ANNALES
POLONICI MATHEMATICI
LXIII.3 (1996)

On the asymptotic behavior of solutions of second order parabolic partial differential equations

by Wei-Cheng Lian and Cheh-Chih Yeh (Chung-Li)

Abstract. We consider the second order parabolic partial differential equation

$$\sum_{i,j=1}^{n} a_{ij}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x,t)u_{x_i} + c(x,t)u - u_t = 0.$$

Sufficient conditions are given under which every solution of the above equation must decay or tend to infinity as $|x| \to \infty$. A sufficient condition is also given under which every solution of a system of the form

$$L^{\alpha}[u^{\alpha}] + \sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)u^{\beta} = f^{\alpha}(x,t),$$

where

$$L^{\alpha}[u] \equiv \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i^{\alpha}(x,t)u_{x_i} - u_t,$$

must decay as $t \to \infty$.

1. Introduction. Let $x = (x_1, \ldots, x_n)$ be a point of the n-dimensional Euclidean space $\mathbb{R}^n$ and let t be a nonnegative number. The distance of the point $x \in \mathbb{R}^n$ from the origin of $\mathbb{R}^n$ is denoted by $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$. Let Ω be an unbounded domain in $\mathbb{R}^n$. The $(n+1)$-dimensional Euclidean domain $D := \Omega \times (0,T)$ is our domain of interest; here $0 < T \le \infty$.

Consider the second order parabolic partial differential equation of the form

$$Lu := \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} = 0 \tag{1}$$

1991 *Mathematics Subject Classification*: Primary 35B40; Secondary 35K40.
Key words and phrases: asymptotic behavior, second order partial differential equation, maximum principles.

in D. We consider only classical solutions of (1), thus we require $u(x,t) \in C^0(\bar{D}) \cap C^2(D)$.

In 1962, Krzyżański [11] proved the existence of the fundamental solution of the following parabolic differential equation:

$$L_0 u := \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} + (-k^2|x|^2 + l)u - \frac{\partial u}{\partial t} = 0, \qquad k > 0,$$

in $\mathbb{R}^n \times (0,\infty)$. Using this fundamental solution, we see that the solution $u(x,t)$ of the above equation with Cauchy data $u(x,0) = M\exp(a|x|^2)$ is given by

$$u(x,t) = M\left(\frac{k}{k\cosh 2kt - 2a\sinh 2kt}\right)^{n/2} \times \exp\left[\frac{k(2a\cosh 2kt - k\sinh 2kt)}{2(k\cosh 2kt - 2a\sinh 2kt)}|x|^2 + lt\right],$$

where $2a < k$. Hence, if $l - kn < 0$, then $u(x,t)$ converges to zero uniformly on every compact set in $\mathbb{R}^n$ as $t \to \infty$. And, if $t > \frac{1}{4k}\ln\frac{2a+k}{k-2a}$, then $u(x,t)$ converges to zero as $|x| \to \infty$.

Results on the asymptotic behavior as $t \to \infty$ of solutions $u(x,t)$ of more general parabolic equations and systems with unbounded coefficients have been obtained by various authors, for example, Chen [2]–[4], Kuroda [12], Kuroda and Chen [13], Kusano [14], [15] and Kusano, Kuroda and Chen [16], [17]. They considered the coefficients of (1) satisfying one of the following two conditions:

(I) There exist constants $K_1 > 0$, $K_2 \geq 0$, $K_3 > 0$, $\mu > 0$ and $\lambda > 0$ such that

$$0 < \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq K_1[\log(|x|^2+1)+1]^{-\lambda}(|x|^2+1)^{1-\mu}|\xi|^2$$

for all nonzero real vectors $\xi = (\xi_1,\ldots,\xi_n)$, and

$$|b_i(x,t)| \leq K_2(|x|^2+1)^{1/2}, \qquad i = 1,\ldots,n,$$
$$c(x,t) \leq K_3[\log(|x|^2+1)+1]^{\lambda}(|x|^2+1)^{\mu};$$

(II) There exist constants $K_1 > 0, K_2 \geq 0, K_3 > 0$, and $\lambda \geq 1$ such that

$$0 < \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq K_1(|x|^2+1)^{1-\lambda}|\xi|^2 \qquad \text{for any nonzero } \xi \in \mathbb{R}^n,$$

$$|b_i(x,t)| \leq K_2(|x|^2+1)^{1/2}, \qquad i = 1,\ldots,n,$$
$$c(x,t) \leq -K_3(|x|^2+1)^{\lambda}.$$

In 1980, Cosner [8] generalized the above results to the more general parabolic equations (1) whose coefficients satisfy the following condition.

