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Some sufficient conditions for solvability of the Dirichlet problem for the complex Monge–Ampère operator

by SŁAWOMIR KOŁODZIEJ (Kraków)

Abstract. We find a bounded solution of the non-homogeneous Monge–Ampère equation under very weak assumptions on its right hand side.

Introduction. In this paper we are interested in solving, under possibly weak assumptions on the measure $d\mu$, the following Dirichlet problem for the complex Monge–Ampère equation in a given strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$:

$$(*)\qquad \begin{aligned} &u \in \mathrm{PSH} \cap L^\infty(\Omega),\\ &(dd^c u)^n = d\mu,\\ &\lim_{z' \to z} u(z') = \phi(z), \quad z \in \partial\Omega,\ \phi \in C(\partial\Omega), \end{aligned}$$

where $d = \partial + \overline{\partial}$, $d^c = i(\overline{\partial} - \partial)$ and so $dd^c = 2\pi i \partial\overline{\partial}$. It has been shown by E. Bedford and B. A. Taylor [BT1] that the wedge product $(dd^c u)^n = dd^c u \wedge \ldots \wedge dd^c u$ is well defined for plurisubharmonic (psh), locally bounded functions u, and that $(*)$ is solvable for measures having continuous densities with respect to the Lebesgue measure (here denoted by $d\lambda$). The equation has attracted attention of a number of authors; we refer to [B] for a more detailed account. In particular, it is known that continuous solutions exist if $d\mu = f\,d\lambda$, where $f \in L^2(\Omega, d\lambda)$ (U. Cegrell–L. Persson [CP]), but for $f \in L^1(\Omega, d\lambda)$ this is not necessarily true [CS]. In Theorem 3 below we show that if $f \in L^p(\Omega, d\lambda)$, $p > 1$, then there exists a continuous solution of $(*)$. This is the answer to the question posed in [CS] and [P] (see also [B], [BL]). For the case of rotation invariant measures in a ball a solution was given in [P]. The result can be extended from $L^p, p > 1$, to some Orlicz spaces as shown

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in Theorem 4. To prove it we use an a priori estimate for the $\|u\|_{L^\infty}$ norm of a solution of $(*)$ if $d\mu$ satisfies a certain integral condition (Theorem 1). E. Bedford [B] conjectured that some such estimate is possible. It is shown that the integral condition cannot be substantially weakened. Combining Theorem 1 with the results of [KO] we solve the Dirichlet problem $(*)$ for a large family of measures $d\mu$.

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Preliminaries. Here we present some notions and results which are used in the paper. The background material can be found in [B], [K], [S]. Ω will denote throughout a strictly pseudoconvex domain in $\mathbb{C}^n$. For a compact subset $K \subset \Omega$ we define the *relative extremal function* and the *relative capacity* [BT2] (see also [B], [K]) by the formulas

$$u_K(z) = \sup\{u(z) : u \in \mathrm{PSH} \cap L^\infty,\ u < 0 \text{ in } \Omega,\ u \le -1 \text{ on } K\},$$
$$\mathrm{cap}(K,\Omega) = \sup\Big\{ \int_K (dd^c u)^n : u \in \mathrm{PSH}(\Omega),\ -1 \le u < 0\Big\}.$$

By [BT2],

$$\mathrm{cap}(K,\Omega) = \int_K (dd^c u_K^*)^n = \int_\Omega (dd^c u_K^*)^n,$$

where $u_K^* := \overline{\lim}_{z' \to z} u_K(z)$. If $u_K^* = u_K$ we say that K is *regular.* For an open subset $U \subset \Omega$ the relative capacity is defined by

$$\mathrm{cap}(U,\Omega) = \sup\{\mathrm{cap}(K,\Omega) : K \subset U,\ K \text{ compact}\}.$$

Another extremal function (of logarithmic growth) and an associated capacity were introduced by J. Siciak (see [S], [AT], [B], [K]):

$$L_K(z) = \sup\{u(z) : u \in \mathrm{PSH}(\mathbb{C}^n),\ u(z) < \log(1+|z|) + O(1),\ u \le 0 \text{ on } K\},$$
$$T_R(K) := \exp(-\sup\{L_K^*(z) : |z| \le R\})$$

for a compact set $K \subset \mathbb{C}^n$ and a given $R > 0$. We extend the definition of T_R to open sets in the same way as the definition of cap above.

