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Growth properties of entire functions depending on a parameter

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Abstract. We study the growth of parameter-dependent entire functions. We are mainly interested in the case where the functions depend holomorphically on the parameter.

1. Introduction. Let $(z, w) \mapsto H(z, w)$ be an entire function in $z \in \mathbb{C}^n$ which depends on a parameter w. We can then study the growth of $H(\cdot, w)$ as a function of w. In this paper we measure the growth with the use of relative order as introduced by C. O. Kiselman in [8], a slight modification of his definition in [7]. Relative order generalizes the classical order and with this notion we can study functions of arbitrarily fast growth. If $\varrho(w)$ denotes the relative order of $H(\cdot, w)$ and H is holomorphic in w we know from [7] that $(-1/\varrho)^*$ is plurisubharmonic. We see in our Theorem 5.2, using a classical result of Bremermann [2], that on a pseudoconvex domain $(-1/\varrho)^*$ can be any negative plurisubharmonic function, while this is not true in general. Sufficient conditions for $(-1/\varrho)^*$ to be pluriharmonic are given in Theorem 5.3 and Corollaries 5.4, 5.5.

In Section 6 we study the continuity properties of the relative order. It is easy to see that the growth can drop suddenly, just take $H(z, w) = F(z)u(w)$, where F is entire and u is a holomorphic function with some zeros. The order will then be constant away from the zeros and will vanish on them. We see in Theorem 6.1 that the opposite can happen, i.e. if Ω is the domain where H is holomorphic in w then the relative order can make a jump up when going from Ω up to any point of the boundary $\partial\Omega$ of Ω even if H extends continuously to the closure $\mathbb{C}^n \times \overline{\Omega}$. On the other hand, we prove in Theorem 6.4 that the relative order is continuous inside Ω if each

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Taylor coefficient of H regarded as a function of $w \in \Omega$ is either non-zero or identically zero. Under the same assumptions on the Taylor coefficients we see in Corollary 6.6 that relative order and supremum commute over relatively compact sets. This is not true for sets which intersect the boundary even if H extends continuously, as seen in Theorem 6.3. We also get a counterexample to Corollary 6.6 if the Taylor coefficients do have zeros. As is seen in Section 7 it is essential in this example that the zeros of the Taylor coefficients accumulate at an infinite number of points. If the relatively compact set is thick enough then we need no conditions at all on the zeros (Theorem 6.7).

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2. Relative order and type. We define relative order and type as in Kiselman [8]. The statements following the definitions are shown there.

DEFINITION 2.1. Let $f, g : E \to [-\infty, +\infty]$ be two functions defined on a real vector space E. We define the *order of f relative to g* as

$$\text{order}(f : g) = \inf[a > 0 : \exists c_a \in \mathbb{R}, \forall x \in E, f(x) \le a^{-1} g(ax) + c_a]. \tag{2.1}$$

If g is convex and $g(0) < +\infty$ then the set above is an interval $]\varrho, +\infty[$ or $[\varrho, +\infty[$, where $0 \le \varrho \le +\infty$.

DEFINITION 2.2. Let f, g be two functions as above. We then define the *type of f relative to g* as

$$\text{type}(f : g) = \inf[b > 0 : \exists c_b \in \mathbb{R},\ \forall x \in E,\ f(x) \le b g(x) + c_b]. \tag{2.2}$$

The set above is an interval $]\sigma, +\infty[$ or $[\sigma, +\infty[$, where $0 \le \sigma \le +\infty$, if g is bounded from below.

3. Growth and coefficient functions. Let F be an entire function in $\mathbb{C}^n$. We then define its *growth function* as

$$f(t) = \sup[\log |F(z)| : z \in \mathbb{C}^n,\ |z| \le e^t], \qquad t \in \mathbb{R}. \tag{3.1}$$

In view of Hadamard's three-circle theorem, f is a convex increasing function.

If F and G are two entire functions, we define the *order of F relative to G* as $\text{order}(F : G) = \text{order}(f : g)$, where f and g are the growth functions defined by (3.1). The order so defined is independent of the norm; see Halvarsson [3]. If G has a zero of a higher degree than F at the origin the order will be infinite. This is not the case in the original definition of Kiselman [7]. Naturally, g need not be a growth function. The interesting choices of "order functions" are those which are convex, bounded from below, increasing and have faster growth than any linear function. As we have seen it is natural to use functions which are bounded from below. If g grows linearly the order is zero for polynomials up to a certain degree and $+\infty$ for other functions. If g is not convex we can replace it with its largest convex minorant. Since a growth function is convex the inequality in (2.1) will still hold then. We will also see in Proposition 4.7 that for each function g' satisfying the conditions discussed there is a growth function g for which $\operatorname{order}(F:g) = \operatorname{order}(F:g')$ for all entire functions F.

We can also define what we will call the *refined growth function* of F, as

$$f_{\mathrm{r}}(t) = \sup[\log|F(z)| : z \in \mathbb{C}^n,\ |z_i| \le e^{t_i}], \qquad t \in \mathbb{R}^n. \tag{3.2}$$

Also this function is convex by Hadamard's theorem.

If F and G are two entire functions in $\mathbb{C}^n$, then $\operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}})$, with $f_{\mathrm{r}}, g_{\mathrm{r}}$ the refined growth functions defined by (3.2), is in general larger than or equal to $\operatorname{order}(f : g)$, with f and g the growth functions defined by (3.1), since we can always take all $t_i = t \in \mathbb{R}$. However, $\operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}})$ is not invariant under linear coordinate changes so it is more natural to define the relative order of two entire functions from $\operatorname{order}(f : g)$. If we take $G = n^{-1}\sum_{i=1}^n e^{z_i}$ we get $\operatorname{order}(f : g) = \operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}})$ equal to the classical order of F. By taking $g(t) = \exp^{[p-1]}(t)$ (where $\exp^{[p-1]}(t) = \exp(\exp^{[p-2]}(t))$, $\exp^{[0]}(t) = t$), we get the $(p,1)$-order of F considered in Sato [15], which has then been generalized to the (p,q)-order introduced in Juneja, Kapoor & Bajpai [5, 6]. We cannot, however, use the methods of the present paper for $q \ne 1$. Note also that if the convex hull $C(F)$ of those multi-indices $k \in \mathbb{N}^n$ for which the Taylor coefficients of F are non-zero is not contained in the likewise defined $C(G)$ then $\operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}})$ equals $+\infty$. See [3], Proposition 7.9.

We can expand F in homogeneous polynomials

$$F(z) = \sum_{j=0}^{\infty} P_j(z), \tag{3.3}$$

where P_j is homogeneous of degree j. We define the norm of the polynomials P_j as

$$\|P_j\| = \sup_{|z|\le 1} |P_j(z)|. \tag{3.4}$$

With this norm we define the *coefficient function* of F as

$$p(j) = \begin{cases} -\log\|P_j\|, & j \in \mathbb{N}, \\ +\infty, & j \in \mathbb{R}\setminus\mathbb{N}. \end{cases} \tag{3.5}$$

If we instead expand $F \in \mathcal{O}(\mathbb{C}^n)$ in a Taylor series

$$F(z) = \sum_k A_k z^k, \qquad z \in \mathbb{C}^n,\ k \in \mathbb{N}^n, \tag{3.6}$$

where k is a multi-index, we define the *refined coefficient function* of F as

$$a(k) = \begin{cases} -\log|A_k|, & k \in \mathbb{N}^n, \\ +\infty, & k \in \mathbb{R}^n \setminus \mathbb{N}^n. \end{cases} \tag{3.7}$$

4. Duality. Let E^* be the algebraic dual of the real vector space E, and E' a fixed linear subspace of E^*. We define the spaces $\mathcal{F}(E, E')$ and $\mathcal{F}(E', E)$ in the following way: $\mathcal{F}(E, E')$ is the space of all functions from E to $[-\infty, +\infty]$ which are convex, lower semicontinuous for the weak topology $\sigma(E, E')$ and take the value $-\infty$ only for the constant function $-\infty$. $\mathcal{F}(E', E)$ is defined similarly for functions from E' to $[-\infty, +\infty]$ but with the weak star topology $\sigma(E', E)$ instead.

Let $f : E \to [-\infty, +\infty]$ be a function on the real vector space E. We define the *Fenchel transform of* f by

$$\widetilde{f}(\xi) = \sup_{x \in E}(\xi \cdot x - f(x)), \qquad \xi \in E'. \tag{4.1}$$

We can apply the transformation twice getting

$$\widetilde{\widetilde{f}}(x) = \sup_{\xi \in E'}(\xi \cdot x - \widetilde{f}(\xi)), \qquad x \in E. \tag{4.2}$$

A direct consequence of the definition is that we have $\widetilde{f} \in \mathcal{F}(E', E)$ and $\widetilde{\widetilde{f}} \in \mathcal{F}(E, E')$. Obviously the transform is dependent on the subspace E' chosen. Some general properties of the Fenchel transform are $\widetilde{\widetilde{f}} \le f$, $\widetilde{\widetilde{\widetilde{f}}} = \widetilde{f}$ and

$$\widetilde{\widetilde{f}} = \sup[v \in \mathcal{F}(E, E') : v \le f]. \tag{4.3}$$

Thus $\widetilde{\widetilde{f}} = f$ if and only if $f \in \mathcal{F}(E, E')$.

Let $f, g : E \to [-\infty, +\infty]$ be two functions on a real vector space E. We then define the *infimal convolution* of f and g by

$$f \mathbin{\Box} g(x) = \inf_y [f(y) \dotplus g(x - y)], \qquad x \in E,$$

where $\dotplus$ is upper addition extending the usual addition to act from $[-\infty, +\infty]^2$ to $[-\infty, +\infty]$, so that $(+\infty) \dotplus (-\infty) = +\infty$. As a general reference on convexity theory we mention Rockafellar [12].

There is a duality theorem connecting the relative order and type via the Fenchel transform.