(A) There exist positive constants μ, K_1, K_2 and K_3 such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \le K_1\phi(1+r^2)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n,$$
$$|b_i(x,t)| \le K_2\phi(1+r^2)\theta(1+r^2)(1+r^2)^{-1/2}, \quad i = 1,\dots,n,$$
$$c(x,t) \le K_3[\theta(1+r^2)]^{\mu},$$

for $(x,t) \in D$, where $r = |x|$ and $\theta(\eta), \phi(\eta)$ satisfy the following condition (H):

(H) $\theta(\eta)$ is a C^2 function on $[1,\infty)$ such that $d\theta(\eta)/d\eta = 1/\phi(\eta)$, $\theta(\eta) \ge 1$, $\phi(\eta)$ is a C^1 positive function of η, and there exist nonnegative constants m_1 and m_2 such that for $\eta \ge 1$, $\eta\phi''(\eta) \le m_1\phi(\eta)\phi'(\eta)$, and $\eta\phi'(\eta) \le m_2[\phi(\eta)]^{2-\mu}$.

He gave some sufficient conditions under which every solution $u(x,t)$ of (1) converges to zero uniformly on every compact set in $\mathbb{R}^n$ as $t \to \infty$.

In 1974, Chen–Lin–Yeh [5] discussed the asympotic behavior of solutions for large $|x|$ of equation (1) whose coefficients satisfy (I) or (II). To our knowledge, there is no other paper discussing the asymptotic behavior for large $|x|$ of solutions of equation (1) whose coefficients satisfy assumption (A).

The purpose of this paper is to give sufficient conditions under which every solution of (1) must decay as $|x| \to \infty$ and to give sufficient conditions under which every solution of (1) must tend to infinity as $|x| \to \infty$. We also generalize the results to a system of the form

$$L^{\alpha}[u^{\alpha}] + \sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)u^{\beta} = 0, \quad \alpha = 1,\dots,N, \tag{2}$$

where

$$L^{\alpha}[u] \equiv \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x,t)u_{x_i x_j} + \sum_{i=1}^{n} b_i^{\alpha}(x,t)u_{x_i} - u_t.$$

A sufficient condition is also given under which every solution of

$$L^{\alpha}[u^{\alpha}] + \sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)u^{\beta} = f^{\alpha}(x,t)$$

must decay as $t \to \infty$, where $\alpha = 1,\dots,N$.

The techniques used in the present article are primarily adapted from those used in Chen, Lin and Yeh [5] and Cosner [7], [8].

2. Main results. In order to prove our main results, we need the following maximum principle which is due to Cosner [7], [8].

LEMMA 1 (Phragmén–Lindelöf principle). *Let $u(x,t) \in C^0(\overline{D}) \cap C^2(D)$ satisfy the inequalities*

$$\begin{cases} L[u] \geq 0 \text{ in } D, \\ u \leq 0 \text{ on } \Sigma := (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,T)). \end{cases} \tag{3}$$

Suppose that the coefficients of L satisfy assumption (A) *in D. If there is a constant $k \geq 1$ such that*

$$\liminf_{r\to\infty} [\max_{\substack{(x,t)\in D\\ |x|=r}} u(x,t)] \exp\{-k[\theta(1+r^2)]^\mu\} \leq 0, \tag{4}$$

then $u(x,t) \leq 0$ in $\overline{D}$.

Remark 1. If (3) and (4) in Lemma 1 are replaced by

$$\begin{cases} L[u] \leq 0 & \text{in } D, \\ u \geq 0 & \text{on } \Sigma, \end{cases}$$

and

$$\limsup_{r\to\infty} [\max_{\substack{(x,t)\in D\\ |x|=r}} u(x,t)] \exp\{-k[\theta(1+r^2)]^\mu\} \geq 0$$

respectively, then $u \geq 0$ in $\overline{D}$. Lemma 1 can be easily generalized to weakly coupled systems (2) (see Cosner [7]).

THEOREM 1. *Suppose that*

(C_1) $u \in C^0(\overline{D}) \cap C^2(D)$ *satisfies $Lu = 0$ in D,*

(C_2) *the coefficients of L satisfy the following condition: There exist constants $k_1 \geq 0$, $K_1 > 0$, $K_2 \geq 0$, $K_3 \geq 0$ and $0 < \mu \leq 1$ such that*

$$k_1\phi(1+r^2)|\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \leq K_1\phi(1+r^2)|\xi|^2 \quad \textit{for } \xi \in \mathbb{R}^n,$$

$$|b_i(x,t)| \leq K_2\phi(1+r^2)\theta(1+r^2)(1+r^2)^{-1/2}, \quad i = 1,\ldots,n,$$

$$c(x,t) \leq K_3[\theta(1+r^2)]^\mu, \ \textit{where } \theta(\eta) \textit{ and } \phi(\eta) \textit{ satisfy condition } \text{(H)},$$

