COLLOQUIUM MATHEMATICUM

VOL. LXX 1996 FASC. 1

COMPARISONS OF SIDON AND I_0 SETS

BY

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Introduction. Let Γ be an arbitrary discrete abelian group. Sidon and I_0 subsets of Γ are interpolation sets in different but quite similar senses. In this paper we establish several similarities and one deeper connection:

(1) $B_d(E)$ and $B(E)$ are isometrically isomorphic for finite $E \subset \Gamma$. $B_d(E) = \ell_\infty(E)$ characterizes I_0 sets E, and $B(E) = \ell_\infty(E)$ characterizes Sidon sets E. [In general, Sidon sets are distinct from I_0 sets. Within the group of integers $\mathbb{Z}$, the set $\{2^n\}_n \cup \{2^n + n\}_n$ is helsonian (hence Sidon) but not I_0.]

(2) Both are F_σ in 2^Γ (as is also the class of finite unions of I_0 sets).

(3) There is an analog for I_0 sets of the sup-norm partition construction used with Sidon sets.

(4) A set E is Sidon if and only if there is some $r \in \mathbb{R}^+$ and positive integer N such that, for all finite $F \subset E$, there is some $H \subset F$ with $|H| \geq r|F|$ and H is an I_0 set of degree at most N. [Here $|S|$ denotes the cardinality of S; two different but comparable definitions of degree for I_0 sets are given below.]

(5) If all Sidon subsets of $\mathbb{Z}$ are finite unions of I_0 sets, the number of I_0 sets required is bounded by some function of the Sidon constant. This is also true in the category of all discrete abelian groups.

This paper leaves open this question: must Sidon sets be finite unions of I_0 sets?

Let G denote the (compact) dual group of Γ. In general, unspecified variables such as j and N denote positive integers. $M(G)$ denotes the Banach algebra under convolution of bounded Borel measures on G; the norm in $M(G)$ is the total mass norm. $M_d(G)$ denotes the Banach subalgebra of $M(G)$ consisting of discrete measures. $b\Gamma$ denotes the Bohr compactification of Γ: $b\Gamma = \widehat{G_d}$, the dual of discretized G. Naturally, Γ is dense in $b\Gamma$. The almost periodic functions on Γ are exactly the functions which extend

1991 *Mathematics Subject Classification*: Primary 43A56.

Key words and phrases: Sidon, I_0 set, almost periodic function.

continuously to $b\Gamma$; they are also the uniform limits of the Fourier transforms of $\mu \in M_d(G)$ [18, p. 32]. For subsets $E \subset \Gamma$, this paper focuses on the relations among several function algebras on E: $B_d(E)$, $B(E)$, $AP(E)$, and $\ell_\infty(E)$. $B_d(E)$ is the space of restrictions to E of Fourier transforms $\widehat{\mu}$ of $\mu \in M_d(G)$, with the following quotient norm:

$$\|f\|_{B_d(E)} = \inf\{\|\mu\| \mid \mu \in M_d(G) \ \&\ \widehat{\mu}|_E = f\}.$$

$B(E)$ is the space of restrictions to E of Fourier transforms $\widehat{\mu}$ of $\mu \in M(G)$, with this quotient norm:

$$\|f\|_{B(E)} = \inf\{\|\mu\| \mid \mu \in M(G) \ \&\ \widehat{\mu}|_E = f\}.$$

$\ell_\infty(E)$ is the space of all bounded functions on E with the supremum norm; $AP(E)$ is the closure in $\ell_\infty(E)$ of $B_d(E)$, and retains the supremum norm (cf. Lemma 1 of the Appendix). The following inclusions hold and are norm-decreasing:

$$(1) \qquad B_d(E) \subset AP(E) \subset \ell_\infty(E) \quad \text{and} \quad B_d(E) \subset B(E) \subset \ell_\infty(E).$$

In general, these inclusions are all strict. When Γ is infinite, equality is rare among all the subsets of Γ (measure zero in 2^Γ) but has been extensively studied. Condition (1) allows six possible equalities among the algebras $B_d(E)$, $AP(E)$, $\ell_\infty(E)$, and $B(E)$. Three of these equalities characterize special sets: Sidon ($B(E) = \ell_\infty(E)$; see [11]), I_0 sets ($AP(E) = \ell_\infty(E)$; see [6]), and helsonian ($B_d(E) = AP(E)$ by Proposition 2 of the Appendix). Kahane resolved one of the remaining possible equalities by proving that I_0 is equivalent to the formally stricter condition $B_d(E) = \ell_\infty(E)$ (see [7]); Kalton's proof of this is in the Appendix. It follows from Kahane's theorem that

$$I_0 \Rightarrow \text{helsonian} \quad \text{and} \quad I_0 \Rightarrow \text{Sidon}.$$