THEOREM 4.1 (Kiselman [8], Theorem 4.3). *Let E be a real vector space and E' a linear subspace of E^*. Assume that $f, g \in \mathcal{F}(E, E')$. Then*

$$\operatorname{order}(\widetilde{g} : \widetilde{f}) = \operatorname{type}(f : g) \quad \textit{and} \quad \operatorname{type}(\widetilde{g} : \widetilde{f}) = \operatorname{order}(f : g). \tag{4.4}$$

R e m a r k. In view of (4.3) and a simple calculation the assumption that $g \in \mathcal{F}(E, E')$ is superfluous. That is, if $f \in \mathcal{F}(E, E')$, then

$$\text{order}(f : \widetilde{\widetilde{g}}) = \text{order}(f : g) \quad \text{and} \quad \text{type}(f : \widetilde{\widetilde{g}}) = \text{type}(f : g). \tag{4.5}$$

The shortest formulation of Theorem 4.1 is $\text{order}(\widetilde{f} : g) = \text{type}(\widetilde{g} : f)$, for all functions f, g.

We will use this theorem to derive a duality between the growth and coefficient functions. We begin by defining the function K as

$$K(t) = \begin{cases} -\log(1 - e^t), & t < 0, \\ +\infty, & t \geq 0, \end{cases} \tag{4.6}$$

and the function K_n as

$$K_n(\xi) = K(\xi_1) + \ldots + K(\xi_n), \qquad \xi \in \mathbb{R}^n, \tag{4.7}$$

with K defined by (4.6). Then we have the following theorems.

THEOREM 4.2 ([8], Theorem 6.1). *Let $F \in \mathcal{O}(\mathbb{C}^n)$ be an entire function. Define f, p by* (3.1) *and* (3.5) *respectively and K by* (4.6). *Then*

$$\widetilde{p} \leq f \leq \widetilde{p} \,\square\, K \qquad \text{on } \mathbb{R}. \tag{4.8}$$

The first inequality is derived from Cauchy's inequalities and the second from the usual upper bound for a series by taking the sum of the modulus of the terms.

COROLLARY 4.3 ([8], Corollary 6.5). *Let F, G be two entire functions in $\mathbb{C}^n$. Let f, g be their growth functions defined by* (3.1) *and p, q be their coefficient functions defined by* (3.5). *Then*

$$\text{order}(f : g) = \text{order}(\widetilde{p} : \widetilde{q}) = \text{type}(\widetilde{\widetilde{q}} : p). \tag{4.9}$$

THEOREM 4.4 ([8], Theorem 6.6). *Let F be an entire function in $\mathbb{C}^n$. Define a, f_{r} by* (3.7), (3.2) *respectively and K_n by* (4.7). *Then*

$$\widetilde{a} \leq f_{\mathrm{r}} \leq \widetilde{a} \,\square\, K_n \qquad \text{on } \mathbb{R}^n. \tag{4.10}$$

COROLLARY 4.5 (Halvarsson [3], Corollary 7.2). *Let F, G be two entire functions in $\mathbb{C}^n$. Let $f_{\mathrm{r}}, g_{\mathrm{r}}$ be defined by* (3.2) *and a, b by* (3.7), *with F, G respectively. Let $E' = \mathbb{R}^n$ in the definition of the Fenchel transform. Then*

$$\text{order}(f_{\mathrm{r}} : g_{\mathrm{r}}) = \text{order}(\widetilde{a} : \widetilde{b}) = \text{type}(\widetilde{\widetilde{b}} : a). \tag{4.11}$$

COROLLARY 4.6. *Let F, G be two entire functions in $\mathbb{C}^n$. Let f, g be defined by* (3.1) *and a, b by* (3.7), *with F, G respectively. Let L be the linear hull of* $(1, 1, \ldots, 1)$. *Then*

$$\text{order}(f : g) = \text{order}(\widehat{a} : \widehat{b}) = \text{type}(\widehat{\widehat{b}} : a), \tag{4.12}$$

where $\widehat{a}$ indicates the Fenchel transform using $E' = L$.

Proof. We note that $f_{\mathrm{r}}(t,t,\ldots,t)=f(t)$ if we use the maximum norm in (3.1). Also, $\widehat{a}(t,t,\ldots,t)=\widetilde{a}(t,t,\ldots,t)$. The proof is now similar to that of Corollaries 4.3 and 4.5. See also the proof of Theorem 6.2 below.

We see that $\widehat{\widehat{a}}$ is constant on hyperplanes orthogonal to $(1,1,\ldots,1)$ and

$$\widehat{\widehat{a}}(s,s,\ldots,s)=\widetilde{\widetilde{m}}(ns),\qquad s\in\mathbb{R}, \tag{4.13}$$

where we define $m(j)=\min_{|k|=j}a(k)$ for $j\in\mathbb{N}$ and $m(j)=+\infty$ otherwise. Moreover, $\widehat{\widehat{a}}\le\widetilde{\widetilde{a}}$.

We will frequently and sometimes tacitly use the easily derived conditions

$$\frac{p(j)}{j}\to+\infty,\ j\to+\infty\Leftrightarrow F\in\mathcal{O}(\mathbb{C}^n) \Leftrightarrow\frac{a(k)}{|k|}\to+\infty,|k|=\sum_{i=1}^{n}k_i\to+\infty, \tag{4.14}$$

for the coefficient and refined coefficient function of F. As a consequence $p(j)=\widetilde{\widetilde{p}}(j)$ in a sequence of points tending to infinity and if we redefine p to $+\infty$ at all points where we have inequality then $\widetilde{\widetilde{p}}$ is unchanged. Similar statements hold for a and $\widetilde{\widetilde{a}}$, $\widehat{\widehat{a}}$. (See for instance Halvarsson [3], Lemma 5.3, Lemma 5.4 and Lemma 7.3.) Also, if $q(j)=\beta_j p(j)$ for all j and some non-zero β_j tending to β as $j\to+\infty$ (or $b(k)=\beta_k a(k)$ for non-zero $\beta_k\to\beta$ as $|k|\to+\infty$) we get $\mathrm{type}(\widetilde{\widetilde{q}}:\widetilde{\widetilde{p}})=\beta$ (or $\mathrm{type}(\widetilde{\widetilde{b}}:\widetilde{\widetilde{a}})=\mathrm{type}(\widehat{\widehat{b}}:\widehat{\widehat{a}})=\beta$). This is by the way a special case of Theorem 5.3 below, using $u_k=\exp((1-\beta_k)a(k))$.

If g is an arbitrary function in $\mathcal{F}(\mathbb{R},\mathbb{R})$ (i.e. $g=\widetilde{\widetilde{g}}$), then it follows from Theorem 4.2 that for f a growth function,

$$\mathrm{order}(f:g)=\mathrm{order}(\widetilde{p}:g)=\mathrm{type}(\widetilde{g}:p). \tag{4.15}$$

This is [8], Corollary 6.4. As a corollary to Theorem 4.4 we similarly get, by defining $g_n(t,\ldots,t)=g(t)$,

$$\mathrm{order}(f:g)=\mathrm{order}(\widehat{a}:g_n)=\mathrm{type}(\widehat{g}_n:a). \tag{4.16}$$

We have $\widehat{g}_n(k)=\widetilde{g}(\sum k_i)$. For $\mathrm{order}(g:f)$ we just switch the arguments in (4.15) and (4.16). It is natural to assume that g is real-valued. From the definition of the Fenchel transformation it follows that if g is also increasing faster than any linear function and is bounded from below then $\widetilde{g}$ is $+\infty$ for negative arguments (since g is increasing), real-valued for non-negative arguments (since g grows faster than any linear function and is bounded from below) and has faster growth than any linear function (since g is real-valued). Therefore we can find for such g by (4.14) an entire function F such that $p(j)=\widetilde{\widetilde{p}}(j)=\widetilde{g}(j)$, $j\in\mathbb{N}$ (or $a(k)=\widehat{\widehat{a}}(k)=\widetilde{g}(\sum k_i)$, $k\in\mathbb{N}^n$). For each growth function h we will then have $\mathrm{order}(h:f)=\mathrm{order}(h:g)$. This follows since if r is the coefficient function that goes with h then we have $\operatorname{order}(h : f) = \operatorname{type}(\widetilde{\widetilde{p}} : r)$ and $\operatorname{order}(h : g) = \operatorname{type}(\widetilde{g} : r)$, but $\widetilde{\widetilde{p}}$ and $\widetilde{g}$ coincide on the set where r is finite, so the two types are the same. If $g_{\mathrm{r}} : \mathbb{R}^n \to \mathbb{R}$ is convex, grows faster than any linear function and is bounded from below then by a similar discussion if we define an entire function F such that its refined coefficient function a satisfies $\widetilde{\widetilde{a}}(k) = \widetilde{g}_{\mathrm{r}}(k)$, $k \in \mathbb{N}^n$, then $\operatorname{order}(h_{\mathrm{r}} : f_{\mathrm{r}}) = \operatorname{order}(h_{\mathrm{r}} : g_{\mathrm{r}})$ for all entire functions H.

PROPOSITION 4.7. *Let $g : \mathbb{R} \to \mathbb{R}$ and $g' : \mathbb{R}^n \to \mathbb{R}$ be two convex functions which are bounded from below and increasing faster than any linear function. Then there exists an entire function $F \in \mathcal{O}(\mathbb{C}^n)$ such that for all entire H,* $\operatorname{order}(H : F) = \operatorname{order}(H : g)$, *and an entire function F' such that for all entire H its refined growth function h_{r} satisfies* $\operatorname{order}(h_{\mathrm{r}} : f'_{\mathrm{r}}) = \operatorname{order}(h_{\mathrm{r}} : g')$, *where f'_{r} is the refined growth function of F'. The function F can be constructed by putting $p(j) = \widetilde{g}(j)$, $j \in \mathbb{N}$, or $a(k) = \widetilde{g}(\sum k_i)$, $k \in \mathbb{N}^n$, and the function F' by putting $a'(k) = \widetilde{g}'(k)$, $k \in \mathbb{N}^n$, with p the growth function of F and a, a' the refined growth functions of F, F' respectively.*

Proof. Already done.

We will see later (in Theorem 6.2, or more directly in its proof), as the reader might believe anyway, that the proposition is true also for a supremum of growth functions $\sup_x h_x$, but we do not need this fact yet.