(C_3) *for every $T > 0$, there exists a constant $k(T) \geq 1$ such that*

$$\lim_{r\to\infty} [\max_{\substack{|x|=r\\ 0\leq t\leq T}} |u|] \exp\{-k(T)[\theta(1+r^2)]^\mu\} = 0.$$

Then:

(a) *If $\theta''(\eta) \geq 0$ for $\eta \geq 1$ and*

$$|u| \leq M\exp\{-k[\theta(1+r^2)]^\mu \varrho^{\tau t}\} \quad \textit{on } \Sigma \tag{5}$$

for some constant M, where

(6) $\tau = -[4k^2 K_1\mu^2 m_2 - 4k\mu(\mu-1)K_1 m_2 + 2k\mu K_2 n + K_3]/(k\ln\varrho)$,

then

$$(\mathrm{R}_1) \qquad |u| \le M\exp\{-k[\theta(1+r^2)]^\mu \varrho^{\tau t}\} \quad \text{in } \overline{D}.$$

(b) *If there exists a constant $m_3 \ge 0$ such that $\eta\theta''(\eta) \ge -m_3\theta'(\eta)$ for $\eta \ge 1$ and $|u| \le M\exp\{-k[\theta(1+r^2)]^\mu\varrho^{\tau t}\}$ on Σ for some constant M, where*

$$\tau = -[4k^2K_1\mu^2m_2 - 4k\mu(\mu-1)K_1m_2 + 4k\mu m_3K_1 + 2k\mu K_2n + K_3]/(k\ln\varrho),$$

then (R_1) *also holds.*

Moreover, if, in addition, $\Omega = \mathbb{R}^n$ and $\theta(\eta) \to \infty$ as $\eta \to \infty$, then the solution u of (1) *decays exponentially to zero as $|x| \to \infty$.*

Proof. (a) Let $\omega(x,t) = M\exp\{-k[\theta(1+r^2)]^\mu\varrho^{\tau t}\}$, where $\varrho > 1$ is a parameter and $\tau = \tau(\varrho)$ is defined in (6). Thus

$$\begin{aligned} L[\omega] &\equiv \sum_{i,j=1}^n a_{ij}\omega_{x_ix_j} + \sum_{i=1}^n b_i\omega_{x_i} + c\omega - \omega_t \\ &= \Big\{4k^2\mu^2\theta^{2\mu-2}(\theta')^2\varrho^{2\tau t}\sum_{i,j=1}^n a_{ij}x_ix_j \\ &\quad - 4k\mu(\mu-1)\theta^{\mu-2}(\theta')^2\varrho^{\tau t}\sum_{i,j=1}^n a_{ij}x_ix_j \\ &\quad - 4k\mu\theta^{\mu-1}\theta''\varrho^{\tau t}\sum_{i,j=1}^n a_{ij}x_ix_j - 2k\mu\theta^{\mu-1}\theta'\varrho^{\tau t}\sum_{i=1}^n a_{ii} \\ &\quad - 2k\mu\theta^{\mu-1}\theta'\varrho^{\tau t}\sum_{i=1}^n b_ix_i + c + k\theta^\mu\tau\varrho^{\tau t}\ln\varrho\Big\}\omega. \end{aligned}$$

By (C_1), (C_2), (C_3) and $\theta''(\eta) > 0$ for $\eta \ge 1$, we obtain

$$\begin{aligned} L[\omega] &\le \{4k^2K_1\mu^2m_2\varrho^{2\tau t}\theta^\mu - 4k\mu(\mu-1)K_1m_2\varrho^{\tau t} \\ &\qquad + 2k\mu K_2\theta^\mu\varrho^{\tau t}n + K_3\theta^\mu + k\theta^\mu\tau\varrho^{\tau t}\ln\varrho\}\omega \\ &\le \{4k^2K_1\mu^2m_2 - 4k\mu(\mu-1)K_1m_2 \\ &\qquad + 2k\mu K_2n + K_3 + k\tau\ln\varrho\}\theta^\mu\varrho^{2\tau t}\omega. \end{aligned}$$

By (6), we have $L[\omega] \le 0$ in D, and hence $L[u-\omega] = L[u]-L[\omega] = -L[\omega] \ge 0$ in D. It follows from (5) that $u-\omega \le 0$ on Σ. Thus, by the Phragmén–Lindelöf principle, we see that $u-\omega \le 0$ in $\Omega\times(0,T)$ for every fixed T. Hence, $u-\omega \le 0$ in D and thus, by continuity, in $\overline{D}$. We can apply Remark 1 to $u+\omega$ in a similar way and conclude that $u+\omega \geq 0$ in $\overline{D}$. Thus $|u| \leq \omega$ in $\overline{D}$, that is, (R_1) holds.