By Proposition 3 of the Appendix, helsonian implies Sidon; thus

$$(2) \qquad I_0 \Rightarrow \text{helsonian} \Rightarrow \text{Sidon}.$$

Bourgain resolved another possible equality by showing that $B_d(E) = B(E)$ implies that E is I_0 (see [1]). By Proposition 4 of the Appendix, $B(E) = AP(E)$ implies that E is I_0, thus disposing of the last possible equality. Example 5 of the Appendix proves that helsonian (Sidon) does not imply I_0. It is unknown whether helsonian (Sidon) sets must be a finite union of I_0 sets [5]. Also unknown is whether Sidon sets must be helsonian. Concerning this last question, there is this theorem by Ramsey: if a Sidon subset of the integers $\mathbb{Z}$ clusters at any member of $\mathbb{Z}$ in $b\mathbb{Z}$, then there is a Sidon set which is dense in $b\mathbb{Z}$ and hence clearly not helsonian [15].

Among the four algebras $B_d(E)$, $B(E)$, $AP(E)$ and $\ell_\infty(E)$, two inclusion relations remain to be explored: $B(E) \subset AP(E)$ and $AP(E) \subset B(E)$. If Γ is an abelian group of bounded order, $B(E) \subset AP(E)$ implies that E is I_0 (see

[17]). (In [17], a hypothesis which is formally weaker than $B(E) \subset AP(E)$ is shown to be sufficient to make E be I_0.) No work has been reported on $AP(E) \subset B(E)$.

Sidon and I_0 sets are F_σ in 2^Γ. David Grow proved that, for finite subsets E of $\mathbb{Z}$, $B(E) = B_d(E)$ *isometrically* [5]. As he rightly concludes, "one cannot determine whether a Sidon set E is a finite union of I_0 sets merely by examining the norms of interpolating discrete measures". This theorem generalizes to Γ (indeed to the dual object of any compact topological group).

THEOREM 1. *The algebras $B_d(E)$ and $B(E)$ are isometric for finite subsets E of a discrete abelian group Γ.*

Proof. Let E be given and $\varepsilon \in \mathbb{R}^+$. Let $f \in B(E)$ and $\mu \in M(G)$ such that $\widehat{\mu}|_E = f$ and $\|\mu\| \le (1+\varepsilon)\|f\|_{B(E)}$. There exists a neighborhood U of $0 \in G$ such that

$$g \in U \quad \text{implies} \quad (\forall x \in E)\left(|x(g) - 1| < \varepsilon' = \frac{\varepsilon}{\|\mu\| + 1}\right).$$

Since G is compact and $\{g + U \mid g \in G\}$ is an open covering of G, there is a finite set $G' = \{g_1, \ldots, g_n\}$ such that $\{g + U \mid g \in G'\}$ covers G. Let $E_1 = g_1 + U$; for $j > 1$ set $E_j = (g_j + U)\setminus(\bigcup_{i<j} E_i)$. Then G is the disjoint union of the E_i's. Let $\nu = \sum_{j=1}^n \mu(E_j)\delta_{g_j}$. Then

$$\|\nu\| = \sum_{j=1}^n |\mu(E_j)| \le \|\mu\| \le (1+\varepsilon)\|f\|_{B(E)}.$$

Also, for $x \in E$, with $|\mu|$ denoting the total variation measure for μ,

$$\begin{aligned}
|\widehat{\nu}(x) - f(x)| = |\widehat{\nu}(x) - \widehat{\mu}(x)| &= \Big|\sum_{j=1}^n \Big[\mu(E_j)x(-g_j) - \int_{E_j} x(-g)\,d\mu(g)\Big]\Big| \\
&= \Big|\sum_{j=1}^n \int_{E_j} [x(-g_j) - x(-g)]\,d\mu(g)\Big| \\
&\le \sum_{j=1}^n \int_{E_j} |x(-g_j) - x(-g)|\,d|\mu|(g) \\
&\le \sum_{j=1}^n \int_{E_j} |x(g - g_j) - 1|\,d|\mu|(g) \le \sum_{j=1}^n \varepsilon'|\mu|(E_j) = \varepsilon'\|\mu\| < \varepsilon.
\end{aligned}$$

