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FOUR MAPPING PROBLEMS OF MAĆKOWIAK

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In his paper "Continuous mappings on continua" [5], T. Maćkowiak collected results concerning mappings on metric continua. These results are theorems, counterexamples, and unsolved problems and are listed in a series of tables at the ends of chapters. It is the purpose of the present paper to provide solutions (three proofs and one example) to four of those problems.

A *compactum* is a compact metric space, a *continuum* is a connected compactum, and a *map* is a continuous function. If $f : X \twoheadrightarrow Y$ is a map (the double arrow indicates a surjection) and X is a continuum, then f is *partially confluent* (respectively, *n-confluent*) if each continuum K in Y is the union of the images of finitely many (respectively, n or fewer) components of $f^{-1}(K)$. If the restriction of f to each subcontinuum of X is partially confluent (respectively, n-confluent), then f is *hereditarily partially confluent* (respectively, *hereditarily n-confluent*). Note that 1-confluent maps are also called weakly confluent and that n-confluent maps are called n-partially confluent in [7, p. 409] where they and partially confluent maps are defined. Example 1 in our paper shows that hereditarily partially confluent maps are not necessarily n-confluent for any positive integer n. A continuum is a *local dendrite* if each of its points is contained in a closed neighborhood which is a dendrite. The first result of this paper, Theorem 1, proves that the image of a local dendrite under a hereditarily partially confluent map is a local dendrite. This yields as a corollary a theorem of Maćkowiak [5, 9.24, p. 82] for hereditarily 1-confluent maps and answers in the affirmative a question raised by him for a subclass of 2-confluent maps [5, 9.26, p. 83].

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A continuum is *hereditarily divisible by points* if every subcontinuum is separated by one of its points. It is easily seen that such a continuum is hereditarily decomposable and hereditarily unicoherent, i.e., that such a continuum is a λ-dendroid. A map $f : X \twoheadrightarrow Y$ is *locally monotone* if each point p in X has a closed neighborhood U such that $f(U)$ is a closed neighborhood of $f(p)$ and $f|U$ is monotone. Maćkowiak asked [5, 7.28, p. 65] whether the property of a continuum being hereditarily divisible by points is preserved by locally monotone maps. Theorem 3 answers this question in the affirmative.

An arbitrary class $\mathcal{A}$ of maps has the (*weak*) *limit property* if, for each pair X, Y of compacta (with Y locally connected), $\{f : X \twoheadrightarrow Y \mid f \in \mathcal{A}\}$ is a closed subset of Y^X, the set of maps from X into Y with the compact-open topology. A map $f : X \twoheadrightarrow Y$ is *atriodic* if, for each subcontinuum K of Y, there are components A and B (not necessarily distinct) of $f^{-1}(K)$ such that $f(A) \cup f(B) = K$ and for each component C of $f^{-1}(K)$, either $f(C) \subseteq f(A)$ or $f(C) \subseteq f(B)$. This is not exactly the definition given in [5, (ix), p. 12] but is easily seen to be equivalent and is better for our purposes.

Atriodic maps are 2-confluent, and every map from a continuum onto an atriodic continuum is an atriodic map [5, 6.12, p. 53]. Maćkowiak asked if the set $\mathcal{A}$ of all atriodic maps has the weak limit property [5, 5.56, p. 45]. Theorem 4 answers this question in the affirmative.

A class $\mathcal{A}$ of maps has the *composition factor property* if the composition $g \circ h$ of maps is a member of $\mathcal{A}$ only if $g \in \mathcal{A}$. A map $f : X \twoheadrightarrow Y$ between continua is *atomic* if, for each subcontinuum K of X such that $f(K)$ is nondegenerate, $K = f^{-1}(f(K))$. Example 2 shows that atomic maps do not have the composition factor property, thus answering in the negative a question of Maćkowiak [5, 5.22, p. 33].