(b) For the same ω and $L[\omega]$ computed as before, we now obtain the estimate

$$\begin{aligned} L[\omega] \leq \{&4k^2K_1\mu^2 m_2 \varrho^{2\tau t}\theta^\mu - 4k\mu(\mu-1)K_1 m_2 \varrho^{\tau t} + 4k\mu\theta^{\mu-1} m_3 K_1 \varrho^{\tau t} \\ &+ 2k\mu K_2 \theta^\mu \varrho^{\tau t} n + K_3\theta^\mu + k\theta^\mu \tau \varrho^{\tau t} \ln \varrho\}\omega \\ \leq \{&4k^2K_1\mu^2 m_2 - 4k\mu(\mu-1)K_1 m_2 \\ &+ 4k\mu m_3 K_1 + 2k\mu K_2 n + K_3 + k\tau \ln \varrho\}\theta^\mu \varrho^{2\tau t}\omega. \end{aligned}$$

Thus $L[\omega] \leq 0$ in D, and we conclude as before that (R_1) holds.

THEOREM 2. *Let* (C_1) *and* (C_3) *hold. Suppose that the coefficients of* L *satisfy the following condition*:

(C_4) *there exist constants* $K_1 > 0, K_2 \geq 0, k_3 > 0, K_3 \geq 0$ *and* $0 < \mu \leq 1$ *such that for all* $(x,t) \in D$,

$$0 \leq \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \leq K_1\phi(1+r^2)|\xi|^2 \quad \textit{for } \xi \in \mathbb{R}^n,$$
$$|b_i(x,t)| \leq K_2\phi(1+r^2)\theta(1+r^2)(1+r^2)^{-1/2}, \quad i=1,\ldots,n,$$
$$-k_3[\theta(1+r^2)]^\mu \leq c(x,t) \leq K_3[\theta(1+r^2)]^\mu,$$

where $\theta(\eta)$ *and* $\phi(\eta)$ *satisfy condition* (H).

Then:

(a) *If* $\theta''(\eta) \geq 0$ *for* $\eta \geq 1$ *and*

$$|u| \geq M \exp\{k[\theta(1+r^2)]^\mu \varrho^{\tau t}\} \quad \textit{on } \Sigma \tag{7}$$

for some constant M, *where*

$$\tau = [4kK_1 m_2 \mu(\mu-1) - 2kK_2\mu n - k_3]/(k\ln\varrho), \tag{8}$$

then

$$|u| \geq M \exp\{k[\theta(1+r^2)]^\mu \varrho^{\tau t}\} \quad \textit{in } \overline{D}. \tag{R_2}$$

(b) *If there exists a constant* $m_3 \geq 0$ *such that* $\eta\theta''(\eta) \geq -m_3\theta'(\eta)$ *for* $\eta \geq 1$ *and* $|u| \geq M\exp\{k[\theta(1+r^2)]^\mu\varrho^{\tau t}\}$ *on* Σ *for some constant* M, *where* $\tau = (4kK_1m_2\mu(\mu-1) - 4kK_1\mu m_3 - 2kK_2\mu n - k_3)/(k\ln\varrho)$, *then* (R_2) *holds.*

Moreover, *if*, *in addition*, $\Omega = \mathbb{R}^n$ *and* $\theta(\eta) \to \infty$ *as* $\eta \to \infty$, *then the solution* $u(x,t)$ *of* (1) *tends to infinity as* $|x| \to \infty$.

Proof. (a) Let $\omega = M\exp\{k[\theta(1+r^2)]^\mu\varrho^{\tau t}\}$, where $\varrho > 1$ is a parameter and $\tau = \tau(\varrho)$ is defined in (8). Then

$$\begin{aligned}
L[\omega] = \Big\{ & 4k^2\mu^2\theta^{2\mu-2}(\theta')^2\varrho^{2\tau t}\sum_{i,j=1}^n a_{ij}x_ix_j \\
& + 4k\mu(\mu-1)\theta^{\mu-2}(\theta')^2\varrho^{\tau t}\sum_{i,j=1}^n a_{ij}x_ix_j \\
& + 4k\mu\theta^{\mu-1}\theta''\varrho^{\tau t}\sum_{i,j=1}^n a_{ij}x_ix_j + 2k\mu\theta^{\mu-1}\theta'\varrho^{\tau t}\sum_{i=1}^n a_{ii} \\
& + 2k\mu\theta^{\mu-1}\theta'\varrho^{\tau t}\sum_{i=1}^n b_ix_i + c - k\theta^\mu\tau\varrho^{\tau t}\ln\varrho\Big\}\omega \\
\geq{}& \{4kK_1m_2\mu(\mu-1)\varrho^{\tau t} - 2kK_2\mu\theta^\mu\varrho^{\tau t}n - k_3\theta^\mu - k\theta^\mu\tau\varrho^{\tau t}\ln\varrho\}\omega \\
\geq{}& \{4kK_1m_2\mu(\mu-1) - 2kK_2\mu n - k_3 - k\tau\ln\varrho\}\theta^\mu\varrho^{\tau t}\omega.
\end{aligned}$$