Important inequalities between cap and T were proved by H. Alexander and B. A. Taylor [AT]. If $B := B(0,R)$ and $K \subset B(0,r), r < R$, is compact, then

$$\exp(-A(r)(\mathrm{cap}(K,B))^{-1}) \le T_R(K) \le \exp(-2\pi(\mathrm{cap}(K,B))^{-1/n}).$$

The main tool in pluripotential theory is the following Comparison Principle of Bedford and Taylor [BT2]:

Comparison Principle. *If* $u, v \in \mathrm{PSH} \cap L^\infty(\Omega)$ *and* $\liminf_{z\to\partial\Omega}(u(z) - v(z)) \geq 0$, *then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

Due to the same authors and presented here in a simplified version, sufficient for our applications, is

Convergence Theorem [BT2]. *If* $u_j \in \mathrm{PSH} \cap L^\infty(\Omega)$, $j = 1, 2, \ldots,$ *and* $u_j \uparrow u$ *a.e. in* Ω *or* $u_j \downarrow u$ *with* $u \in \mathrm{PSH} \cap L^\infty_{\mathrm{loc}}(\Omega)$ *then*

$$(dd^c u_j)^n \to (dd^c u)^n$$

in the sense of currents.

An a priori estimate. We begin with proving an a priori estimate for the L^∞ norm of a solution to the Dirichlet problem $(*)$ when $d\mu$ is assumed to satisfy a certain integral condition.

Theorem 1. *Let* Ω *be a strictly pseudoconvex domain in* $\mathbb{C}^n$ *and let* μ *be a Borel measure in* Ω *such that* $\int_\Omega d\mu \leq 1$. *Consider an increasing function* $h : \mathbb{R} \to (1, \infty)$ *satisfying*

$$\int_1^\infty (y h^{1/n}(y))^{-1}\, dy < \infty.$$

If μ *satisfies the integral condition*

$$(**) \qquad \int_\Omega |v|^n h(|v|)\, d\mu \leq A$$

whenever

$$v \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega}), \quad v = 0 \text{ on } \partial\Omega, \quad \int_\Omega (dd^c v)^n \leq 1,$$

then the norm $\|u\|_{L^\infty}$ *of a solution of the Dirichlet problem* $(*)$ *is bounded by a constant* $B = B(h, A)$ *which does not depend on* μ.

Proof. It is no restriction to assume that $\phi = 0$ in $(*)$: the general case will follow by the Comparison Principle [BT2]. Let u be a solution of $(*)$. For $s < 0$ denote by U_s the open set $\{u < s\}$ and put

$$a(s) := \mathrm{cap}(U_s, \Omega) = \mathrm{cap}(U_s), \qquad b(s) := \mu(U_s).$$

Our proof rests on the following two propositions.

Proposition 1. $b(s) \leq A a(s) h^{-1}([a(s)]^{-1/n})$.

Proposition 2. $t^n a(s) \leq b(s+t)$ *if* $t > 0$ *and* $s + t < 0$.

P r o o f o f P r o p o s i t i o n 1. Consider $v = (ra(s))^{-1/n} u_K$, where $K \subset U_s$ is a compact regular set with $\operatorname{cap}(K) = ra(s)$ $(r < 1)$. Then $\int (dd^c v)^n = 1$ and so the integral condition $(**)$ applies, giving

$$A \geq \int_{\Omega} |v|^n h(|v|)\, d\mu \geq \int_K |v|^n h(|v|)\, d\mu = (ra(s))^{-1} h([ra(s)]^{-1/n}) \mu(K),$$

which is just the desired estimate as $r \to 1$ (and so $\mu(K) \to b(s)$).

P r o o f o f P r o p o s i t i o n 2. We apply the Comparison Principle [BT2] to the pair of functions u_K and $v := (rt)^{-1}(u - s - t)$, where K, r are defined as above. Note that $K \subset \{v < u_K\} \subset U_{s+t}$. Hence

$$\begin{aligned} ra(s) = \int_{\{v<u_K\}} (dd^c u_K)^n &\leq (rt)^{-n} \int_{\{v<u_K\}} (dd^c u)^n \\ &\leq (rt)^{-n} \mu(U_{s+t}) = (rt)^{-n} b(s+t). \end{aligned}$$

The proposition follows if we let $r \to 1$.