1. Hereditarily partially confluent maps on local dendrites

THEOREM 1. *If f is a hereditarily partially confluent map defined on a local dendrite X, then $Y = f(X)$ is a local dendrite.*

Proof. Any local dendrite is regular [4, Theorem 1, p. 303] and so is hereditarily locally connected [4, Theorem 2, p. 283]. Hence X is hereditarily locally connected, and so is Y, since partially confluent maps preserve that property [6, Theorem II.4, p. 566].

Assume the theorem is false, i.e., assume X and f satisfy the hypotheses and $Y = f(X)$ is not a local dendrite. Then Y, since it is not a local dendrite, contains infinitely many distinct simple closed curves $J_1, J_2, \ldots$ [4, Lemma 3, p. 303]. Since Y is hereditarily locally connected, the simple closed curves can be chosen so that they converge to a point s. They can also be chosen so that none is contained in the union of the others. Let Y' be a subcontinuum of Y containing $\bigcup_{j=1}^{\infty} J_j$ such that, for $i = 1, 2, \dots$, there is an arc $[r_i, s_i, t_i]$ in $J_i \setminus \bigcup_{j \neq i} J_j$ that is in the interior of J_i relative to Y'.

Since f is partially confluent and $Y' \setminus [\bigcup_{j=1}^{\infty}(s_j, t_j)]$ is a subcontinuum of Y, there is a continuum H in $f^{-1}(Y' \setminus [\bigcup_{j=1}^{\infty}(s_j, t_j)])$ such that $f(H) \cap \{s_1, s_2, \dots\}$ is infinite. Assume without loss of generality that $f(H) \supseteq \{s_1, s_2, \dots\}$. Let $\widetilde{r}$ be a point of the limiting set of $H \cap f^{-1}(s_1)$, $H \cap f^{-1}(s_2), \dots$ Then $\widetilde{r} \in f^{-1}(s)$, since $s_1, s_2, \dots$ converges to s. Let R_1 be an arc in H that is irreducible from $f^{-1}(s_1)$ to $\widetilde{r}$. Since R_1 is an arc in H and $f|R_1$ is partially confluent, $f(R_1)$ is a graph [6, Theorem III.2, p. 569] in $Y' \setminus [\bigcup_{j=1}^{\infty}(s_j, t_j)]$. Any graph in that continuum has an end point in each of the arcs $[r_1, s_1]$, $[r_2, s_2], \dots$ that it intersects. Since no graph has infinitely many endpoints, $f(R_1)$ intersects at most a finite number of those arcs. Hence, assume without loss of generality that $f(R_1) \cap (\bigcup_{j=1}^{\infty}[r_j, s_j]) = [r_1, s_1]$.

Since H is locally connected, there is, relative to H, a connected, open neighborhood C_1 of $\widetilde{r}$ with $\operatorname{diam} C_1 \leq 1$ such that $f(C_1) \cap [r_1, s_1] = \emptyset$. Assume without loss of generality that $C_1 \cap f^{-1}(s_2) \neq \emptyset$, and let R_2 be an arc in C_1 that is irreducible from $f^{-1}(s_2)$ to $\widetilde{r}$. Without loss of generality $f(R_2) \cap (\bigcup_{j=1}^{\infty}[r_j, s_j]) = [r_2, s_2]$. Let C_2 be a connected, relatively open subset of H containing $\widetilde{r}$, with $\operatorname{diam} C_2 \leq 1/2$, such that $f(C_2) \cap \{[r_1, s_1] \cup [r_2, s_2]\} = \emptyset$. Assume without loss of generality that $C_2 \cap f^{-1}(s_3) \neq \emptyset$, and let R_3 be an arc in C_2 that is irreducible from $f^{-1}(s_3)$ to $\widetilde{r}$. Without loss of generality $f(R_3) \cap (\bigcup_{j=1}^{\infty}[r_j, s_j]) = [r_3, s_3]$. Continuing with this construction (choosing $C_1, C_2, \dots$ so that $\operatorname{diam} C_i \to 0$) yields a sequence $R_1, R_2, \dots$ of arcs in H that converges to $\widetilde{r}$ and has the property that for $i = 1, 2, \dots$, R_i is irreducible from $f^{-1}(s_i)$ to $\widetilde{r}$ and $f(R_i) \cap (\bigcup_{j=1}^{\infty}[r_j, s_j]) = [r_i, s_i]$.