It follows from (8) that $L[\omega] \geq 0$ in D. By (7), we have

$$|u| \geq M\exp\{k[\theta(1+r^2)]^\mu\varrho^{\tau t}\} = \omega \quad \text{on } \Sigma.$$

Case 1. If $u \geq 0$, then $u-\omega \geq 0$ on Σ and $L[u-\omega] = L[u] - L[\omega] = -L[\omega] \leq 0$ in D. Thus, by the Phragmén–Lindelöf principle, we have $u-\omega \geq 0$ in $\Omega\times(0,T)$ for each fixed $T>0$. Hence, $u-\omega \geq 0$ in D and, by continuity, $u \geq \omega$ in $\overline{D}$.

Case 2. If $u \leq 0$, then $u+\omega \leq 0$ on Σ and $L[u+\omega] \geq 0$ in D. Thus, by the Phragmén–Lindelöf principle, we have $u+\omega \leq 0$ in $\Omega\times(0,T)$ for each fixed $T>0$. Hence, $u+\omega \leq 0$ in D and, by continuity, in $\overline{D}$. Thus, $|u| \geq \omega$ in $\overline{D}$, that is, (R_2) holds.

(b) For the same ω and $L[\omega]$ computed as before, we now obtain the estimate

$$\begin{aligned}
L[\omega] \geq{}& \{4kK_1m_2\mu(\mu-1)\varrho^{\tau t} - 4kK_1\mu\theta^{\mu-1}\varrho^{\tau t}m_3 \\
& - 2kK_2\mu\theta^\mu\varrho^{\tau t}n - k_3\theta^\mu - k\theta^\mu\tau\varrho^{\tau t}\ln\varrho\}\omega \\
\geq{}& \{4kK_1m_2\mu(\mu-1) - 4kK_1\mu m_3 - 2kK_2\mu n - k_3 - k\tau\ln\varrho\}\varrho^{\tau t}\theta^\mu\omega.
\end{aligned}$$

Thus $L[\omega] \geq 0$ in D. As in the proof of case (a), we easily see that (R_2) holds.

Similarly, we can obtain the following results:

THEOREM 3. *Let* (C_1), (C_2) *and* (C_3) *hold with* $\mu \geq 1$. *Then*:

(a) *If* $\theta''(\eta) \geq 0$ *for* $\eta \geq 1$ *and* $|u| \leq M\exp\{-k[\theta(1+r^2)]^\mu\varrho^{\tau t}\}$ *on* Σ *for some constant* M, *where* $\tau = -[4k^2K_1\mu^2m_2 + 2k\mu K_2n + K_3]/(k\ln\varrho)$, *then*

$$(\mathrm{R}_3) \qquad |u| \leq M\exp\{-k[\theta(1+r^2)]^\mu\varrho^{\tau t}\} \quad \text{in } \overline{D}.$$

(b) *If there exists a constant* $m_3 \geq 0$ *such that* $\eta\theta''(\eta) \geq -m_3\theta'(\eta)$ *for* $\eta \geq 1$ *and* $|u| \leq M\exp\{-k[\theta(1+r^2)]^\mu\varrho^{\tau t}\}$ *on* Σ *for some constant* M, *where* $\tau = -[4k^2K_1\mu^2m_2 + 4k\mu m_3K_1 + 2k\mu K_2n + K_3]/(k\ln\varrho)$, *then* (R_3) *also holds.*

Moreover, if, in addition, $\Omega = \mathbb{R}^n$ *and* $\theta(\eta) \to \infty$ *as* $\eta \to \infty$, *then the solution* $u(x,t)$ *decays exponentially to zero as* $|x| \to \infty$.

THEOREM 4. *Let* (C_1), (C_3) *and* (C_4) *hold with* $\mu \geq 1$. *Then*:

(a) *If* $\theta''(\eta) \geq 0$ *for* $\eta \geq 1$ *and* $|u| \geq M\exp\{k[\theta(1+r^2)]^\mu\varrho^{\tau t}\}$ *on* Σ *for some constant* M, *where* $\tau = (-2kK_2\mu n - k_3)/(k\ln\varrho)$, *then*

$$(\mathrm{R}_4) \qquad |u| \geq M\exp\{k[\theta(1+r^2)]^\mu\varrho^{\tau t}\} \quad \text{in } \overline{D}.$$

(b) *If there exists a constant* $m_3 \geq 0$ *such that* $\eta\theta''(\eta) \geq -m_3\theta'(\eta)$ *for* $\eta \geq 1$ *and* $|u| \geq M\exp\{k[\theta(1+r^2)]^\mu\varrho^{\tau t}\}$ *on* Σ *for some constant* M, *where* $\tau = \tau(\varrho) = (-4kK_1\mu m_3 - 2kK_2\mu n - k_3)/(k\ln\varrho)$, *then* (R_4) *holds.*

Moreover, if, in addition, $\Omega = \mathbb{R}^n$ *and* $\theta(\eta) \to \infty$ *as* $\eta \to \infty$, *then the solution* $u(x,t)$ *of* (1) *tends to infinity as* $|x| \to \infty$.