E n d o f t h e p r o o f o f T h e o r e m 1. Fix s_0 so that $a = a(s_0) \neq 0$. We need to find a lower bound for s_0. To this end we first define an increasing sequence $s_0, s_1, \ldots, s_N$ by

$$s_j := \sup\{s : a(s) \leq \lim_{t \to s_{j-1}+} ea(t)\}.$$

Then

$$\lim_{t \to s_j -} a(t) \leq \lim_{t \to s_{j-1}+} ea(t) \quad \text{and} \quad a(s_j) \geq ea(s_{j-2}).$$

We continue this process till

$$1 \leq a(s_N). \tag{1}$$

For fixed s and s' such that $a(s) \leq ea(s')$ and $t := s - s'$ we have by the above two propositions

$$\begin{aligned} a(s') \leq t^{-n} b(s) &\leq At^{-n} a(s) h^{-1}([a(s)]^{-1/n}) \\ &= Aet^{-n} a(s') h^{-1}([a(s)]^{-1/n}). \end{aligned}$$

Hence

$$t \leq (Ae)^{1/n} h_1(a(s))$$

where $h_1(x) := h^{-1/n}(x^{-1/n})$. Letting $s \to s_{j+1}-$ and $s' \to s_j+$ we thus get

$$t_j := s_{j+1} - s_j \leq (Ae)^{1/n} h_1(a(s_{j+1})).$$

Since the function $h_2(x) := h_1(e^x) = h^{-1/n}(e^{-x/n})$ is increasing we can further estimate

$$\sum_{j=0}^{N-1} t_j \le (Ae)^{1/n} \sum_{j=0}^{N-1} h_2(\log a(s_{j+1})) \tag{2}$$
$$\le (Ae)^{1/n}\Big(\sum_{j=0}^{N-2} \int\limits_{\log a(s_j)}^{\log a(s_{j+2})} h_2(x)\,dx + 2h_2(\log a(s_N))\Big)$$
$$\le 2(Ae)^{1/n}\Big(\int\limits_{-\infty}^{0} h_2(x)\,dx + h_2(\infty)\Big).$$

By our hypothesis on h, we have $h_2(\infty) \le 1$ and

$$\int\limits_{-\infty}^{0} h_2(x)\,dx = \int\limits_{-\infty}^{0} h^{-1/n}(e^{-x/n})\,dx$$
$$= n\int\limits_{1}^{\infty} h^{-1/n}(y)y^{-1}\,dy =: nc(h) < \infty.$$

These remarks combined with (2) give

$$s_N - s_0 = \sum_{j=0}^{N-1} t_j \le 2(Ae)^{1/n}(nc(h)+1) =: c.$$

This means that for $s' \ge s_0 + c$ we have $a(s') > 1$ (see (1)). So fixing $s' = s_0 + c + 1$ we conclude that $s' \ge 0$ because otherwise, by applying Proposition 2, we would get a contradiction with the assumptions:

$$\mu(U_{s'}) > 1.$$

Thus $s_0 \ge -c - 1 =: B$. The proof is complete.

Remark. The hypothesis that μ satisfies $(**)$ can be replaced by

$$\mu(K) \le A\,\mathrm{cap}(K)h^{-1}((\mathrm{cap}(K))^{-1/n})$$

for any $K \subset \Omega$ compact and regular. The above proof still works.

It turns out that the integral condition $(**)$ is not far from being sharp. From [BL, Corollary 2.2] (see also [D, Th. 2.2]) it follows that any bounded solution of $(*)$ satisfies $(**)$ with $h \equiv 1$ and $A = n!\|u\|_{L^\infty}^n \int_\Omega d\mu$. However, if we let $h \equiv 1$ then $(**)$ ceases to be a sufficient condition for boundedness of u (when $n > 1$). This can be seen by considering radial psh functions in a ball $B = B(0,R)$. In that case we have a characterization of bounded solutions of $(*)$ given in [P] (see also [M]). A radial psh function u is bounded if and only if

$$\int\limits_0^R r^{-1}F^{1/n}(r)\,dr < \infty, \tag{3}$$

where $F(r) = \int_{B(0,r)} (dd^c u)^n$.