Since f is partially confluent and $Y' \setminus [\bigcup_{j=1}^{\infty}(r_j, s_j)]$ is a subcontinuum of Y, there is a continuum K in $f^{-1}(Y' \setminus [\bigcup_{j=1}^{\infty}(r_j, s_j)])$ such that $f(K) \cap \{s_1, s_2, \dots\}$ is infinite. Without loss of generality $f(K) \supseteq \{s_1, s_2, \dots\}$. A construction similar to that in H yields $\widetilde{t}$ in $f^{-1}(s)$ and a sequence $T_1, T_2, \dots$ of arcs in K that converges to $\widetilde{t}$ and has the property that, for $i = 1, 2, \dots$, T_i is irreducible from $f^{-1}(s_i)$ to $\widetilde{t}$ and $f(T_i) \cap (\bigcup_{j=1}^{\infty}[s_j, t_j]) = [s_i, t_i]$.

Let $H' = \bigcup_{j=1}^{\infty} R_j$ and $K' = \bigcup_{j=1}^{\infty} T_j$. For $i = 1, 2, \dots$, let $\widetilde{r}_i$ and $\widetilde{t}_i$ be the endpoints of R_i and T_i, respectively, that are in $f^{-1}(s_i)$. Then $\widetilde{r}_i = \widetilde{t}_i$ for at most a finite number of values of i, since X does not contain infinitely many simple closed curves [4, Theorem 4(i), pp. 303, 304]. Assume without loss of generality that $\widetilde{r}_i \neq \widetilde{t}_i$, for $i = 1, 2, \dots$ If $H' \cap K' = \emptyset$, let A be an arc in X that is irreducible between H' and K'. If $H' \cap K' \neq \emptyset$, let $A = \{p\}$ for some p in $H' \cap K'$. In either case $f|(A \cup H' \cup K')$ is partially confluent, since f is hereditarily partially confluent and $A \cup H' \cup K'$ is a subcontinuum of X. Consider the subcontinuum $V = f(K') \cup (\bigcup_{j=1}^{\infty}[r'_j, s_j])$ of $f(A \cup H' \cup K')$, where $r'_j \in (r_j, s_j)$, for $j = 1, 2, \ldots$ There is some component L of $f^{-1}(V) \cap (A \cup H' \cup K')$ such that $f(L) \cap \{r'_1, r'_2, \ldots\}$ is infinite. Since any continuum in $f^{-1}(V) \cap (H' \cup K')$ that intersects $f^{-1}(r'_i)$ is contained in R_i, for $i = 1, 2, \ldots$, L is not a subset of $H' \cup K'$. Hence $L \cap [A \setminus (H' \cup K')] \neq \emptyset$ and A is an arc that is irreducible between H' and K'. So L is the union of closed connected sets (some perhaps void) C_A, C_H, and C_K, where $C_A \subseteq A$, $C_H \subseteq H'$ and $C_K \subseteq K'$. Then $f(C_H) \cap \{r'_1, r'_2, \ldots\}$ contains at most one point, and $f(C_K) \cap \{r'_1, r'_2, \ldots\} = \emptyset$, so $f(C_A) \cap \{r'_1, r'_2, \ldots\}$ is infinite. But $f(C_A)$ is a graph [6, Theorem III.2, p. 569] in V and no graph has infinitely many endpoints. Hence $f(C_A) \cap \{r'_1, r'_2, \ldots\}$ is finite. This contradiction establishes the theorem.

The following result, due to Maćkowiak [5, 9.24, p. 82], is a corollary of Theorem 1.

COROLLARY. *If f is a hereditarily weakly confluent map defined on a local dendrite X, then $Y = f(X)$ is a local dendrite.*

Proof. This follows from Theorem 1 because weakly confluent maps are 1-confluent.