Now if we have found an entire function F as in the proposition, is it true that also $\operatorname{order}(f : h) = \operatorname{order}(g : h)$ for all growth functions h? This holds if there exists a coefficient function p such that $\operatorname{type}(\widetilde{\widetilde{p}} : \widetilde{g}) = \operatorname{type}(\widetilde{g} : \widetilde{\widetilde{p}}) = 1$, but not if we have $\operatorname{type}(\widetilde{\widetilde{p}} : \widetilde{g}) > 1$. If for example $g(t) = C_\alpha t_+^\alpha$, $C_\alpha > 0$, $\alpha > 1$, then $\widetilde{g}(\tau) = D_\alpha \tau^{\alpha/(\alpha-1)}$ for $\tau \ge 0$ and $\widetilde{g}(\tau) = +\infty$ for $\tau < 0$, and we can find a coefficient function p such that both of the types equal one, but this is not the case for $g(t) = t(\log t - 1)$ for $t > 1$ and $g(t) = -1$ for $t \le 1$ when $\widetilde{g}(\tau) = e^\tau$ for $\tau \ge 0$ and $\widetilde{g}(\tau) = +\infty$ for $\tau < 0$. It is enough to check this for the function p in the proposition. See also Kiselman [8], Theorem 9.3.

We can characterize those functions G for which for all F, $\operatorname{order}(f : g) = \operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}})$. For any set A we define its *indicator function* i_A as

$$i_A(x) = \begin{cases} 0, & x \in A, \\ +\infty, & x \notin A. \end{cases} \tag{4.17}$$

We denote by $\operatorname{dom} \phi$ the *effective domain* of a function ϕ with values in $[-\infty, +\infty]$, that is, the set of all points x such that $\phi(x) < +\infty$.

PROPOSITION 4.8. *Let $G \in \mathcal{O}(\mathbb{C}^n)$ and define g by* (3.1), *g_{r} by* (3.2) *and b by* (3.7). *Let A be the convex hull of a set in $\mathbb{N}^n$ and let i_A be the indicator* *function of A. If* $\operatorname{type}(\widetilde{\widetilde{b}} : \widehat{\widehat{b}} + i_A) = 1$ *then* $\operatorname{order}(f : g) = \operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}})$ *for every entire function F with Taylor coefficients A_k which are non-zero only for $k \in A$. If* $\operatorname{type}(\widetilde{\widetilde{b}} : \widehat{\widehat{b}} + i_A) > 1$ *then there exists a function $H \in \mathcal{O}(\mathbb{C}^n)$ with Taylor coefficients C_k which are non-zero only for $k \in A$ such that* $\operatorname{order}(h : g) < \operatorname{order}(h_{\mathrm{r}} : g_{\mathrm{r}})$.

Proof. First assume that $\operatorname{type}(\widetilde{\widetilde{b}} : \widehat{\widehat{b}} + i_A) = 1$ and let F have non-zero Taylor coefficients only in A. We already know that $\operatorname{order}(f : g) \le \operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}})$. To see the opposite inequality we use Corollary 4.5 and the submultiplicativity of the type:

$$\begin{aligned} \operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}}) &= \operatorname{type}(\widetilde{\widetilde{b}} : \widetilde{\widetilde{a}}) \\ &\le \operatorname{type}(\widetilde{\widetilde{b}} : \widehat{\widehat{b}} + i_A)\operatorname{type}(\widehat{\widehat{b}} + i_A : \widehat{\widehat{a}} + i_A)\operatorname{type}(\widehat{\widehat{a}} + i_A : \widetilde{\widetilde{a}}); \end{aligned} \tag{4.18}$$

we will see that the right-hand side of (4.18) equals $\operatorname{order}(f : g)$. By assumption $\operatorname{type}(\widetilde{\widetilde{b}} : \widehat{\widehat{b}} + i_A) = 1$. This also implies that A is unbounded. Otherwise we would have $\operatorname{type}(\widetilde{\widetilde{b}} : \widehat{\widehat{b}} + i_A) = 0$. Since $\widehat{\widehat{a}}, \widehat{\widehat{b}}$ are constant on hyperplanes orthogonal to $(1, 1, \ldots, 1)$ and A is unbounded we have, by Corollary 4.6, $\operatorname{type}(\widehat{\widehat{b}} + i_A : \widehat{\widehat{a}} + i_A) = \operatorname{order}(f : g)$. It follows from the fact that $\widehat{\widehat{a}} \le \widetilde{\widetilde{a}}$ and from our relations in (4.14) that $\operatorname{type}(\widehat{\widehat{a}} : \widetilde{\widetilde{a}}) = 1$, unless F is a polynomial, but in this case we anyway have $\operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}}) = \operatorname{order}(f : g) = 0$, so if i_A is finite (zero) in the set where $\widetilde{\widetilde{a}}$ is finite, i.e. $\operatorname{dom} \widetilde{\widetilde{a}} \subset A$, then we are done. But this is the case since F was assumed to have non-zero Taylor coefficients only in A. (Actually, $\operatorname{dom} \widetilde{\widetilde{a}}$ equals the convex hull of those points in $\mathbb{N}^n$ for which the Taylor coefficients of F are non-zero; see the proof of Halvarsson [3], Proposition 7.9.) Now assume that $\operatorname{type}(\widetilde{\widetilde{b}} : \widehat{\widehat{b}} + i_A) > 1$. If we define a function H such that $H(z) = \sum_k C_k z^k$, with $c(k) = \widehat{\widehat{b}}(k)$ on $A \cap \mathbb{N}^n$ and $c(k) = +\infty$ otherwise, it will certainly be entire and we will get $\operatorname{type}(\widetilde{\widetilde{b}} : \widetilde{\widetilde{c}}) > 1$ by construction. Moreover, we get $\operatorname{type}(\widehat{\widehat{b}} : \widehat{\widehat{c}}) = 1$. By Corollaries 4.5 and 4.6 we then get $1 = \operatorname{order}(h : g) < \operatorname{order}(h_{\mathrm{r}} : g_{\mathrm{r}})$.

Note that in view of Corollary 4.5 we must have $\operatorname{dom} \widetilde{\widetilde{a}} \subset \operatorname{dom} \widetilde{\widetilde{b}}$ if $\operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}}) < +\infty$. The condition $\operatorname{type}(\widetilde{\widetilde{b}} : \widehat{\widehat{b}} + i_{\mathbb{R}^n_+}) = 1$ is always satisfied for instance by functions $G = \sum_{i=1}^n G_i(z_i)$ if $\operatorname{order}(G_i : G_j) = 1$ for all i, j. If $F = \sum_{i=1}^{n-1} G_i(z_i)$ we get $\operatorname{order}(F : G) = \operatorname{order}(G : F) = \operatorname{order}(f_{\mathrm{r}} : g_{\mathrm{r}}) = 1$ but $\operatorname{order}(g_{\mathrm{r}} : f_{\mathrm{r}}) = +\infty$, which can be seen by fixing the last variable or the first $n-1$ variables respectively. Thus $\operatorname{order}(g_{\mathrm{r}} : f_{\mathrm{r}}) \ne \operatorname{order}(G : F)$ even though $\operatorname{type}(\widetilde{\widetilde{b}} : \widehat{\widehat{b}} + i_{\mathbb{R}^n_+}) = 1$.

5. Plurisubharmonicity. In the following f will always denote the growth function of the entire function F and h_w, $w \in \Omega$, will denote the partial growth function of $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$, unless otherwise stated.

As an introduction we will make a simple construction.

THEOREM 5.1. *Let $F \in \mathcal{O}(\mathbb{C}^n)$ be a transcendental entire function and $u \in \mathcal{O}(\Omega)$ be a holomorphic function on some analytic manifold Ω such that $|u(w)| < 1$. Then there exists a holomorphic function $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$ such that*

$$-\operatorname{order}(h_w : f)^{-1} = -\operatorname{order}(f : h_w) = \log|u(w)| \tag{5.1}$$

and $H(z,w) = F(z)$ at all points where $u(w) = e^{-1}$.

Proof. Put

$$H(z,w) = \sum_j P_j(z)(eu(w))^{m_j},$$

where m_j is equal to the integer part of $\max(p(j), 1)$ for $p(j) < +\infty$ and zero otherwise. If we denote the partial coefficient function of H by r_w we have

$$r_w(j) = p(j) - m_j \log|eu(w)| = -p(j)\log|u(w)| + \theta_j \log|eu(w)|, \tag{5.2}$$

for j so large that $p(j) > 1$, where $0 \le \theta_j < 1$. To see that H is holomorphic we must show that $|P_j(z)(eu(w))^{m_j}| \to 0$ as $j \to +\infty$ on compact subsets of $\mathbb{C}^n \times \Omega$. Using the homogeneity of the polynomials $\{P_j\}$ we see that this will happen if and only if $r_w(j)/j \to +\infty$ as $j \to +\infty$ locally uniformly in Ω. Since F is entire $p(j)/j \to +\infty$ and since $|u(w)| < 1$ the series defining H will converge locally uniformly. When $u \neq 0$, we also get, by the discussion preceding Proposition 4.7,

$$\operatorname{type}(\widetilde{\widetilde{p}} : \widetilde{\widetilde{r}}_w) = -\log|u(w)|^{-1}, \qquad \operatorname{type}(\widetilde{\widetilde{r}}_w : \widetilde{\widetilde{p}}) = -\log|u(w)|. \tag{5.3}$$

Using Corollary 4.3 we get the desired orders, since if $u(w) = 0$ then $H(z,w) = 0$ and $\operatorname{order}(f : h_w) = +\infty$, $\operatorname{order}(h_w : f) = 0$.

Although in this construction we have $\operatorname{order}(f : h_w) = \operatorname{order}(h_w : f)^{-1}$ this is not true in general. With f the exponential function, $1/\operatorname{order}(f : h_w)$ corresponds to the classical lower order of H_w. See Kiselman [8] for a discussion also involving Whittaker's decomposition theorem.