It is easy to see that for the rotation invariant measure $d\mu=(dd^c u)^n$ the integral in $(**)$ assumes its maximal value for $v(z) = (2\pi)^{-n}\log|z|$. Suppose that

$$(4)\qquad (2\pi)^n \int_B |v|^n\, d\mu = \int_0^R |\log r|^n F'(r)\, dr < \infty.$$

Via integration by parts this is equivalent to

$$\int_0^R |\log r|^{n-1} r^{-1} F(r)\, dr < \infty.$$

Write $F(r) = |\log r|^{-n} g^{-1}(r)$. Then (4) takes the form

$$\int_0^R [|\log r| r g(r)]^{-1}\, dr < \infty,$$

whereas (3) now says

$$\int_0^R [|\log r| r g^{1/n}(r)]^{-1}\, dr < \infty.$$

Taking g such that the former inequality is satisfied but the latter is not, e.g. $g(r) = (\log|\log(r)|)^n$, we arrive at the desired conclusion.

Coupling Theorem 1 above with Theorem 1 from [KO] we obtain a fairly general class of measures for which the Dirichlet problem $(*)$ is solvable. For the definition of a measure locally dominated by capacity which we need in the statement of the next theorem we refer to [KO]. Essentially we require from such a measure (say μ) that there exists $c > 0$ such that given two concentric balls $B_1 := B(a, r) \subset B_2 := B(a, 2r) \subset \Omega$ and a compact subset $E \subset B_1$, the following estimate holds:

$$\mu(E) \le c\mu(B_1)\,\mathrm{cap}(E, B_2).$$

(The actual definition is a bit less restrictive.)

THEOREM 2. *If a measure μ in Ω is locally dominated by capacity and satisfies the condition $(**)$ from Theorem 1 with h such that*

$$h(ax) \le bh(x), \qquad x > 0,$$

for some $a > 1$ and $b > 1$, then there exists a solution of $(*)$.

Proof. For a while we assume that μ has compact support in Ω. Define a regularizing sequence of measures μ_t by fixing a radial non-negative function $\omega \in C_0^\infty(B)$ with $\int \omega\, d\lambda = 1$ (here B is the unit ball in $\mathbb{C}^n$) and setting

$$\mu_t = \omega_t * \mu, \quad \text{where} \quad \omega_t(z) = t^{-2n}\omega(z/t), \quad t > 0.$$

By Theorem 1 and Remark following it, it is enough to find $t_0 > 0$ and $A > 0$ such that for any compact set $K \subset \Omega$,

$$(\iota) \qquad \mu_t(K) \le A \operatorname{cap}(K,\Omega) h^{-1}((\operatorname{cap}(K,\Omega))^{-1/n}), \qquad t < t_0.$$

PROPOSITION 3. *If $E \Subset \Omega$ is regular then for any $d > 1$ there exists t_0 such that*

$$\operatorname{cap}(K_y,\Omega) \le d \operatorname{cap}(K,\Omega), \qquad |y| < t_0,$$

where $K \subset E$ is regular and $K_y := \{x : x - y \in K\}$.

Proof. For $K \subset E$ define $w_y := u_{K_y}(x+y)$, where u_{K_y} is the extremal function of K_y. For any c such that $0 < c < 1/2$ define $\Omega_c = \{u_E < -c\}$. By continuity of u_E one can fix $t_0 > 0$ such that if $|y| \le t_0$ and $x \in \Omega_{c/2}$ then $x + y \in \Omega$. Therefore

$$g(x) := \begin{cases} \max(w_y - c, (1+2c)u_E)(x), & x \in \Omega_{c/2}, \\ (1+2c)u_E(x), & x \notin \Omega_{c/2}, \end{cases}$$

is a well defined plurisubharmonic function in Ω. Since $K \subset E$ and $w_y = -1$ on K one concludes that $g = w_y - c$ in a neighbourhood of K. Hence

$$\begin{aligned} \operatorname{cap}(K,\Omega) &\ge (1+2c)^{-n} \int_K (dd^c g)^n = (1+2c)^{-n} \int_K (dd^c w_y)^n \\ &= (1+2c)^{-n} \int_{K_y} (dd^c u_{K_y})^n = (1+2c)^{-n} \operatorname{cap}(K_y,\Omega). \end{aligned}$$

Thus the proposition is proved.