Maćkowiak asked [5, 9.26, p. 83] whether hereditarily atriodic maps preserve the property of being a local dendrite and conjectured that they do. Since hereditarily atriodic maps are hereditarily 2-confluent, an affirmative answer to Maćkowiak's question follows from Theorem 1.

The following example shows that the hypothesis, in Theorem 1, that f is hereditarily partially confluent is weaker than the property that there is a positive integer n such that f is hereditarily n-confluent.

EXAMPLE 1. Planar continua X and Y that are of Menger order 3 and are homeomorphic to each other and a map f from X onto Y that is hereditarily partially confluent but is not n-confluent for any positive integer n.

For i and j any positive integers with $i \geq j$, let a_{ij} and b_{ij} be points in the interval $(0, 1)$ on the x-axis in the coordinate plane in the following order: $a_{11} < b_{11} < a_{21} < a_{22} < b_{21} < b_{22} < a_{31} < a_{32} < a_{33} < b_{31} < \ldots$ That is, for every positive integer i, (1) $a_{ij} < a_{ik}$ and $b_{ij} < b_{ik}$ if $j < k \leq i$, and (2) $a_{ij} < b_{ik} < a_{(i+1)j}$ if j and k are in $\{1, \ldots, i\}$. Also, for i and j any positive integers with $i \geq j$, let $[a_{ij}, b'_{ij}, c'_{ij}]$ and $[b_{ij}, c_{ij}]$ be intervals, of length $1/i$, perpendicular to $[0, 1]$. Let $X = [0, 1] \cup [\bigcup_{i=1}^{\infty} (\bigcup_{j=1}^{i} [a_{ij}, c'_{ij}])]$ and $Y = [0, 1] \cup [\bigcup_{i=1}^{\infty} (\bigcup_{j=1}^{i} [b_{ij}, c_{ij}])]$. Let f be the identity on $[0, 1]$ and map $[a_{ij}, b'_{ij}]$ linearly onto $[a_{ij}, b_{ij}]$ and $[b'_{ij}, c'_{ij}]$ linearly onto $[b_{ij}, c_{ij}]$, for all appropriate i and j. It is easily verified that X, Y, and f have the stated properties.

2. Locally monotone maps on continua that are hereditarily divisible by points

LEMMA 1. *Let $f : X \twoheadrightarrow Y$ be a map, and let $Z = \{X_p \mid p \in X\}$ where X_p is the p-component of $f^{-1}(f(p))$. Then the projection function $g : X \twoheadrightarrow Z$ is a monotone map.*

Proof. It is easily seen that the decomposition is upper semi-continuous, since f is continuous. It then follows that the projection function g is continuous. From the definition of Z it is clear that the preimage of any member of Z is connected. Hence g is a monotone map.

LEMMA 2. *Let $f : X \twoheadrightarrow Y$ be a locally monotone map, $Z = \{X_p \mid p \in X\}$ with the decomposition topology where X_p is the p-component of $f^{-1}(f(p))$, and $h : Z \twoheadrightarrow Y$ be defined by $h(X_p) = f(p)$. Then h is a finite-to-one, open map.*

Proof. Locally monotone maps are confluent [5, Table II, p. 28] and have the property that the collection of components of $f^{-1}(f(p))$ is finite for all p in X [5, 4.32, p. 22]. Hence the map h is finite-to-one. Suppose h is not open. Then there is an element K of Z and an open set V in Z such that $K \in V$ but $h(K)$ is not an interior point of $h(V)$. Let $p \in K$ and let U be a closed neighborhood of p (in X) such that $f(p) \in \mathrm{Int}\,(f(U))$ and $f|U$ is monotone. Then there is a sequence $q'_1, q'_2, \ldots$ of points in $\{Y \setminus h(V)\} \cap f(U)$ that converges to $h(K) = f(p)$. For $i = 1, 2, \ldots$, let $q_i \in U \cap f^{-1}(q'_i)$ and assume without loss of generality that $q_1, q_2, \ldots$ converges to a point q in U. Then $f(q) = f(p)$. Since $\{p, q\} \subseteq U \cap f^{-1}(f(p))$ and $U \cap f^{-1}(f(p))$ is a subcontinuum of U, it follows that $\{p, q\} \subseteq K$. Let g be the projection map from X to Z. Then $g^{-1}(V)$ is an open set in X containing K and, therefore, containing q. Hence $q_i \in g^{-1}(V)$ for large i, since $q_1, q_2, \ldots$ converges to q and, therefore, $q'_i = f(q_i) = h(g(q_i)) \in h(V)$ for large i. This contradiction proves the lemma.