We see that $-1/\operatorname{order}(h_w : f) = -\operatorname{order}(f : h_w)$ is a negative plurisubharmonic function of w, pluriharmonic for w such that $u(w) \neq 0$. In general we know that the upper regularization

$$\left(w \mapsto \frac{-1}{\operatorname{order}(h_w : g)}\right)^* \tag{5.4}$$

is plurisubharmonic for an arbitrary increasing convex function g which has faster growth than any linear function (Kiselman [7], Theorem 4.1). (This is also true for another kind of relative order; see Lelong [10], Theorem 6.6.2.) Can it be any negative plurisubharmonic function? This is true if we allow h to be not just a partial growth function but any plurisubharmonic function on $\Omega\times\mathbb{C}$ with $h_w(t) = h_w(\operatorname{Re} t)$, $t \in \mathbb{C}$ (Kiselman [7], Theorem 4.2). Now let $H(z,w) = \sum_j P_j(z)u_j(w)$ be a function such that $F = \sum_j P_j$ is a transcendental entire function. We may assume that $p(j) = \widetilde{\widetilde{p}}(j)$ for all $j \in \mathbb{N}$. By Corollary 4.3 this will have no effect on the relative order. The partial coefficient function of H will then be

$$r_w = p(j) - \log|u_j(w)| = p(j)\left(1 - \frac{1}{p(j)}\log|u_j(w)|\right).$$

Thus we get for all $\varepsilon > 0$ the lower bound

$$\widetilde{\widetilde{r}}_w(t) \geq \widetilde{\widetilde{p}}(t) \liminf_{j\to+\infty}\left(1 - \frac{1}{p(j)}\log|u_j| - \varepsilon\right), \qquad \forall t > N,$$

where N depends on ε. If the lower limit happens to be $+\infty$ we replace it by a positive number R_N, which is increasing and tends to $+\infty$ with N. Since $p(j) = \widetilde{\widetilde{p}}(j)$ on $\mathbb{N}$, there exists no larger lower bound, hence $\operatorname{type}(\widetilde{\widetilde{p}} : \widetilde{\widetilde{r}}_w) = 1/(1 - \limsup_{j\to+\infty}(1/p(j))\log|u_j(w)|)$ and therefore

$$\frac{-1}{\operatorname{order}(h_w : f)} = \limsup_{j\to+\infty}\frac{1}{p(j)}\log|u_j(w)| - 1. \tag{5.5}$$

The conditions on $\{u_j\}$ for H to be entire in each w make (5.5) non-positive. In the general case $H(z,w) = \sum_{k\in\mathbb{N}^n} A_k z^k u_k(w)$. This case can be treated similarly assuming $a(k) = \widehat{\widehat{a}}(k)$, which by Corollary 4.6 does not alter the order. The only difference in the result is an exchange of j to k and $p(j)$ to $a(k)$. This is hence a new proof of Kiselman's result. (Recall also Proposition 4.7.) We state this as a theorem:

THEOREM 5.2. *Let $\mathbb{C}^n \times \Omega \ni (z,w) \mapsto H(z,w)$ be a function which for each w is an entire function*:

$$H(z,w) = \sum_{k\in\mathbb{N}^n} C_k(w)z^k, \tag{5.6}$$

and let F be a transcendental entire function satisfying $F(0) \neq 0$. Then

$$\begin{aligned}\frac{-1}{\operatorname{order}(h_w : f)} &= \limsup_{|k|\to+\infty}\frac{1}{\widehat{\widehat{a}}(k)}\log|u_k(w)| - 1\\ &= \limsup_{|k|\to+\infty}\frac{1}{\widehat{\widehat{a}}(k)}\log|C_k(w)|,\end{aligned} \tag{5.7}$$

where $u_k = C_k/\exp(-\widehat{\widehat{a}}(k))$ and $\widehat{\widehat{a}}$ is the twofold Fenchel transform of the refined coefficient function of F as in Corollary 4.6. If H is in addition *holomorphic then*

$$\left(w \mapsto \frac{-1}{\mathrm{order}(h_w : f)}\right)^*$$

is plurisubharmonic in Ω and if Ω is pseudoconvex this can be any non-positive plurisubharmonic function including $-\infty$ identically.

Proof. The first part is already done. The conditions on F just prevent the order from being $+\infty$ trivially. We could also have deduced this part directly from Corollary 4.6. It is a fact that every plurisubharmonic function on a pseudoconvex domain can be expressed as the upper regularization of

$$w \mapsto \limsup_{\nu \to +\infty} \frac{1}{\nu} \log |u_\nu(w)|, \tag{5.8}$$

for some sequence $\{u_\nu\}$ of holomorphic functions (Bremermann [2]). If we use (4.14) we see that this applies to (5.7). On the other hand, in the same reference it is shown that there exist domains where there are plurisubharmonic functions which cannot be expressed by (5.7). This depends on the fact that the functions in (5.7) can be extended to the envelope of holomorphy of the domain, whereas not all plurisubharmonic functions can. For a nice description of this in non-convex tubular domains see Lelong [11].

A natural way to construct functions $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$ with orders relative to F satisfying some condition is to multiply the homogeneous polynomials of F by holomorphic functions in such a way that the partial coefficient function $r_w(j)$ equals $\beta_j(w)p(j)$, where β_j tends to some limit function β as $j \to +\infty$. As seen by the following theorem the limit function will be very special.

THEOREM 5.3. *Assume $\Omega \subset \mathbb{C}^m$ is simply connected*, $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$, $H(z,w) = \sum_{k \in \mathbb{N}^n} A_k z^k u_k(w)$, *where $F(z) = \sum_{k \in \mathbb{N}^n} A_k z^k$ is an entire function and $u_k \in \mathcal{O}(\Omega)$, $u_k \neq 0$ everywhere. If*

$$m(w) = \lim_{\substack{k \to \infty \\ a(k) \neq +\infty}} |u_k(w)|^{1/a(k)} \tag{5.9}$$

exists for all $w \in \Omega$, then m is the modulus of a holomorphic function $u \in \mathcal{O}(\Omega)$. Either $0 < |u| < e$ or u is identically zero or u is a constant of modulus e and

$$\mathrm{order}(h_w : f)^{-1} = \mathrm{order}(f : h_w) = 1 - \log m(w). \tag{5.10}$$

Proof. Since Ω is simply connected and $u_k \neq 0$ everywhere there exist holomorphic roots

$$u'_k = u_k^{1/[a(k)]} \tag{5.11}$$

(if $0 < [a(k)] < +\infty$), where $[\cdot]$ denotes the integer part. For the partial refined coefficient function of H we have

$$r_w(j) = a(k) - \log|u_k(w)| = a(k) - [a(k)]\log|u'_k(w)|, \tag{5.12}$$

if $|k|$ is large and $a(k) \neq +\infty$. We see that $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$ only if

$$\lim_{k\to\infty} \log|u'_k| \le 1$$

locally uniformly so that $\{u'_k\}$ is a normal family. By the Arzelà–Ascoli theorem there exists a locally uniformly convergent subsequence of $\{u'_k\}$ so that the limit function m is the modulus of a holomorphic function u and since $u'_k \neq 0$ either $u \neq 0$ or u is identically zero. Also, by the maximum principle either $|u| < e$ or u is of modulus e identically.

Remark. Equation (5.10) holds also if we take the order between the refined growth functions of F and $H(\cdot, w)$. The case when $H(z,w) = \sum P_j(z)u_j(w)$, $\sum P_j = F$, can be treated similarly. (It is the case when $u_k = u_j$, $|k| = j$.) The conclusion of the theorem also holds true for $H(z,w) = \sum_k \exp(-\widehat{\widehat{a}}(k))z^k u_k(w)$, where the sum is taken over all $k \in \mathbb{N}^n$ such that $A_k \neq 0$, if we replace a by $\widehat{\widehat{a}}$ in (5.9). For the refined order this is true if we use $\widetilde{\widetilde{a}}$ instead of $\widehat{\widehat{a}}$ in this definition of H and in (5.9).

Corollary 5.4. *Let Ω be a connected analytic manifold. Assume $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$ can be written as $H(z,w) = \sum_{k\in\mathbb{N}^n} A_k z^k u_k(w)$, where $\{A_k\}$ are coefficients such that $\sum_k A_k z^k = F(z)$, $F \in \mathcal{O}(\mathbb{C}^n)$ and $u_k \in \mathcal{O}(\Omega)$. Let α be the set of all points in Ω such that for all neighbourhoods of the point infinitely many of the functions u_k have a zero and let $F(0) \neq 0$, $u_0 \neq 0$. If*

$$m(w) = \lim_{\substack{k\to\infty \\ a(k)\neq+\infty}} |u_k(w)|^{1/a(k)} \tag{5.13}$$

exists and $0 < m(w) < e$ for all $w \in \Omega \setminus \alpha$, then

$$\operatorname{order}(h_w : f)^{-1} = \operatorname{order}(f : h_w) = 1 - \log m(w)$$

is a positive pluriharmonic function on $\Omega \setminus \alpha$.

Proof. Let w be a point in $\Omega \setminus \alpha$. Then there exists a neighbourhood of w where only a finite number of the functions u_k have a zero. We can take this neighbourhood small enough to be able to work in a coordinate patch. The result now follows from Theorem 5.3 after the observation that if only a finite number of the functions $\{u_k\}$ have a zero at a point this will not affect the order since by assumption $A_0u_0(w) \neq 0$ for all $w \in \Omega$.

We remark that in the refined case to prevent $\operatorname{order}(f_{\mathrm{r}} : h_{\mathrm{r},w})$ from being $+\infty$ trivially we must make more assumptions than just $A_0u_0 \neq 0$. See the discussion just before equation (3.3).

It can of course happen that $\alpha = \Omega$. We also make the following observation:

COROLLARY 5.5. *Assume that the hypothesis of Corollary* 5.4 *holds. Then also* $\operatorname{order}(h_w : h_{w_0})^{-1} = \operatorname{order}(h_{w_0} : h_w)$ *for all* $w, w_0 \in \Omega \setminus \alpha$ *and this is a pluriharmonic function of* w. *Consequently*, *we see that* $\operatorname{order}(h_w : g)^{-1} = \operatorname{order}(h_w : h_{w_0})^{-1} \operatorname{order}(h_{w_0} : g)^{-1}$ *and also that* $\operatorname{order}(g : h_w) = \operatorname{order}(g : h_{w_0}) \operatorname{order}(h_{w_0} : h_w)$ *for all* $w, w_0 \in \Omega \setminus \alpha$ *and these are pluriharmonic functions of* w *for any function* g.

