

and $q > -3p/4 - 1/3$, *or if*

$$p^2/24 - 7p/12 - 1/4 + \frac{|p+2|}{24}((p+2)^2+4)^{1/2} < q < p^2/8 - p/4 + 1/12,$$

and $1 + 4q + 3p > 0$, *then* $|D_4| \le 4 + 6p + 8q$ *with the Koebe function being extremal* (*the point* $(0, 0)$ *belongs to the last domain*).

References

[1] Z. J. Jakubowski, H. Siejka and O. Tammi, *On the maximum of* $a_4 - 3a_2a_3 + \mu a_2$ *and some related functionals for bounded real univalent functions*, Ann. Polon. Math. 46 (1985), 115–128.

[2] J. Ławrynowicz and O. Tammi, *On estimating of a fourth order functional for bounded univalent functions*, Ann. Acad. Sci. Fenn. Ser. AI 490 (1971), 1–18.

[3] N. A. Lebedev, *Area Principle in the Theory of Univalent Functions*, Nauka, Moscow, 1975 (in Russian).

[4] P. Lehto, *On fourth-order homogeneous functionals in the class of bounded univalent functions*, Ann. Acad. Sci. Fenn. Ser. AI Math. Dissertationes 48 (1984).

[5] K. Włodarczyk, *On certain non-homogeneous combinations of coefficients of bounded univalent functions*, Demonstratio Math. 16 (1983), 919–924.

Larisa Gromova
DEPARTMENT OF PHYSICS&MATH.
SARATOV PEDAGOGICAL INSTITUTE
92 MICHURIN ST.
SARATOV 410071, RUSSIA

Alexander Vasil'ev
DEPARTMENT OF MATH.&MECH.
SARATOV STATE UNIVERSITY
83 ASTRAKHANSKAYA ST.
SARATOV 410071, RUSSIA

Reçu par la Rédaction le 12.10.1993
Révisé le 15.3.1995