LEMMA 3. *Let $f : X \twoheadrightarrow Y$ be a locally monotone map. Then there exist a continuum Z and maps $g : X \twoheadrightarrow Z$ and $h : Z \twoheadrightarrow Y$ such that $f = h \circ g$, where g is monotone and h is finite-to-one and open.*

Proof. Let Z be the continuum and g the map defined in Lemma 1, and let h be the map defined in Lemma 2. The conclusion of the lemma follows from Lemmas 1 and 2.

THEOREM 2. *Let $f : X \twoheadrightarrow Y$ be a finite-to-one, open map and X be a continuum that is hereditarily divisible by points. Then Y is a continuum that is hereditarily divisible by points.*

Proof. Since X is a λ-dendroid, and the image of a λ-dendroid under an open map is a λ-dendroid [5, 7.24, p. 64], Y is a λ-dendroid, and, hence, is hereditarily unicoherent. Let K be a subcontinuum of Y. Since f is finite-to-one, there is a positive integer n and an uncountable subset K_n of K such that $f^{-1}(y)$ has fewer than n points, for all y in K_n.

Let H be a component of $f^{-1}(K)$. Since f is confluent [5, 3.6, p. 13] and finite-to-one, each component of $f^{-1}(K)$ maps onto K, and there are only finitely many such components. It then follows that $f|H$ is an open map [5, 3.16, p. 14]. We wish to show that H contains a finite set F such that $H\setminus F$ is the union of n or more separated sets, i.e., disjoint sets that are both open and closed relative to $H\setminus F$. We will then show that $f(F)$ separates K, and, therefore, since K is unicoherent, that some point of $f(F)$ separates K.

If H contains a point p such that $H\setminus\{p\}$ has n or more components, then let p_1 be such a point and let $F=\{p_1\}$. Otherwise, let p_1 be a separating point of H. Then $H\setminus\{p_1\}$ is the union of separated sets B_1 and A_1 where A_1 is connected. Let p_2 be a separating point of $\overline{A}_1=A_1\cup\{p_1\}$. Either $F=\{p_1,p_2\}$ has the desired property or $H\setminus\{p_1,p_2\}$ is the union of separated sets B_1, B_2 and A_2 where A_2 is connected. Continuing this process we construct a finite subset $F=\{p_1,\ldots,p_m\}$ of H such that $H\setminus F$ is the union of n or more separated sets. Since these sets are disjoint, there is no point y of K_n such that $f^{-1}(y)$ intersects all of them. Let $y_1\in K_n\setminus f(F)$, and let A' be one of the separated sets in $H\setminus F$ that does not intersect $f^{-1}(y_1)$. Then A' is a nonvoid subset of $H\setminus F$ that is open and closed relative to $H\setminus F$ and $f(A')$ does not contain $K\setminus f(F)$, since $y_1\notin f(A')$.

Since $f|H$ is an open map and A' is open relative to H, it follows that $f(A')$ is open relative to K. Hence $f(A')\setminus f(F)$ is open relative to $K\setminus f(F)$. Also, $f(A')\cup f(F)=f(A'\cup F)=f(\overline{A'}\cup F)=f(\overline{A'})\cup f(F)$ is closed. Therefore, $f(A')\setminus f(F)$ is a nonvoid, proper subset of $K\setminus f(F)$ that is open and closed relative to $K\setminus f(F)$. It follows that $K\setminus f(F)$ is not connected, so $f(F)$ is a finite subset of K that separates K.