By the previous paragraph, there is a sequence of discrete measures ν_j such that $\|\nu_j\| \le (1 + 1/j)\|f\|_{B(E)}$ and $\|\widehat{\nu}_j|_E - f\|_\infty \le (1/j)$. Thus $\widehat{\nu}_j|_E$ converges to f in $\ell_\infty(E)$. By [16, p. 222] any finite subset of Γ is an I_0 set.

By Theorem 7 of the Appendix, the $\ell_\infty(E)$ and $B_d(E)$ norms are equivalent: there is a constant K such that, for all $g \in \ell_\infty(E)$,

$$\|g\|_{B_d(E)} \le K\|g\|_\infty.$$

Thus $\widehat{\nu}_j|_E$ converges to f in $B_d(E)$, and hence

$$\|f\|_{B_d(E)} = \lim_{j\to\infty} \|\widehat{\nu}_j|_E\|_{B_d(E)} \le \limsup_{j\to\infty} \|\nu_j\| \le \|f\|_{B(E)}.$$

That proves isometry, since $\|f\|_{B_d(E)} \le \|f\|_{B(E)}$ always holds.

There is a more elementary way to see this, without using [16]. Since E is finite, $B_d(E)$ is a finite-dimensional vector subspace of $\ell_\infty(E)$. Due to the finite-dimensionality of $B_d(E)$, $B_d(E)$ is a closed subspace of $\ell_\infty(E)$ and norm equivalence holds for $g \in B_d(E)$. Since $\widehat{\nu}_j$ is from $B_d(E)$ and converges to $f \in \ell_\infty(E)$, the closedness of $B_d(E)$ puts f in $B_d(E)$. By the norm equivalence, $\widehat{\nu}_j$ converges to f in $B_d(E)$, and the rest of the proof is valid. ∎

Sidon sets are "finitely describable" by norm comparisons. Following [11], the Sidon constant of a set $E \subset \Gamma$ is the minimum constant $\alpha(E) \ge 0$ such that, for all $f \in \ell_\infty(E)$, $\|f\|_{B(E)} \le \alpha(E)\|f\|_\infty$. As in [11], this is the same minimum constant such that $\|\tau\|_{A(G)} \le \alpha(E)\|\tau\|_{C(G)}$ for all $\tau \in \mathrm{Trig}_E(G)$, the trigonometric polynomials on G with spectrum in E. This is true because, viewing $\mathrm{Trig}_E(G)$ as a closed subspace of $C(G)$, one has $\mathrm{Trig}_E(G)^* = B(E)$ (isometrically) while $A(G)$ is isometric to $\ell_1(\Gamma)$ and hence $A(G)^*$ is isometric to $\ell_\infty(\Gamma)$.

It follows that

$$E_1 \subset E_2 \quad \text{implies} \quad \alpha(E_1) \le \alpha(E_2) \tag{3}$$

and that

$$\alpha(E) = \sup\{\alpha(F) \mid F \subset E \ \&\ F \text{ is finite}\}. \tag{4}$$

These observations lead to the next lemma:

Lemma 2. *Let $\mathcal{S}_r = \{E \subset \Gamma \mid \alpha(E) \le r\}$. Then $\mathcal{S}_r$ is closed in 2^Γ.*

Proof. In this proof, we identify $A \subset \Gamma$ with $\chi_A \in 2^\Gamma$. Let E_β be a net in $\mathcal{S}_r$ which converges to $E \subset \Gamma$. Let F be any finite subset of E. Because the convergence in 2^Γ is pointwise, there is some β_0 for which $\beta \ge \beta_0$ implies $F \subset F_\beta$. By (3) above, $\alpha(F) \le \alpha(F_\beta) \le r$. Since this holds for all finite $F \subset E$, $\alpha(E) \le r$ by (4) above. ∎

Proposition 3. *For discrete abelian groups Γ, the class of Sidon sets is an F_σ subset of 2^Γ: it is $\bigcup_n \mathcal{S}_n$ with $\mathcal{S}_n$ as in Lemma 2.*

David Grow's theorem makes clear that only making norm comparisons will not extend Proposition 3 to I_0 sets. The following definition provides appropriate tools.