3. Further results. In this section, we generalize the results of Section 2 to weakly coupled systems of the form

$$L^\alpha[u^\alpha] + \sum_{\beta=1}^{N} c^{\alpha\beta}u^\beta = 0, \qquad \alpha = 1,\ldots,N,$$

where

$$L^\alpha[u] \equiv \sum_{i,j=1}^{n} a_{ij}^\alpha u_{x_ix_j} + \sum_{i=1}^{n} b_i^\alpha u_{x_i} - u_t.$$

THEOREM 5. *Suppose that*

(C_5) *the functions* $u^\alpha, \alpha = 1,\ldots,N$, *satisfy*

$$L^\alpha[u^\alpha] + \sum_{\beta=1}^{N} c^{\alpha\beta}u^\beta = 0 \quad \text{in } D$$

and $u^\alpha \in C^0(\overline{D}) \cap C^2(D)$ *for each* $\alpha = 1,\ldots,N$,

(C_6) *for* $\alpha,\beta = 1,\ldots,N$, *the operators* L^α *and the functions* $c^{\alpha\beta}$ *satisfy the following conditions*: *There exist constants* $k_1 \geq 0$, $K_1 > 0$, $K_2 > 0$, $K_3 > 0$ *and* $0 < \mu \leq 1$ *such that for* $\alpha = 1,\ldots,N$ *and* $(x,t) \in D$,

$$k_1\phi(1+r^2)|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^\alpha(x,t)\xi_i\xi_j \leq K_1\phi(1+r^2)|\xi|^2,$$

$$|b_i^\alpha(x,t)| \leq K_2\phi(1+r^2)\theta(1+r^2)(1+r^2)^{-1/2}, \qquad i = 1,\ldots,n,$$

$$\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t) \le K_3[\theta(1+r^2)]^{\mu},$$

where $\theta(\eta)$ and $\phi(\eta)$ satisfy condition (H),

(C_7) *for each $\alpha = 1,\ldots,N$ and for every $T > 0$, there exists a constant $k(T) \ge 1$ such that*

$$\lim_{r\to\infty} [\max_{\substack{|x|=r\\|t|<T}} |u^\alpha|] \exp\{-k(T)[\theta(1+r^2)]^{\mu}\} = 0.$$

Then:

(a) *If $\theta''(\eta) \ge 0$ for $\eta > 1$, and $|u^\alpha| \le M\exp\{-k[\theta(1+r^2)]^{\mu}\varrho^{\tau t}\}$ on Σ for some constant M and $\alpha = 1,\ldots,N$, where $\tau = -[4k^2K_1\mu^2 m_2 - 4k\mu(\mu-1)K_1m_2 + 2k\mu K_2 n + K_3]/(k\ln\varrho)$, then*

(R_5) $$|u^\alpha| \le M\exp\{-k[\theta(1+r^2)]^{\mu}\varrho^{\tau t}\} \quad \text{in } \overline{D} \text{ for } \alpha = 1,\ldots,N.$$

(b) *If there exists a constant $m_3 \ge 0$ such that $\eta\theta''(\eta) \ge -m_3\theta'(\eta)$ for $\eta \ge 1$ and $|u^\alpha| \le M\exp\{-k[\theta(1+r^2)]^{\mu}\varrho^{\tau t}\}$ on Σ for some constant M and for $\alpha = 1,\ldots,N$, where $\tau = -[4k^2K_1\mu^2m_2 - 4k\mu(\mu-1)K_1m_2 + 4k\mu m_3K_1 + 2k\mu K_2 n + K_3]/(k\ln\varrho)$, then* ($R_5$) *also holds.*

Moreover, if, in addition, $\Omega = \mathbb{R}^n$ and $\theta(\eta) \to \infty$ as $\eta \to \infty$, then the solution $u^\alpha(x,t)$ of (2) *decays exponentially to zero as $|x| \to \infty$, for $\alpha = 1,\ldots,N$.*

Remark 6. Similarly, if the functions u^α, $c^{\alpha\beta}$ and the coefficients of the operator L^α $(\alpha,\beta = 1,\ldots,N)$ satisfy the hypotheses of Theorems 2–4, then results of the above-mentioned theorems are true with respect to u^α, $\alpha = 1,\ldots,N$.