Proof. We rewrite H as

$$H(z,w) = \sum_k B_k z^k v_k(w),$$

where $\sum_k B_k z^k = G \in \mathcal{O}(\mathbb{C}^n)$ and $v_k \in \mathcal{O}(\Omega)$ satisfies $v_k(w_0) = 1$. We then have $B_k = A_k \exp(-\beta_k a(k) - i\phi_k)$ and $v_k(w) = u_k(w) \exp(\beta_k a(k) + i\phi_k)$ for some real numbers β_k, ϕ_k. If we consider

$$\begin{aligned} |v_k(w)|^{1/b(k)} &= |u_k(w) e^{\beta_k a(k)}|^{1/((1+\beta_k)a(k))} \\ &= e\biggl(\frac{|u_k(w)|^{1/a(k)}}{e}\biggr)^{1/(1+\beta_k)}, \end{aligned} \tag{5.14}$$

we see that since $v_k(w_0) = 1$ and since the limit in (5.13) exists and is strictly between 0 and e, β_k will tend to some finite number $\beta > -1$. From this we see that also the limit in (5.14) exists for all $w \in \Omega \setminus \alpha$ as $k \to \infty$ and we can apply Corollary 5.4 using $G = H(\cdot, w_0)$ and $\{v_k\}$ instead of F and $\{u_k\}$ to conclude that $\operatorname{order}(h_w : h_{w_0})^{-1} = \operatorname{order}(h_{w_0} : h_w)$ and that this is a pluriharmonic function. The last statement of the theorem follows by the submultiplicativity of the order. For instance,

$$\begin{aligned} \operatorname{order}(h_w : g) &\le \operatorname{order}(h_w : h_{w_0}) \operatorname{order}(h_{w_0} : g) \\ &\le \operatorname{order}(h_w : h_{w_0}) \operatorname{order}(h_{w_0} : h_w) \operatorname{order}(h_w : g) \\ &= \operatorname{order}(h_w : g). \end{aligned}$$

Note that in general $\operatorname{order}(h_{w_0} : g) \operatorname{order}(g : h_{w_0}) > 1$.

We give some examples of (pluri-)harmonicity which will also be used as motivation for the next section.

EXAMPLE 5.6. We take $u_j(w) = \sin(\beta_j p(j) w)$, for $p(j) \neq +\infty$, where $\beta_j \to \beta \ge 0$ and $\Omega = \{w \in \mathbb{C} : |\operatorname{Im} w| < 1/\beta\}$. If $H(z,w) = \sum_j P_j(z) u_j(w)$ we get

$$\operatorname{order}(h_w : f)^{-1} = \operatorname{order}(f : h_w) = 1 - \beta |\operatorname{Im} w|, \qquad \operatorname{Im} w \neq 0,\ w \in \Omega.$$

In this case α is the whole real axis and we get harmonicity outside. Note that we cannot extend harmonically to any neighbourhood across α. We get $\operatorname{order}(h_w : f) \le 1$ for $\operatorname{Im} w = 0$, with a zero at the origin. If we take for instance $\beta_j = 1$ and $p(j) = \pi 2^j$ we get zeros on a countable dense subset of α. On the other hand, without any effort, using the local integrability of $\log|\sin s|$, $s \in \mathbb{R}$, and Fatou's Lemma on (5.5) we see that $\operatorname{order}(h_w : f) = 1$ almost everywhere $[ds]$. The reason for this is explained in Section 6.

EXAMPLE 5.7. We take

$$u_j(w) = w^{[\beta_j p(j)]} \pm \frac{1}{w^{[\beta_j p(j)]}}, \qquad \text{for } p(j) \neq +\infty,$$

where $\beta_j \to \beta \geq 0$ and

$$\Omega = \{w \in \mathbb{C} : e^{-1/\beta} < |w| < e^{1/\beta}\}.$$

If $H(z,w) = \sum_j P_j(z) u_j(w)$ we get

$$\operatorname{order}(h_w : f)^{-1} = \operatorname{order}(f : h_w) = \begin{cases} 1 - \beta \log|w|, & |w| > 1, \\ 1 + \beta \log|w|, & |w| < 1. \end{cases}$$

In this case α is the unit circle and we get harmonicity outside. We can make the same discussion as in Example 5.6.

6. Continuity and commutativity. It is natural to ask what continuity properties the functions $w \mapsto \operatorname{order}(h_w : f)$ and $w \mapsto \operatorname{order}(f : h_w)$ can have. We get some information from Theorem 5.2. We can also easily see that at zeros of the coefficients of H, $w \mapsto \operatorname{order}(h_w : f)$ can suddenly make a jump down and the other order can jump up. Can it go the other way? The answer is yes on the boundary of the domain of definition.

THEOREM 6.1. *For all $\sigma = \varrho^{-1} > 1$, and transcendental entire functions $F \in \mathcal{O}(\mathbb{C}^n)$, there exists a function*

$$H \in \mathcal{O}(\mathbb{C}^n \times \{w \in \mathbb{C} : \operatorname{Re} w < 1\}) \cap C(\mathbb{C}^n \times \{w \in \mathbb{C} : \operatorname{Re} w \leq 1\}) \tag{6.1}$$

such that $\operatorname{order}(h_w : f)^{-1} = \operatorname{order}(f : h_w) = \sigma$ *if* $\operatorname{Re} w < 1$, *but* $H(z,1) = F(z)$, $\operatorname{order}(h_w : f) = \operatorname{order}(f : h_w) = 1$ *if* $\operatorname{Re} w = 1$.

Proof. Expand F in homogeneous polynomials $F(z) = \sum_j P_j(z)$. Put

$$R_j(z,w) = \begin{cases} P_j(z) \|P_j\|^{\sigma - 1}(1 + 2^{-1} e^{\beta_j (w - \delta_j)}), & j \geq j_0, \\ P_j(z), & j < j_0, \end{cases} \tag{6.2}$$

where $\delta_j \nearrow 1$ as $j \to +\infty$ and

$$\beta_j = \begin{cases} \dfrac{\log(2(\|P_j\|^{1-\sigma} - 1))}{1 - \delta_j}, & \|P_j\| > 0, \\ 0, & \|P_j\| = 0. \end{cases} \tag{6.3}$$

We choose j_0 so large that (6.3) makes sense giving $\beta_j > 0$, $j \geq j_0$, and so that $R_j(z,w) = P_j(z)$ for the first non-zero polynomial. Then we have $R_j(z,1) = P_j(z)$, $\|R_j(\cdot,w)\| = \|P_j\| + O(\|P_j\|^\sigma)$ for $\operatorname{Re} w = 1$ and

$$\frac{1}{2}\|P_j\|^\sigma \leq \|R_j(\cdot,w)\| \leq \frac{3}{2}\|P_j\|^\sigma, \tag{6.4}$$

if $j \geq j_0$ and $\operatorname{Re} w < \delta_j$. If we now put $H(z,w) = \sum_j R_j(z,w)$ we get the desired orders. By construction (6.1) holds. That H is not holomorphic for $\operatorname{Re} w > 1$ can be shown by inspection if we choose some sequence $\{\delta_j\}$ but to see that we cannot make a clever choice we will show this by defining the function

$$h(t,s) = \sup_w (h_w(t) : w \in \Omega,\ |w| = e^s), \qquad s \in \mathbb{R}, \tag{6.5}$$

which is convex in (t,s). Now h extends continuously to $h(t,0)$ and since

$$\operatorname{order}(h(\cdot,0) : h(\cdot,-\delta)) = \sigma,$$

for all $\delta > 0$, $h(\cdot,s)$ cannot be real-valued for $s > 0$ by Kiselman [8], Theorem 7.2.

We see that the order makes a jump on the whole line $\operatorname{Re} w = 1$. Compare also with Example 5.6.

We will now generalize Corollary 4.3:

THEOREM 6.2. *Let $\{F_x\}_{x\in X}$ and $\{G_y\}_{y\in Y}$ be two families of entire functions in $\mathbb{C}^n$. Let f_x, p_x, a_x be the partial growth, coefficient and refined coefficient functions of F_x and g_y, q_y, b_y be the partial growth, coefficient and refined coefficient functions of G_y respectively. Assume for simplicity only that none of the families consists of polynomials of bounded degree and that both $\sup_x f_x$ and $\sup_y g_y$ are real-valued. Then*

$$\begin{aligned} &\operatorname{order}(\sup_{x\in X} f_x : \sup_{y\in Y} g_y) \\ &\qquad = \operatorname{order}(\sup_{x\in X} \widetilde{p}_x : \sup_{y\in Y} \widetilde{q}_y) = \operatorname{type}((\inf_{y\in Y} q_y)^{\approx} : \inf_{x\in X} p_x) \\ &\qquad = \operatorname{order}(\sup_{x\in X} \widehat{a}_x : \sup_{y\in Y} \widehat{b}_y) = \operatorname{type}((\inf_{y\in Y} b_y)^{\widehat{\ }} : \inf_{x\in X} a_x). \end{aligned} \tag{6.6}$$

Proof. We will show the first line of (6.6). The other line can be shown in a similar manner. By Theorem 4.2 we have for all $x \in X$, $\widetilde{p}_x \leq f_x \leq \widetilde{p}_x \square K$. This implies $\widetilde{p}_x(t) \leq f_x(t) \leq \widetilde{p}_x(t+1) + K(-1)$, which in turn implies

$$\sup_x \widetilde{p}_x(t) \leq \sup_x f_x(t) \leq \sup_x \widetilde{p}_x(t+1) + K(-1). \tag{6.7}$$

Now $\sup_x f_x$ is convex and if it is also real-valued then the order is translation invariant (Kiselman [8], Lemma 3.2). If $\{F_x\}$ does not consist of polynomials of bounded degree, $\sup_x f_x$ will grow faster than any linear function so that

$$\begin{aligned} 1 &= \operatorname{order}(\sup_x f_x : \sup_x f_x) \\ &\leq \operatorname{order}(\sup_x f_x : \sup_x \widetilde{p}_x) \operatorname{order}(\sup_x \widetilde{p}_x : \sup_x f_x) \leq 1, \end{aligned} \tag{6.8}$$

where the first inequality is submultiplicativity of the order and the second inequality comes from (6.7). Hence

$$\operatorname{order}(\sup_x f_x : \sup_x \widetilde{p}_x) = \operatorname{order}(\sup_x \widetilde{p}_x : \sup_x f_x) = 1.$$

We get a similar equality for the other family. By another submultiplicativity argument we get

$$\operatorname{order}(\sup_x f_x : \sup_y g_y) = \operatorname{order}(\sup_x \widetilde{p}_x : \sup_y \widetilde{q}_y). \tag{6.9}$$

We can easily deduce that $\sup_x \widetilde{p}_x = (\inf_x p_x)^{\sim}$. If we now apply Theorem 4.1 and the remark following it, we are done.