To complete the proof of Theorem 2 let us fix a set E and a positive number t_0 such that the above proposition holds with $E := \bigcup_{t<t_0} \operatorname{supp} \mu_t \Subset \Omega$ and $d = a^n$. By the assumptions there exists $A_0 > 0$ such that

$$\mu(K) \le A_0 \operatorname{cap}(K) h^{-1}((\operatorname{cap}(K))^{-1/n}).$$

Hence for $t < t_0$ we have by Proposition 3 and the extra assumption on h,

$$\begin{aligned} \mu_t(K) &\le \sup_{|y|<t} \mu(K_y) \le A_0 \sup_{|y|<t} \operatorname{cap}(K_y) h^{-1}((\operatorname{cap}(K_y))^{-1/n}) \\ &\le A_0 d \operatorname{cap}(K) h^{-1}((d \operatorname{cap}(K))^{-1/n}) \\ &\le A_0 d b^{1/n} \operatorname{cap}(K) h^{-1}((\operatorname{cap}(K))^{-1/n}). \end{aligned}$$

Setting $A := A_0 a^n b^{1/n}$ we verify this way that μ_t satisfies (ι) for $t < t_0$, with the constant A independent of t. Thus by Theorem 1 the family of solutions of $(*)$ for μ_t, $t < t_0$, is uniformly bounded. So one can apply [KO, Th. 1] to get the conclusion.

To verify the statement for an arbitrary measure μ note that by the above argument the solutions exist for $\chi_j \, d\mu$, where χ_j is a non-decreasing sequence of smooth cut-off functions with $\chi_j \uparrow 1$ in Ω. Moreover, the L^∞

norms of those solutions are uniformly bounded by a constant depending only on A. Hence the result follows by applying the monotone convergence theorem of [BT2].

Solutions for measures having densities in $L^p, p > 1$. In Theorem 3 we are going to prove that for $d\mu = f\,d\lambda$, $f \in L^p(\Omega)$, $p > 1$, the Dirichlet problem $(*)$ has a continuous solution. To this end we shall use the following

Lemma 1. *Suppose $v \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$, $v=0$ on $\partial\Omega$ and $\int (dd^c v)^n = 1$. Then the Lebesgue measure $\lambda(U_s)$ of the set $\{v < s\}$ is bounded from above by $c\exp(-2\pi|s|)$, where c does not depend on v.*

Proof. The proof is a variation of the proof of Proposition 2 of [KO]. First we shall estimate $\mathrm{cap}(U_s) = \mathrm{cap}(U_s, \Omega)$ applying the Comparison Principle [BT2]. For $t > 1$ and a regular compact set $K \subset U_s$ we have

$$\mathrm{cap}(K) = \int_K (dd^c u_K)^n = \int_{\{-ts^{-1}v < u_K\}} (dd^c u_K)^n \le t^n s^{-n} \int_\Omega (dd^c v)^n \le t^n s^{-n}.$$

Hence

$$\mathrm{cap}(U_s) \le |s|^{-n}. \tag{5}$$

Write $(z_1, z') \in \mathbb{C} \times \mathbb{C}^{n-1}$ and set $U_s(z') := \{z_1 \in \mathbb{C} : (z_1, z') \in U_s\}$. Let $V_{z'}$ (resp. V) be the extremal function of logarithmic growth of $U_s(z')$ (resp. U_s). Then (see [TS])

$$\lambda(U_s(z')) \le C_1 T_R(U_s(z')),$$

where λ denotes the Lebesgue measure in $\mathbb{C}$, C_1 is an independent constant and

$$T_R(U_s(z')) := \exp(-\sup_{|z_1|<R} V_{z'}),$$

with R chosen so that $\Omega \subset B(0, R)$. Thus

$$\begin{aligned}\lambda(U_s) &= \int \lambda(U_s(z'))\,d\lambda(z') \le C_1 \int T_R(U_s(z'))\,d\lambda(z') \\ &= C_1 \int \exp(-\sup_{|z_1|<R} V(z_1, z'))\,d\lambda(z').\end{aligned} \tag{6}$$

A simple argument using a result of Alexander [A] shows that the right hand side of (6) is dominated by

$$C_2 \exp(-\sup_{|z|<R} V(z)) = C_2 T_R(U_s)$$

(see [KO] for details). Finally, we apply an inequality between the capacities cap and T proved in [AT] to obtain

$$\lambda(U_s) \le C_2 \exp[-2\pi(\mathrm{cap}(U_s, B(0,R)))^{-1/n}] \le C_2 \exp[-2\pi(\mathrm{cap}(U_s, \Omega))^{-1/n}].$$

Hence by (5) we get

$$\lambda(U_s) \le C_2 \exp(-2\pi|s|),$$

which was to be proved.