4. Exponential decay of solutions as $t \to \infty$. In [1], Chabrowski discussed the decay as $t \to \infty$ of solutions of a single parabolic equation

$$Lu = f(x,t)$$

with bounded coefficients in $\mathbb{R}^n \times [0,\infty)$. In this section, we extend Chabrowski's result to the system

(9) $$L^\alpha[u^\alpha] = f^\alpha(x,t), \quad \alpha = 1,\ldots,N,$$

with unbounded coefficients. Here L and L^α are defined as in (1) and (2) respectively. To do this, we need the following maximum principle which is an easy extension of the maximum principle stated in Kusano–Kuroda–Chen [16].

LEMMA 7. *Suppose that the coefficients of* (9) *in* $\mathbb{R}^n \times [0,\infty)$ *satisfy*

$$(\mathrm{C}_8)\quad \begin{cases} 0 \le \sum_{i,j=1}^{n} a_{ij}^{\alpha}(x,t)\xi_i\xi_j \le K_1\phi(1+|x|^2)|\xi|^2 \quad \textit{for all } \xi \in \mathbb{R}^n, \\ |b_i^{\alpha}(x,t)| \le K_2\phi(1+|x|^2)\theta(1+|x|^2)(1+|x|^2)^{-1/2}, \quad i=1,\ldots,n, \\ c^{\alpha\beta}(x,t) \ge 0,\ \alpha \ne \beta,\ \sum_{\beta=1}^{n} c^{\alpha\beta}(x,t) \le K_3[\theta(1+|x|^2)]^{\mu}, \end{cases}$$

for $\alpha = 1,\ldots,N$, *where* $K_1 > 0, K_2 \ge 0, K_3 > 0$ *and* $\mu > 0$ *are constants, and* $\theta(\eta)$ *and* $\phi(\eta)$ *satisfy condition* (H). *Let* $u^{\alpha}(x,t)$, $\alpha = 1,\ldots,N$, *satisfy*

$$L^{\alpha}[u^{\alpha}] + \sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)u^{\beta} \ge 0, \qquad \alpha = 1,\ldots,N,$$

in $\mathbb{R}^n \times [0,\infty)$ *with the properties* $u^{\alpha}(x,0) \le 0$ *for* $x \in \mathbb{R}^n$, *and* $u^{\alpha}(x,t) \le M\exp\{k\theta(1+|x|^2)^{\mu}\}$ *for* $(x,t) \in \mathbb{R}^n \times (0,\infty)$, *where* $\alpha = 1,\ldots,N$, *and* M *and* k *are some positive constants. Then* $u^{\alpha}(x,t) \le 0$ *in* $\mathbb{R}^n \times (0,\infty)$ *for* $\alpha = 1,\ldots,N$.

THEOREM 8. *Let the coefficients of* (9) *satisfy condition* (C_8) *and* $\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t) \le -K_3$ *for* $\alpha = 1,\ldots,N$. *Suppose* $u^{\alpha}(x,t), \alpha = 1,\ldots,N$, *are bounded solutions of* (9). *If* $\lim_{t\to\infty} f^{\alpha}(x,t) = 0$, $\alpha = 1,\ldots,N$, *uniformly with respect to* $x \in \mathbb{R}^n$, *then* $\lim_{t\to\infty} u^{\alpha}(x,t) = 0$, $\alpha = 1,\ldots,N$, *uniformly with respect to* $x \in \mathbb{R}^n$.

Proof. Let $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$|f^{\alpha}(x,t)| \le \varepsilon, \qquad \alpha = 1,\ldots,N,$$

for $x \in \mathbb{R}^n$ and $t \ge \delta$. Put

$$M^{\alpha} = \sup_{(x,t)\in\mathbb{R}^n\times[0,\infty)} |u^{\alpha}(x,t)|, \qquad \alpha = 1,\ldots,N.$$

Define

$$\omega_{\pm}^{\alpha}(x,t) = -2\frac{\varepsilon}{K_3} - M^{\alpha}e^{-h(t-\delta)} \pm u^{\alpha}(x,t), \qquad \alpha = 1,\ldots,N,$$

where h is a positive constant such that $0 < h < K_3$. Hence

$$\begin{aligned} L^{\alpha}[\omega_{\pm}^{\alpha}] + \sum_{\beta=1}^{N} c^{\alpha\beta}(x,t)u^{\beta} = &-\frac{2\varepsilon}{K_3}\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t) - M^{\alpha}e^{-h(t-\delta)}\sum_{\beta=1}^{N} c^{\alpha\beta}(x,t) \\ &- hM^{\alpha}e^{-h(t-\delta)} \pm f^{\alpha}(x,t) \\ &\ge \varepsilon + M^{\alpha}e^{-h(t-\delta)}(K_3 - h) > 0, \quad \alpha = 1,\ldots,N. \end{aligned}$$