DEFINITION. Let $D(N)$ denote the set of discrete measures μ on G for which

$$\mu = \sum_{j=1}^{N} c_j \delta_{t_j},$$

where $|c_j| \le 1$ and $t_j \in G$ for each j. For $E \subset \Gamma$ and $\delta \in \mathbb{R}^+$, let $AP(E, N, \delta)$ be the set of $f \in \ell_\infty(E)$ for which there exists $\mu \in D(N)$ such that

$$\|f - \widehat{\mu}|_E\|_\infty \le \delta.$$

E is said to be $I(N, \delta)$ if the unit ball in $\ell_\infty(E)$ is a subset of $AP(E, N, \delta)$. Further, $N(E)$, the I_0 degree of a set E, is the minimum m for which E is $I(m, 1/2)$ if such an m exists, and ∞ otherwise. [By Theorem 7 of the Appendix, E is I_0 if and only $N(E) < \infty$.]

The analog of condition (3) is immediate from the preceding definitions:

$$\text{(3I)} \qquad E_1 \subset E_2 \quad \text{implies} \quad N(E_1) \le N(E_2).$$

The next lemma is the analog of condition (4).

LEMMA 4. *For $E \subset \Gamma$,*

$$\text{(4I)} \qquad N(E) = \sup\{N(F) \mid F \text{ is a finite subset of } E\}.$$

Proof. Set J equal to the right-hand side of (4I). By condition (3I), $J \le N(E)$. If $J = \infty$, then $N(E) = \infty$ and hence $J = N(E)$. So suppose that J is finite. Let $f \in \ell_\infty(E)$ such that $\|f\|_\infty \le 1$. For each finite $F \subset E$, interpolate $f|_F$ within $1/2$ by a discrete measure $\mu^F \in D(J)$; write μ^F as

$$\mu^F = \sum_{j=1}^{J} c_j^F \delta_{g_j^F}$$

with $|c_j^F| \le 1$. The finite subsets of E form a net, ordered by increasing inclusion. By the compactness of G (from which g_j^F comes), and the compactness of the unit disc in $\mathbb{C}$, one may choose $2J$ subnets successively so that, for the final net $\{F_\alpha\}_\alpha$, one has

$$\lim_\alpha g_j^{F_\alpha} = g_j \ \& \ \lim_\alpha c_j^{F_\alpha} = c_j \qquad \text{for all } 1 \le j \le J.$$

Necessarily, $|c_j| \le 1$. Set $\mu = \sum_{j=1}^{J} c_j \delta_{g_j}$. Let $\gamma \in E$. There is some α_0 in the subnet such that $\gamma \in F_\alpha$ for all $\alpha \ge \alpha_0$. Also for $\alpha \ge \alpha_0$,

$$|f(\gamma) - \widehat{\mu^{F_\alpha}}(\gamma)| \le 1/2.$$

However, $\lim_\alpha \gamma(g_j^{F_\alpha}) = \gamma(g_j)$ for $1 \le j \le N$ because γ is a continuous character on G. It follows that

$$\lim_\alpha \widehat{\mu^{F_\alpha}}(\gamma) = \lim_\alpha \sum_{j=1}^{J} c_j^{F_\alpha} \gamma(-g_j^{F_\alpha}) = \sum_{j=1}^{J} c_j \gamma(-g_j) = \widehat{\mu}(\gamma).$$

Thus $|f(\gamma)-\widehat{\mu}(\gamma)| \leq 1/2$. That establishes $f \in AP(E, J, 1/2)$. So $N(E) \leq J$. ∎

The proof of the next proposition is the same as that of Lemma 2 and Proposition 3.

PROPOSITION 5. *The I_0 sets are an F_σ in 2^Γ: they are $\bigcup_n \{E \subset \Gamma \mid N(E) \leq n\}$ where $\{E \subset \Gamma \mid N(E) \leq n\}$ is closed in 2^Γ.*

The author first realized that I_0 sets and Sidon sets are F_σ in 2^Γ, when studying $A = \widetilde{A}$ sets: those sets for which $A(E) = B(E) \cap c_0(E)$ [4, p. 364]. Whether $A = \widetilde{A}$ sets are F_σ in 2^Γ is not known. Equally unknown is the status of sets E such that $A(E) = B_0(E)$, where

$$B_0(E) = \{f|_E \mid f \in B(\Gamma) \cap c_0(\Gamma)\}.$$

Both of these properties, to a naive view, seem to "live at infinity" and thus fail to be "finitely describable". If it could be proved that they are *not* F_σ in 2^Γ, then questions (1) and (1′) of [4, p. 369] would have negative answers. An open question which is closer to the focus of this paper is this: do helsonian sets constitute an F_σ class?