Let F' be a subset of $f(F)$ that is minimal with respect to separating K, and let $K\setminus F'=R\cup S$, a separation. Since no proper subset of F' separates K, both $\overline{R}$ and $\overline{S}$ contain F'. Similarly, any nonvoid subset of $\overline{R}$ (or $\overline{S}$) that is open and closed relative to $\overline{R}$ (or $\overline{S}$) must contain F'. Therefore $\overline{R}=R\cup F'$ and $\overline{S}=S\cup F'$ are continua whose intersection is F'. Since Y is hereditarily unicoherent, F' is connected. Hence F' consists of one point, which is a separating point of K. It follows that Y is hereditarily divisible by points.

THEOREM 3. *Let $f:X\twoheadrightarrow Y$ be a locally monotone map and X be a continuum that is hereditarily divisible by points. Then Y is hereditarily divisible by points.*

Proof. By Lemma 3, $f=h\circ g$ where g is monotone and h is finite-to-one and open. Monotone maps preserve the property of being hereditarily divisible by points [5, 7.26, p. 65]. By Theorem 2, finite-to-one, open maps also preserve this property. Therefore $f = h \circ g$ also preserves it and the theorem is proved.

This theorem answers a question of Maćkowiak [5, 7.28, p. 65].

3. Atriodic maps and the weak limit property. Maćkowiak gave an example [5, 5.57, p. 45] showing that the class of atriodic maps does not have the limit property. In the example, the domain compactum is not connected and the range is not locally connected, so the example does not answer either of the following two questions that he raised [5, 5.56, p. 45]. Does the class of atriodic maps have the limit property on continua? Does it have the weak limit property? The first question is still unanswered, and Theorem 4 below answers the second in the affirmative.

Atriodic maps are 2-confluent and, for each positive integer n, the class of n-confluent maps has the limit property [5, 5.55, p. 45], but, unfortunately, that was not helpful in proving Theorem 4.

THEOREM 4. *The class of atriodic maps has the weak limit property.*

Proof. Let X and Y be compacta with Y locally connected, and let Y^X be the space of all maps from X into Y with the compact-open topology. Let $\mathcal{A} = \{g \in Y^X \mid g \text{ is an atriodic map from } X \text{ onto } Y\}$, and let $f \in \bar{\mathcal{A}}$. We wish to show that f is an atriodic map from X onto Y, thus showing that $\mathcal{A}$ is closed, which establishes the theorem.

Let K be a subcontinuum of Y, and let $N_1, N_2, \ldots$ be a sequence of connected open sets closing down on K (i.e., $N_1 \supseteq \bar{N}_2 \supseteq N_2 \supseteq \bar{N}_3 \supseteq \ldots$ and $K = \bigcap_{i=1}^{\infty} N_i$). Since Y^X is metrizable [2, 8.2(3), p. 270] and has the compact-open topology, there is a sequence $f_1, f_2, \ldots$ in $\mathcal{A}$ that converges to f and has the property that $f_i(f^{-1}(K)) \subseteq N_i$, for $i = 1, 2, \ldots$ If $p \in \limsup f_i^{-1}(\bar{N}_i)$, then $f(p) \in K$, since $f_1, f_2, \ldots$ converges to f uniformly [2, Theorem 7.2, p. 268]. Hence, not only do the terms of $f_1^{-1}(\bar{N}_1), f_2^{-1}(\bar{N}_2), \ldots$ contain $f^{-1}(K)$ but the sequence converges to $f^{-1}(K)$. It also follows from the uniform convergence of $f_1, f_2, \ldots$, and the continuity of the functions, that f is continuous [2, 7.3, p. 268].