With the function h defined by (6.5) in mind we state

Theorem 6.3. *For all $\varrho > 1$ and transcendental $F \in \mathcal{O}(\mathbb{C}^n)$ there exists a function*

$$H \in \mathcal{O}(\mathbb{C}^n \times \{w \in \mathbb{C} : \operatorname{Re} w < 1\}), \tag{6.10}$$

which can be extended continuously as a non-tangential limit to $\mathbb{C}^n \times \{1\}$ such that $\operatorname{order}(h_w : h_{w'}) = 1$, *for all w, w' on the unit circle* $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$,

$$\operatorname{order}(\sup_{w\in\mathbb{T}} h_w : f) = \operatorname{order}(f : \sup_{w\in\mathbb{T}} h_w) = 1,$$

but

$$\operatorname{order}(\sup_{w\in\mathbb{T}} h_w : h_{w'}) = \operatorname{order}(h_{w'} : \sup_{w\in\mathbb{T}} h_w)^{-1} = \varrho, \qquad \forall w' \in \mathbb{T}. \tag{6.11}$$

Proof. Expand F in homogeneous polynomials as $F = \sum_j P_j$. Put

$$R_j(z,w) = \begin{cases} P_j(z)\|P_j\|^{\varrho-1}(1 + 2^{-1}e^{\beta_j(e^{i\phi_j}w - \delta_j)}), & j \ge j_0, \\ P_j(z), & j < j_0, \end{cases} \tag{6.12}$$

where

$$\beta_j = \begin{cases} \dfrac{\log(2(\|P_j\|^{1-\varrho} - 1))}{1-\delta_j}, & \|P_j\| > 0, \\ 0, & \|P_j\| = 0. \end{cases} \tag{6.13}$$

We choose j_0 so large so that β_j is defined and positive for $j \ge j_0$ and so that $R_j(z,w) = P_j(z)$ for the first non-zero polynomial. Let $\phi_j = 2^{-j}$ and $\delta_j = \cos(\phi_j/4) = \cos 2^{-j-2}$. Then we have $\inf_{w\in\mathbb{T}} r_w(j) = p(j)$ for all j, but

$$-\log\frac{3}{2} + \varrho p(j) \le r_w(j) \le \log 2 + \varrho p(j), \qquad \forall j > N(w),\ w \in \mathbb{T}. \tag{6.14}$$

We can now use Theorem 6.2 and go on as in the proof of Theorem 6.1.

We will now give the main continuity theorem. Discontinuities in the order can only arise if the Taylor coefficients have zeros.

THEOREM 6.4. *Assume $\Omega \subset \mathbb{C}^m$ to be simply connected. Let $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$ be a holomorphic function which is transcendental for fixed w in Ω and is given by*

$$H(z,w) = \sum_{k\in\mathbb{N}^n} C_k(w) z^k, \tag{6.15}$$

where $C_k \in \mathcal{O}(\Omega)$ are either non-zero or identically zero. Then

$$\lim_{w\to w_0} \operatorname{order}(h_w : h_{w_0}) = \lim_{w\to w_0} \operatorname{order}(h_{w_0} : h_w) = 1, \quad \forall w_0 \in \Omega. \tag{6.16}$$

As a direct consequence, for all $w_0 \in \Omega$ and any function g,

$$\lim_{w\to w_0} \operatorname{order}(h_w : g) = \operatorname{order}(h_{w_0} : g), \quad \lim_{w\to w_0} \operatorname{order}(g : h_w) = \operatorname{order}(g : h_{w_0}).$$

Proof. By assumption, for the non-zero terms we can write

$$C_k(w) = C_k(w_0)\frac{C_k(w)}{C_k(w_0)} = A_k u_k(w), \tag{6.17}$$

where $A_k = C_k(w_0)$ and $u_k(w) = C_k(w)/C_k(w_0)$. Trivially we see that $\sum_{k\in\mathbb{N}^n} A_k z^k$ defines an entire function and $\{u_k\}$ is a sequence of holomorphic functions in Ω. We then have for the refined coefficient function c_w of H for $|k|$ large enough and for $C_k \neq 0$,

$$c_w(k) = a(k) - \log|u_k(w)| = a(k) - [a(k)]\log|u'(w)|, \tag{6.18}$$

where $[a(k)]$ denotes the integer part of $a(k)$ and u'_k is an $[a(k)]$th holomorphic root of u_k. By holomorphy of H we must have

$$\frac{c_w(k)}{|k|} \to +\infty \quad \text{as } |k| \to +\infty, \tag{6.19}$$

uniformly on compact subsets of Ω. This happens only if

$$\limsup_{|k|\to+\infty} \log|u'_k(w)| \le 1, \tag{6.20}$$

locally uniformly. That is, $\{u'_k\}$ must be locally uniformly bounded and hence by the Cauchy integral formula be an equicontinuous family. Note that we tacitly assume k to avoid values for which $C_k = 0$. By (6.18) we have a lower bound for $\operatorname{type}(\widehat{\widehat{c}}_w : \widehat{\widehat{a}})$ as well as for $\operatorname{type}(\widehat{\widehat{a}} : \widehat{\widehat{c}}_w)^{-1}$ (and $\operatorname{type}(\widetilde{\widetilde{c}}_w : \widetilde{\widetilde{a}})$, $\operatorname{type}(\widetilde{\widetilde{c}}_w : \widetilde{\widetilde{a}})^{-1}$):

$$\begin{aligned} \operatorname{type}(\widehat{\widehat{c}}_w : \widehat{\widehat{a}}), \operatorname{type}(\widehat{\widehat{a}} : \widehat{\widehat{c}}_w)^{-1} &\ge \liminf_{|k|\to+\infty} (1 - \log|u'_k(w)|) \\ &= 1 - \log u^{(s)}(w), \end{aligned} \tag{6.21}$$

where

$$u^{(s)}(w) = \limsup_{|k|\to+\infty} |u_k'(w)|. \tag{6.22}$$

We also get an upper bound

$$\begin{aligned}\operatorname{type}(\widehat{\widehat{c}}_w : \widehat{\widehat{a}}),\ \operatorname{type}(\widehat{\widehat{a}} : \widehat{\widehat{c}}_w)^{-1} &\le \limsup_{|k|\to+\infty}(1 - \log|u_k'(w)|) \\ &= 1 - \log u^{(i)}(w),\end{aligned} \tag{6.23}$$

where

$$u^{(i)}(w) = \liminf_{|k|\to+\infty} |u_k'(w)|. \tag{6.24}$$

In general, $u^{(s)}$ and $u^{(i)}$ would be just semicontinuous but since $\{u_k'\}$ is an equicontinuous family $u^{(s)}$ and $u^{(i)}$ will be locally uniformly continuous functions. By construction they also satisfy $u^{(s)}(w_0) = u^{(i)}(w_0) = 1$. So by elementary calculus and Corollary 4.6 the theorem now follows.

Remark. We see using similar estimates and Corollary 4.5 that also the refined order is continuous under the hypotheses of Theorem 6.4.

Under the same conditions on the Taylor coefficients we see that the operations of taking supremum and relative order commute and as a preparation we state the following lemma:

Lemma 6.5. *Let I be a finite index set and $\{f_i\}_{i\in I}$, g be functions from a real vector space E to the extended real line $[-\infty, +\infty]$. If g is convex and $g(0) < +\infty$ then*

$$\operatorname{order}(\max_{i\in I} f_i : g) = \max_{i\in I} \operatorname{order}(f_i : g). \tag{6.25}$$

Proof. It is obvious that $\operatorname{order}(\max_i f_i : g) \ge \max_i \operatorname{order}(f_i : g)$ By the remark following Definition 2.1,

$$f_i(x) \le \frac{1}{a} g(ax) + c_{a,i}, \tag{6.26}$$

for all $i \in I$ and $x \in E$ if $a > \max_i \operatorname{order}(f_i : g)$. It follows that

$$\max_i f_i(x) \le \frac{1}{a} g(ax) + \max_i c_{a,i}. \tag{6.27}$$

Since $\max_i c_{a,i}$ is finite we are done.

Remark. We also have $\operatorname{order}(g : \max_i f_i) \le \min_i \operatorname{order}(g : f_i)$, but with inequality in general. We can for example take $g(x) = x^2$ and $f_1(x) = x_+^3$, $f_2(x) = (-x^3)_+$. Then $\max f_i(x) = |x|^3$ and $\operatorname{order}(g : f_i) = +\infty$ but $\operatorname{order}(g : \max f_i) = 0$.

Corollary 6.6. *Let Ω be an analytic manifold of dimension m. Let $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$ be given by*

$$H(z,w) = \sum_{k \in \mathbb{N}^n} C_k(w) z^k, \tag{6.28}$$

where except for a finite number of coefficients $C_k \in \mathcal{O}(\Omega)$ are either non-zero or identically zero. Then for any relatively compact set $K \subset \Omega$ and convex function g bounded from below and increasing faster than any linear function we have

$$\sup_{w \in K} \operatorname{order}(h_w : g) = \operatorname{order}(\sup_{w \in K} h_w : g). \tag{6.29}$$

Proof. The corollary is clearly true if $H(\cdot, w)$ is a polynomial for all $w \in K$. Assume therefore that $H(\cdot, w)$ is transcendental for some $w \in K$. We can cover K with a finite number of simply connected coordinate charts $\{\Omega_i\}_{i \in I}$ and decompose K in a finite union $K = \bigcup_{i \in I} K_i$, where $K_i \subset \Omega_i$. By Lemma 6.5 it then suffices to treat the case when Ω is a simply connected subdomain of $\mathbb{C}^m$. By Theorem 5.2 and Proposition 4.7,

$$\frac{-1}{\sup_{w \in K} \operatorname{order}(h_w : g)} = \sup_{w \in K} \limsup_{|k| \to +\infty} \frac{1}{\widetilde{g}(|k|)} \log |C_k(w)| \tag{6.30}$$

and by Theorem 6.2, Theorem 5.2 and Proposition 4.7,

$$\frac{-1}{\operatorname{order}(\sup_{w \in K} h_w : g)} = \limsup_{|k| \to +\infty} \frac{\sup_{w \in K} \log |C_k(w)|}{\widetilde{g}(|k|)}. \tag{6.31}$$

It follows that

$$\begin{aligned} \sup_{w \in K} \frac{-1}{\operatorname{order}(h_w : g)} &\le \frac{-1}{\operatorname{order}(\sup_{w \in K} h_w : g)} \\ &\le \sup_{w \in K} \left(\frac{-1}{\operatorname{order}(h_w : g)} \right)^*, \end{aligned} \tag{6.32}$$

where the last inequality follows from Hartogs' Lemma since the functions $\log |C_k(\cdot)| / \widetilde{g}(|k|)$ are locally uniformly bounded. But by Theorem 6.4 the function $w \mapsto \operatorname{order}(h_w : g)$ is continuous so we have equality all the way in (6.32).