"Finitely described", again. In [6], two other equivalent formulations of being I_0 are established. First, a set E is I_0 if and only if every function on E taking values 0 and 1 can be extended to a continuous almost periodic function over Γ [6, p. 25]. Second, a set E is an I_0 set if and only if, for every subset $F \subset E$, the sets F and $E \backslash F$ have disjoint closures in $b\Gamma$. These formulations permit a weakening of the sufficient conditions listed in Theorem 7 of the Appendix (a very similar and yet weaker condition is in [12]).

DEFINITION. Let C_1 and C_2 be closed subsets of $\mathbb{C}$. For $E \subset \Gamma$, E is said to be $J(N, C_1, C_2)$ if and only if, for all $F \subset E$, there is some $\mu \in D(N)$ such that $\widehat{\mu}(F) \subset C_1$ and $\widehat{\mu}(E\backslash F) \subset C_2$. When $C_1 = \{z \mid \Im(z) \geq \delta\}$, and $C_2 = \{z \mid \Im(z) \leq -\delta\}$, $J(N, C_1, C_2)$ is abbreviated as $J(N, \delta)$. $S(E)$ is the minimum m such that E is $J(m, 1/2)$ if such an m exists, and ∞ otherwise. [By Proposition 6 below, E is I_0 if and only if $S(E) < \infty$.]

PROPOSITION 6. *The following are equivalent*:

(1) *E is an I_0 set.*
(2) *E is $J(N, C_1, C_2)$ for some N and some disjoint subsets C_1 and C_2.*
(3) *For all $0 < \delta < 1$, there is some N such that E is $J(N, \delta)$.*

Proof. (3)⇒(2) is immediate.

(2)⇒(1). Assume that E is $J(N, C_1, C_2)$ for some disjoint C_1 and C_2 and some N. For $F \subset E$, let $\mu_F \in D(N)$ satisfy condition (2) for F. By [18, p. 32], the group $b\Gamma$ is the maximal ideal space of $M_d(G)$ and the Gelfand transform is just the Fourier–Stieltjes transform. Because $D(N) \subset M_d(G)$, $\widehat{\mu_F}$ is a continuous function on $b\Gamma$. Because C_1 is a closed subset of $\mathbb{C}$, $H_1 = \widehat{\mu_F}^{-1}(C_1)$ is a closed subset of $b\Gamma$ with $F \subset H_1$. Likewise, $H_2 = \widehat{\mu_F}^{-1}(C_2)$ is a closed subset of $b\Gamma$ with $(E\backslash F) \subset H_2$. Because C_1 and C_2 are disjoint, H_1 and H_2 are disjoint; thus F and $E\backslash F$ have disjoint closures in $b\Gamma$. Because this holds for all $F \subset E$, E is an I_0 set by [6].

(1)⇒(3). Now suppose that E is an I_0 set and consider any δ such that $0 < \delta < 1$. By Theorem 7 of the Appendix, there is some N such that E is $I(N, 1-\delta)$. Let $F \subset E$; the function h which is i on F and $-i$ on $E\backslash F$ is in the unit ball of $\ell_\infty(E)$. By the definition of $I(N, 1-\delta)$, there is some $\mu \in D(N)$ such that

$$\|\widehat{\mu}|_E - h\|_\infty \le 1 - \delta.$$

For $\gamma \in F$, $h(\gamma) = i$ and hence $\Im(\widehat{\mu}(\gamma)) \ge 1 - (1-\delta) = \delta$. For $\gamma \in (E\backslash F)$, $h(\gamma) = -i$ and hence $\Im(\widehat{\mu}(\gamma)) \le -1 + (1-\delta) \le -\delta$. ■

The proof of Proposition 6 provides the following corollary.

COROLLARY 7. *For $E \subset \Gamma$, $S(E) \le N(E)$.*

Bounding $N(E)$ by some function of $S(E)$ is the purpose of the next theorem.