For $i = 1, 2, \ldots$, $\bar{N}_i$ is a subcontinuum of Y and f_i is atriodic, so there are components A_i and B_i (not necessarily distinct) of $f_i^{-1}(\bar{N}_i)$ such that $f_i(A_i) \cup f_i(B_i) = \bar{N}_i$ and, if C_i is a component of $f_i^{-1}(\bar{N}_i)$, then $f_i(C_i) \subseteq f_i(A_i)$ or $f_i(C_i) \subseteq f_i(B_i)$. Assume without loss of generality that $A_1, A_2, \ldots$ and $B_1, B_2, \ldots$ converge to continua A' and B', respectively. Since $A_1 \cup B_1$, $A_2 \cup B_2, \ldots$ converges to $A' \cup B'$, and $f_1, f_2, \ldots$ converges to f uniformly, $f(A' \cup B') = K$.

Let A and B be the components of $f^{-1}(K)$ that contain A' and B', respectively. Then $K \supseteq f(A) \cup f(B) \supseteq f(A') \cup f(B') = f(A' \cup B') = K$, so

$f(A) \cup f(B) = K$. Let C be a component of $f^{-1}(K)$. Then for $i = 1, 2, \ldots$, there is a component C_i of $f_i^{-1}(\bar{N}_i)$ that contains C. Assume without loss of generality that $C_1, C_2, \ldots$ converges to a continuum C' which, of course, contains C. Also assume that $f_i(C_i) \subseteq f_i(A_i)$, for $i = 1, 2, \ldots$

Let $p \in C$. Then $f_i(p) \in f_i(C) \subseteq f_i(C_i) \subseteq f_i(A_i)$ for $i = 1, 2, \ldots$, so there is a point p_i in A_i such that $f_i(p) = f_i(p_i)$. Assume without loss of generality that $p_1, p_2, \ldots$ converges to a point p' in $A' \subseteq A$. Then $f(p) = f(p')$, since $f_1, f_2, \ldots$ converges uniformly to f. Hence $f(C) \subseteq f(A)$, which implies that $f \in \mathcal{A}$, and, hence, that $\mathcal{A}$ is closed. This establishes the theorem.

4. Atomic maps and the composition factor property. Maćkowiak asked if the class of atomic maps has the composition factor property [5, 5.22, p. 33] and conjectured that it does. Example 2 shows that the property fails, even on continua.

Example 2. An atomic map $f : X \twoheadrightarrow Z$ that is the composite of maps $h : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$ but g is not atomic.

In [1, Section 7, pp. 189–191] a continuous collection G of disjoint pseudoarcs is described such that $M = \bigcup G$ is a homogeneous continuum and G with the decomposition topology is a simple closed curve. Also, if $A \in G$ and K is a subcontinuum of M that contains a point of A and a point of $M \setminus A$, then $A \subseteq K$ (see [3, Theorem 2, p. 739]). Let f be the projection map from M onto G with the decomposition topology. Then f is clearly an atomic map onto a simple closed curve.

Let $X = M$ and $Z = G$ with the decomposition topology. We wish to define a continuum Y (a decomposition space of X) and maps $h : X \twoheadrightarrow Y$ and $g : Y \twoheadrightarrow Z$ such that $f = g \circ h$ and g is not atomic. The construction of G referred to above uses a circle W_2 in the plane that has the properties that (1) $W_2 \cap M$ is a topological Cantor set, (2) both endpoints of each component of $W_2 \setminus M$ belong to the same member of G, and (3) W_2 intersects each member of G [1, Section 7, pp. 189–191]. Let Y be the decomposition space of M that results from identifying the endpoints of each component of $W_2 \setminus M$. Let h be the projection map from X onto Y. Now $h(M \cap W_2)$ is a simple closed curve that intersects each member of $\{h(A) \mid A \in G\}$. Let $g : Y \twoheadrightarrow Z$ be such that $f = g \circ h$, i.e., for each $x \in A \in G$, let $g(h(x)) = A$, as a point of Z. Clearly g is not atomic since $K = h(M \cap W_2)$ is a subcontinuum of Y and $g^{-1}(g(K)) = Y \neq K$.

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