for $x \in \mathbb{R}^n$ and $t > \delta$. Moreover,

$$\omega_{\pm}^{\alpha}(x,\delta) = -2\frac{\varepsilon}{K_3} - M^{\alpha} + u^{\alpha}(x,\delta) < 0, \qquad \alpha = 1,\ldots,N,$$

for $x \in \mathbb{R}^n$. From Lemma 7, we see that $\omega_{\pm}^{\alpha}(x,t) \le 0$, $\alpha = 1,\ldots,N$, for $x \in \mathbb{R}^n$ and $t > \delta$. Hence

$$-2\frac{\varepsilon}{K_3} - M^{\alpha}e^{-h(t-\delta)} \le u^{\alpha}(x,t) \le 2\frac{\varepsilon}{K_3} + M^{\alpha}e^{-h(t-\delta)}$$

for $x \in \mathbb{R}^n, t > \delta$ and $\alpha = 1,\ldots,N$. Therefore

$$-2\frac{2\varepsilon}{K_3} \le \lim_{t\to\infty}\inf u^{\alpha}(x,t) \le \lim_{t\to\infty}\sup u^{\alpha}(x,t) \le \frac{2\varepsilon}{K_3},$$

which proves our theorem.

References

[1] P. J. Chabrowski, *Propriétés asymptotiques d'une mesure associée à l'équation différentielle aux dérivées partielles de type parabolique*, Funkc. Ekvac. 13 (1970), 35–43.

[2] L. S. Chen, *On the behavior of solutions for large $|x|$ of parabolic equations with unbounded coefficients*, Tôhoku Math. J. 20 (1968), 589-595.

[3] —, *Note on the behavior of solutions of parabolic equations with unbounded coefficients*, Nagoya Math. J. 37 (1970), 1–4.

[4] —, *Remark on behavior of solutions of some parabolic equations*, Tôhoku Math. J. 22 (1970), 511–516.

[5] L. S. Chen, J. S. Lin and C. C. Yeh, *Asymptotic behavior of solutions for large $|x|$ of weakly coupled parabolic systems with unbounded coefficients*, Hiroshima Math. J. 4 (1974), 477–490.

[6] L. S. Chen, C. C. Yeh and H. Y. Chen, *On the behavior of solutions of the Cauchy problem for parabolic equations with unbounded coefficients*, Hiroshima Math. J. 1 (1971), 145–153.

[7] C. Cosner, *A Phragmén–Lindelöf principle and asymptotic behavior for weakly coupled systems of parabolic equations with unbounded coefficients*, Dissertation, University of California, Berkeley, 1977.

[8] —, *Asymptotic behavior of solutions of second order parabolic partial differentical equations with unbounded coefficients*, J. Differential Equations 35 (1980), 407–428.

[9] S. D. Èĭdel'man and F. O. Porper, *Properties of solutions of second-order parabolic equations with dissipation*, Differential Equations 7 (1971), 1280–1288.

[10] —, —, *On the asymptotic behavior of solutions of parabolic systems with dissipation*, Soviet Math. Dokl. 12 (1971), 471–475.

[11] M. Krzyżański, *Evaluations des solutions de l'équation linéaire de type parabolique à coefficients non bornés*, Ann. Polon. Math. 11 (1962), 253–260.

[12] T. Kuroda, *Asymptotic behavior of solutions of parabolic equations with unbounded coefficients*, Nagoya Math. J. 37 (1970), 5–12.

[13] T. Kuroda and L. S. Chen, *On the behavior of solutions of parabolic equations with unbounded coefficients*, Ann. Polon. Math. 23 (1970), 57–64.

[14] T. Kusano, *On the decay for large $|x|$ of solutions of parabolic equations with unbounded coefficients*, Publ. Res. Inst. Math. Sci. Kyoto Univ. Ser. A 3 (1967), 203–210.
[15] —, *On the behavior of large $|x|$ for solutions of parabolic equations with unbounded coefficients*, Funkc. Ekvac. 11 (1968), 169–174.
[16] T. Kusano, T. Kuroda and L. S. Chen, *Weakly coupled parabolic systems with unbounded coefficients*, Hiroshima Math. J. 3 (1973), 1–14.
[17] —, —, —, *Some parabolic equations with unbounded coefficients*, Funkc. Ekvac. 16 (1973), 1–28.
[18] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1967.

Department of Mathematics
National Central University
Chung-Li
32054 Taiwan, Republic of China
E-mail: yeh@wangwei.math.ncu.edu.tw

Reçu par la Rédaction le 16.6.1994
Révisé le 31.8.1995