If the conditions on H in the corollary are not satisfied we can get a counterexample. Let for instance $K = \{0\} \cup \bigcup_{j=1}^{\infty} \{w_j\}$ in $\mathbb{C}$, where $w_j = 1/j$. Let $F = \sum_j P_j$ be entire and let $\{u_j\} \subset \mathcal{O}(\Omega)$ be a set of uniformly bounded functions on $\Omega \subset \mathbb{C}$, $K \subset \Omega$, such that $u_0 = 1$ identically and the zero-sets for the other functions are $Z(u_m) = \{0\} \cup \bigcup_{j=1}^{m-1} \{w_j\}$. Define $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$

by $H(z,w)=\sum_j P_j(z)u_j(w)$. We then have, for $m\geq 0$,

$$H(z,w_m)=\sum_{j=0}^{m} u_j(w_m)P_j(z), \tag{6.33}$$

where $u_j(w_m)\neq 0$. For each $w\in K$, h_w will then be the growth function of a polynomial, but $\sup_{w\in K} h_w$ will be like the growth function of a transcendental function. If we take $g=\sup_{w\in K} h_w$ we will thus get

$$\sup_{w\in K} \operatorname{order}(h_w:g)=0, \quad \text{but} \quad \operatorname{order}(\sup_{w\in K} h_w:g)=1.$$

We see that in this example K is polar. Also, we see that the zeros accumulate at an infinite number of points.

It follows from the plurisubharmonicity and Bedford & Taylor [1], Corollary 7.3, that the set of points where $\operatorname{order}(h_w:g)\neq \operatorname{order}(h_w:g)^*$ is pluripolar. So if the set K is thick enough then supremum over K and relative order should commute. This is indeed the case. We recall that a set K is called *thin* at a point $p\in \overline{K}$ if there exists a plurisubharmonic function u such that

$$\limsup_{\substack{w\to p\\ w\in K\setminus\{p\}}} u(w)<u(p). \tag{6.34}$$

If a set is thin at all points of its closure then the set is called *thin*. A set is called *negligible* if it is of the form $\{\sup u_\alpha<(\sup u_\alpha)^*\}$ for a family of plurisubharmonic functions uniformly bounded from above. Negligible is the same as pluripolar and a thin set is always negligible. In one variable negligible sets are thin but this is not always the case in several variables.

In [3] and [4] we take the supremum over polycircles. These are thick sets of a type covered by the following theorem. See also Section 8.

THEOREM 6.7. *Let $\Omega\subset\mathbb{C}^m$ and let $H\in\mathcal{O}(\mathbb{C}^n\times\Omega)$ be given by* (6.28). *Then for any relatively compact set $K\subset\Omega$ of the form $K=K_1\times\ldots\times K_m$, where K_j are nowhere thin subsets of $\mathbb{C}$, and for any convex function g bounded from below and increasing faster than any linear function we have*

$$\sup_{w\in K} \operatorname{order}(h_w:g)=\operatorname{order}(\sup_{w\in K} h_w:g). \tag{6.35}$$

Proof. We show the case $m=2$. It is then easy to prove the general case by induction. So we have $K=K_1\times K_2$, where the sets K_1, K_2 are nowhere thin. Let $u_k=\log|C_k(w)|/\widetilde{g}(|k|)$. Then $\{u_k\}$ is a family of plurisubharmonic functions uniformly bounded from above and this is all that we shall need to know about them. For fixed $w_1\in K_1$ we have, as in (6.32),

$$\sup_{w_2\in K_2}\limsup_{k\to\infty} u_k(w_1,w_2) \le \limsup_{k\to\infty}\sup_{w_2\in K_2} u_k(w_1,w_2) \le \sup_{w_2\in K_2}(\limsup_{k\to\infty} u_k(w_1,w_2))^*, \tag{6.36}$$

where the star means upper regularization in the second variable. The set of points w_2 where $\limsup_{k\to\infty} u_k \neq (\limsup_{k\to\infty} u_k)^*$ is negligible, hence thin. The set which remains if we remove this set from K_2 must then be nowhere thin, because the union of two sets which are thin at a point is thin at the point. Hence we have equalities in (6.36). Next consider the inequalities

$$\begin{aligned}\sup_{w_1\in K_1}\limsup_{k\to\infty}\sup_{w_2\in K_2} u_k(w_1,w_2) &\le \limsup_{k\to\infty}\sup_{w_1\in K_1}(\sup_{w_2\in K_2} u_k(w_1,w_2))^* \\ &\le \sup_{w_1\in K_1}(\limsup_{k\to\infty}(\sup_{w_2\in K_2} u_k(w_1,w_2))^*)^*.\end{aligned} \tag{6.37}$$

We have

$$(\sup_{w_2\in K_2} u_k(w_1,w_2))^* = \sup_{w_2\in K_2} u_k(w_1,w_2), \tag{6.38}$$

except on a negligible set, and a countable union of negligible sets is negligible. Since K_1 is nowhere thin we get equality all the way in (6.37).

7. Commutativity, one variable. In the case of one variable we can weaken the hypotheses of Corollary 6.6.

We shall say that a family of uniformly bounded point-sets $\{\alpha_k\}_{k\in I}$, $I\subset\mathbb{N}^n$, $\alpha_k = \{\alpha_{kj}\}_{j\in I_k}\subset\mathbb{C}$, *has p as an accumulating point* if there exists an infinite subset $J\subset I$ such that for each $k\in J$ there is a point $\alpha_{kj}\in\alpha_k$ such that

$$\lim_{\substack{k\to\infty\\ k\in J}} |\alpha_{kj}-p| = 0. \tag{7.1}$$

If $\{\alpha_k\}$ has exactly one accumulating point p we say that $\{\alpha_k\}$ *tends to p*. This is then, by the uniform boundedness of the family, the same as

$$\lim_{k\to\infty}\sup_{j\in I_k} |\alpha_{kj}-p| = 0. \tag{7.2}$$

If the zero-sets of the Taylor coefficients of the function H in Corollary 6.6 tend to a point, we know that if we remove from the set K everything within a small circle centred at this point then the operations of taking supremum and order commute. As the example following the corollary shows we cannot deduce from this that the same is true if we take all of K. But again since the zero-sets accumulate at one point and nowhere else we should loosely speaking be away from this point if we want large order. This is the motivation for the theorem to come. By Lemma 6.5 the result can be extended to the case of finitely many accumulating points.

THEOREM 7.1. *Let Ω be a domain in $\mathbb{C}$ containing the closure of the unit disk $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$. Let $H \in \mathcal{O}(\mathbb{C}^n \times \Omega)$ be given by*

$$H(z, w) = \sum_{k \in \mathbb{N}^n} C_k(w) z^k, \tag{7.3}$$

where the Taylor coefficients $C_k \in \mathcal{O}(\Omega)$ which are not identically zero have zero-sets tending to the origin. Then for any relatively compact set $K \subset \mathbb{D}$ and convex function g bounded from below and increasing faster than any linear function we have

$$\sup_{w \in K} \operatorname{order}(h_w : g) = \operatorname{order}(\sup_{w \in K} h_w : g). \tag{7.4}$$

Proof. If $\operatorname{order}(\sup_{w \in K} h_w : g) = 0$ (which is the case for instance when $H(\cdot, w)$ is a polynomial for each w) then $\operatorname{order}(h_w : g) = 0$ for all w, so the theorem follows in this case. Assume therefore that $\operatorname{order}(\sup_{w \in K} h_w : g) > 0$. We may also assume that there are points in K which are arbitrarily close to and distinct from the origin. Otherwise we can use Corollary 6.6 and Lemma 6.5 if $0 \in K$ to obtain the theorem. We will use this assumption in the estimate (7.8). We recall the formula for $\operatorname{order}(\sup_{w \in K} h_w : g)$ in (6.31). To simplify the analysis we choose an index set $I \subset \mathbb{N}^n$ such that

$$\frac{-1}{\operatorname{order}(\sup_{w \in K} h_w : g)} = \varlimsup_{\substack{|k| \to +\infty \\ k \in I}} \sup_{w \in K} \frac{1}{\widetilde{g}(|k|)} \log |C_k(w)|. \tag{7.5}$$

Those C_k which are identically zero do not contribute to the order, so they need not be in I and in the following we shall ignore them. Now factorize C_k into $C_k = B_k v_k$, where B_k is a Blaschke product of the zeros of C_k in $\mathbb{D}$ and $v_k \in \mathcal{O}(\mathbb{D})$ is non-zero. We denote the zero-set of C_k in $\mathbb{D}$ by α_k not counting multiplicities and we shall index the zeros by α_{kj} and call the multiplicity of each zero N_{kj}, without specifying the finite index set to which j belongs. Then we have

$$B_k(w) = \prod_j B_{kj}(w)^{N_{kj}} = \prod_j \left(\frac{w - \alpha_{kj}}{1 - \overline{\alpha}_{kj} w} \right)^{N_{kj}}. \tag{7.6}$$

We omit here the usual unimodular constants in the Blaschke factors B_{kj} and agree that the product over the empty set is one. It is well known that $|B_k(w)| = 1$ on the unit circle $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$. We have

$$\text{(7.7)} \qquad \frac{1}{\widetilde{g}(|k|)}\log|C_k(w)| = \frac{1}{\widetilde{g}(|k|)}\log|B_k(w)| + \frac{1}{\widetilde{g}(|k|)}\log|v_k(w)| \\ = \sum_j \frac{N_{kj}}{\widetilde{g}(|k|)}\log|B_{kj}(w)| + \log|v_k'(w)|,$$

where we have defined holomorphic roots $v_k'(w) = v_k(w)^{1/\widetilde{g}(|k|)}$. These roots are well defined when $\widetilde{g}(|k|) \neq 0$, i.e. for large $|k|$. Now intuitively the supremum of $\log|C_k|$ should not be attained near the origin. This is not entirely true but by examining each of the terms in (7.7) we will see what is going on there.