COROLLARY. *If* $v \in \mathrm{PSH}(\Omega) \cap C(\overline{\Omega})$, $v = 0$ *on* $\partial\Omega$ *and* $\int_\Omega (dd^c v)^n \le 1$, *then* $\|v\|_{L^p} \le c(p)$.

Proof. By the lemma,

$$\int |v|^p \, d\lambda \le \int_\Omega d\lambda + \sum_{s=1}^{\infty} \int_{\{-s-1<v<-s\}} |v|^p \, d\lambda \le c \sum_{s=1}^{\infty} (s+1)^p e^{-2\pi s} =: c(p) < \infty.$$

Now we are in a position to prove

THEOREM 3. *If* $f \in L^p(\Omega, d\lambda)$, $p > 1$, $f \ge 0$ *then the Dirichlet problem* $(*)$ *has a continuous solution for* $d\mu = f\, d\lambda$.

Proof. Set $f_j := \min(f, j)$. Let u_j be the continuous solution of

$$(dd^c u)^n = f_j \, d\lambda,$$
$$\lim_{z' \to z} u(z') = \phi(z), \qquad z \in \partial\Omega$$

(see [C], [CP]). Then by the convergence theorem of [BT2], $u = \lim u_j$ is the desired solution provided u_j is uniformly bounded. This is the case if the integral condition $(**)$ in Theorem 1 is satisfied for $d\mu = f\, d\lambda$ and some suitable h. Let us verify this condition for $h(x) = \max(1, x)$. By Hölder's inequality we have

$$\int |v|^n h(|v|) f \, d\lambda = \int_{\{v \ge -1\}} + \int_{\{v<-1\}} \le \|f\|_{L^1} + \Big(\int |v|^{(n+1)q} \, d\lambda\Big)^{1/q} \|f\|_{L^p},$$

where $p^{-1} + q^{-1} = 1$. Since by the Corollary above,

$$\int |v|^{(n+1)q} \, d\lambda \le c(q(n+1)),$$

one can apply Theorem 1 to conclude that $u = \lim u_j$ is bounded.

Now, if u_{jk} solves $(dd^c u)^n = |f_j - f_k| \, d\lambda$, $u = 0$ on $\partial\Omega$, then by the Comparison Principle and the above argument,

$$\|u_j - u_k\| \le -u_{jk} \le c_p \|f_j - f_k\|_{L^p}^{1/n}.$$

So u_j is uniformly convergent and u is continuous.

The last result readily extends to cover densities belonging to some Orlicz spaces. As an example (which can be refined yet) we give the following

THEOREM 4. *Let* $L^\varphi(\Omega, d\lambda)$ *denote the Orlicz space corresponding to* $\varphi(t) = |t|(\log(1+|t|))^n h(\log(1+|t|))$ *with* h *satisfying the hypothesis of Theorem* 1. *If* $f \in L^\varphi(\Omega, d\lambda)$ *then* $(*)$ *is solvable with* $d\mu = f\, d\lambda$.

Proof. As in the preceding proof, it is enough to verify the condition $(**)$. We apply Young's inequality for the function $g(\log(1+r)) = (\log(1+r))^n h(\log(1+r))$ and its inverse. Then

$$\begin{aligned} g(|v(x)|)f(x) &\le \int_0^{f(x)} g(\log(1+r))\,dr + \int_0^{g(|v(x)|)} [\exp(g^{-1}(t))-1]\,dt \\ &\le f(x)g(\log(1+f(x))) + \int_0^{|v(x)|} e^s g'(s)\,ds \\ &\le \|f\|_{L^\varphi} + g(|v(x)|)e^{|v(x)|}. \end{aligned}$$

When integrated over Ω, the right hand side remains bounded since by the lemma,

$$\int_\Omega g(|v(x)|)e^{|v(x)|}\,dx \le c\sum_{s=1}^{\infty} e^{s(1-2\pi)} g(s+1) < \infty.$$

Thus the result follows from Theorem 1.

Example. If $\varphi(t) = |t|(\log(1+|t|))^n(\log(\log(1+|t|)))^m, m > n$, then Theorem 4 applies. On the other hand, if $\varphi(t) = |t|(\log(1+|t|))^m, m < n$, it is no longer true; a suitable counterexample is given in [P].

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Institute of Mathematics
Jagiellonian University
Reymonta 4
30-059 Kraków, Poland
E-mail: kolodzie@im.uj.edu.pl

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