Since the order is non-negative and $|B_k| = 1$ on $\mathbb{T}$ we see that the roots v_k' are uniformly bounded from above on $\mathbb{T}$ and hence also in $\mathbb{D}$. Thus by the Arzelà–Ascoli Theorem we can find a subset J of I such that $v_k' \to v \in \mathcal{O}(\mathbb{D})$ locally uniformly when $|k| \to +\infty$, $k \in J$. Taking J instead of I will of course not change the limit in (7.5). It is a well-known fact that if a sequence of non-zero holomorphic functions tends locally uniformly to a holomorphic limit function, then this function is either non-zero or identically zero. In the latter case we must have $\operatorname{order}(\sup_{w\in K} h_w : g) = 0$, which we have ruled out already. Otherwise the family $\{v_k'\}_{k\in J}$ must also be locally uniformly bounded from below. We know that a uniformly bounded family of holomorphic functions is equi-continuous. From this and the bound from below we can conclude that also the family $\{\log|v_k'|\}_{k\in J}$ is equi-continuous and locally uniformly convergent. So we have good behaviour on the second term. We now go to the first term.

Let J_M denote the subset of J consisting of those $k \in J$ for which $|k| \geq M$. First we will see that the number of zeros cannot grow too rapidly. Let $\varepsilon > 0$ and $0 < c < 1-2\varepsilon$ be given, such that K is contained in $c\mathbb{D}$. There then exists a number $\delta > 0$ such that if $|\alpha_{kj}| < \delta$ then $|B_{kj}(w)| < 1-\varepsilon$ on $c\mathbb{T}$. Since the zero-sets tend to the origin there exists a number M such that for all $k \in J_M$ we have $|\alpha_{kj}| < \delta$. This implies that $N_k/\widetilde{g}(|k|) \leq C < +\infty$ uniformly for all $k \in J$, where $N_k = \sum_j N_{kj}$. If this were not the case the limit in (7.5) would be $-\infty$ and again we have assumed this not to be the case. Thus we may assume, perhaps by taking an infinite subset of J (which we still denote by J), that $N_k/\widetilde{g}(|k|) \to \eta$, $0 \leq \eta < +\infty$, as $k \to \infty$, $k \in J$. We will use this later together with the fact that each Blaschke factor tends locally uniformly to the identity function $w \mapsto w$. Now we start to estimate the size of the terms around the origin. We begin with the second term.

By assumption K has points arbitrarily close to the origin. Hence by equi-continuity we can find $\delta_1 > 0$ such that

$$\text{(7.8)} \qquad 0 \leq \lim_{\substack{k\to\infty \\ k\in J}} \sup_{w\in K} \log|v_k'(w)| - \lim_{\substack{k\to\infty \\ k\in J}} \sup_{w\in K_1} \log|v_k'(w)| < \frac{\varepsilon}{2},$$

where $K_1 = K \setminus \{|w| < \delta_1\}$. We know that the limits in (7.8) exist because of the uniform convergence.

We divide the study of the first term into two cases: $\eta = 0$ and $\eta > 0$. If $\eta = 0$ we can find M so large that

$$\sum_j \frac{N_{kj}}{\widetilde{g}(|k|)} \log |B_{kj}(w)| > -\frac{\varepsilon}{2}, \qquad |w| \ge \delta_1,\ k \in J_M. \tag{7.9}$$

Since $\log |B_k| \le 0$ on all of $\mathbb{D}$ we can conclude by (7.5), (7.7), (7.8) and (7.9) that

$$\begin{aligned} &\frac{-1}{\operatorname{order}(\sup_{w\in K} h_w : g)} - \frac{-1}{\operatorname{order}(\sup_{w\in K_1} h_w : g)} \\ &\quad \le \lim_{\substack{k\to\infty\\ k\in J}} \sup_{w\in K} \log |v_k'(w)| - \Big(\lim_{\substack{k\to\infty\\ k\in J}} \sup_{w\in K_1} \log |v_k'(w)| - \frac{\varepsilon}{2}\Big) < \varepsilon. \end{aligned} \tag{7.10}$$

Now we can apply Corollary 6.6 to K_1 and a small disk removed from Ω to get

$$\frac{-1}{\operatorname{order}(\sup_{w\in K} h_w : g)} - \frac{-1}{\sup_{w\in K_1} \operatorname{order}(h_w : g)} < \varepsilon, \tag{7.11}$$

and the supremum over K_1 is of course less than the supremum over K. Since always $\operatorname{order}(\sup_{w\in K} h_w : g) \ge \sup_{w\in K} \operatorname{order}(h_w : g)$, we are done with the case when $\eta = 0$.

If $\eta > 0$ we will find a number $\delta_2 < \delta_1$ such that the supremum of $\log |C_k|$ can never be attained inside the circle of radius δ_2 for any $k \in J_M$ provided M is large. This would imply that $\operatorname{order}(\sup_{w\in K} h_w : g) = \operatorname{order}(\sup_{w\in K_2} h_w : g)$, $K_2 = K \setminus \{|w| < \delta_2\}$ and we can apply Corollary 6.6 to K_2 and $\Omega \setminus \overline{\delta_3\mathbb{D}}$, $\delta_3 < \delta_2$ to finish the proof. By the estimate in (7.8) there exists M so large that

$$0 \le \sup_{w\in K} \log |v_k'(w)| - \sup_{w\in K_1} \log |v_k'(w)| < \frac{3\varepsilon}{4}, \qquad k \in J_M. \tag{7.12}$$

We then take δ_2 so small that

$$\frac{N_k}{\widetilde{g}(|k|)} \log \frac{\delta_2}{\delta_1} = \frac{N_k}{\widetilde{g}(|k|)} \log \frac{\sup_{|w|\le\delta_2} |w|}{\inf_{|w|\ge\delta_1} |w|} < -2\varepsilon, \qquad \forall k \in J_M. \tag{7.13}$$

The apparently silly expression in (7.13) will be clear in a moment. By perhaps choosing a larger M we can make the difference $|B_{kj}(w) - w|$ uniformly small in $\mathbb{D} \setminus \{|w| < \delta_2\}$ for $k \in J_M$ so that

$$\sum_j \frac{N_{kj}}{\widetilde{g}(|k|)} \log \frac{\sup_{|w|\le\delta_2} |B_{kj}(w)|}{\inf_{|w|\ge\delta_1} |B_{kj}(w)|} < -\varepsilon, \qquad k \in J_M. \tag{7.14}$$

By the estimate in (7.12) we conclude that the supremum of $\log |C_k|$ cannot be attained inside the disk with radius δ_2 and so we are done.

8. A note on functions of regular growth. The *lower order* λ *of* F *relative to* g is defined as $\lambda = 1/\mathrm{order}(g : F)$. A function is said to have *regular growth with respect to* g if $\lambda = \varrho$, where $\varrho = \mathrm{order}(F : g)$. See also Kiselman [8]. In general, $\varrho \geq \lambda$, provided g is of more than linear growth. Otherwise we may have $\varrho = 0$, $\lambda = +\infty$.

THEOREM 8.1. *Let* $\{H(\cdot, w)\}_{w\in K} \subset \mathcal{O}(\mathbb{C}^n)$ *be a set of entire functions and let* $g : \mathbb{R} \to \mathbb{R}$ *be a function of more than linear growth. Assume that*

$$\mathrm{order}(\sup_{w\in K} h_w : g) \leq \varrho \tag{8.1}$$

and that there exists some $w \in K$ *such that* $\mathrm{order}(g : h_w) \leq 1/\varrho$. *Then we have equality in* (8.1) *and*

$$\inf_{w\in K} \mathrm{order}(g : h_w) = \mathrm{order}(g : \sup_{w\in K} h_w) = 1/\varrho. \tag{8.2}$$

If there exists a subset K_1 *of* K *such that*

$$\mathrm{order}(g : \sup_{w\in K_1} h_w) \leq 1/\varrho, \tag{8.3}$$

then we have equality in (8.1) *and*

$$\mathrm{order}(g : \sup_{w\in K} h_w) = 1/\varrho. \tag{8.4}$$

Proof. The first statement follows from the inequalities

$$\inf_{w\in K} \mathrm{order}(g : h_w) \geq \mathrm{order}(g : \sup_{w\in K} h_w) \tag{8.5}$$

and

$$1 \leq \mathrm{order}(\sup_{w\in K} h_w : g)\, \mathrm{order}(g : \sup_{w\in K} h_w). \tag{8.6}$$

The order in (8.3) also majorizes the order in (8.4), so the second statement follows from (8.6).

EXAMPLE 8.2. In Theorem 6.3 we assume that $\mathrm{order}(h_w : h_{w'}) = 1$ on all of $\mathbb{T}$. However, we know that $\mathbb{T}$ is nowhere thin and if $\mathbb{T}$ lies inside the domain then by Theorem 6.7 supremum and order commute, so that

$$\mathrm{order}(\sup_{w\in\mathbb{T}} h_w : h_{w'}) = \mathrm{order}(h_{w'} : \sup_{w\in\mathbb{T}} h_w) = 1, \qquad \forall w' \in \mathbb{T}. \tag{8.7}$$

We see that Theorem 6.3 is false if $\mathbb{T} \Subset \Omega$. In fact, we then only need the assumption $\mathrm{order}(h_w : h_{w'}) \leq 1$, $w \in \mathbb{T}$, to conclude that the equalities in (8.7) hold since trivially $\mathrm{order}(h_{w'} : h_{w'}) = 1$ (unless $H(\cdot, w')$ is a polynomial).

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