THEOREM 8. *There is a non-decreasing function ϕ with $\phi(\mathbb{Z}^+) \subset \mathbb{Z}^+$ such that, for all discrete abelian groups Γ and all $E \subset \Gamma$, $N(E) \le \phi(S(E))$.*

Some lemmas will help in proving Theorem 8. Lemma 9 follows immediately from the definitions of $N(E)$ and $S(E)$.

LEMMA 9. *For $E \subset \Gamma$ and $\gamma \in \Gamma$, $N(E) = N(E+\gamma)$ and $S(E) = S(E+\gamma)$.*

LEMMA 10. *For any N, let S be a finite set which is $1/(8N)$ dense in $\mathbb{T}$ and let $E \subset \Gamma$ with $S(E) \le N$. Then, for all subsets $F \subset E$, there are N points $t_j \in G$, integers $r_j \in [0, 8N]$, and $s_j \in S$ such that*

$$(\forall \gamma \in F)[\Im(\widehat{\mu}(\gamma)) \ge 1/4] \quad \textit{and} \quad (\forall \gamma \in E\backslash F)[\Im(\widehat{\mu}(\gamma)) \le -1/4],$$

where

$$\mu = (8N)^{-1} \sum_{j=1}^{N} s_j r_j \delta_{t_j}.$$

P r o o f. By the definition of $S(E)$, E is $J(S(E), 1/2)$ and hence $J(N, 1/2)$. Thus, for any $F \subset E$, there is a discrete measure $\nu \in D(N)$ such that

$$(\forall \gamma \in F)[\Im(\widehat{\nu}(\gamma)) \ge 1/2] \quad \text{and} \quad (\forall \gamma \in E\backslash F)[\Im(\widehat{\nu}(\gamma)) \le -1/2],$$

where $\nu = \sum_{j=1}^{N} c_j \delta_{t_j}$ for some t_j's in G and c_j's in the unit disc of $\mathbb{C}$. Write c_j as $d_j|c_j|$ with $|d_j| = 1$. Since S is $1/(8N)$ dense in $\mathbb{T}$, one may choose

$s_j \in S$ such that $|d_j - s_j| < 1/(8N)$. Let $r_j = \lfloor 8N|c_j| \rfloor$. Then, if

$$\mu = (8N)^{-1} \sum_{j=1}^{N} s_j r_j \delta_{t_j},$$

it follows that

$$\begin{aligned}
\|\nu - \mu\|_{M(G)} &\le \sum_{j=1}^{N} |c_j - s_j r_j/(8N)| \\
&\le \sum_{j=1}^{N} |c_j - |c_j| s_j| + \sum_{j=1}^{N} |s_j |c_j| - s_j r_j/(8N)| \\
&= \sum_{j=1}^{N} |c_j||d_j - s_j| + \sum_{j=1}^{N} |s_j| \cdot ||c_j| - r_j/(8N)| \\
&\le \sum_{j=1}^{N} |d_j - s_j| + \sum_{j=1}^{N} ||c_j| - r_j/(8N)| \\
&\le N/(8N) + N/(8N) = 1/4.
\end{aligned}$$

It next follows that, for $\gamma \in F$,

$$\Im(\widehat{\mu}(\gamma)) = \Im[(\widehat{\nu}(\gamma)) - \{\widehat{\nu}(\gamma) - \widehat{\mu}(\gamma)\}] \ge \Im[\widehat{\nu}(\gamma)] - \|\nu - \mu\|_{M(G)} \ge 1/4.$$

Likewise, for $\gamma \in (E \backslash F)$, $\Im(\widehat{\mu}(\gamma)) \le -1/4$. ∎

LEMMA 11. *For any N, let S be a finite set which is $1/(8N)$ dense in $\mathbb{T}$. Assume that $S(E) \le N$ and $E \subset \{1\} \times \Gamma \subset \mathbb{Z}_2 \times \Gamma$. For $F \subset E$ and $s \in S$ there are $8N^2$ points of G, here labeled as $t_{s,j}$, such that*

$$(\forall \gamma \in F)[\Im(\widehat{\tau}(\gamma)) \ge 1/8] \quad \textit{and} \quad (\forall \gamma \in (E \backslash F))[\Im(\widehat{\tau}(\gamma)) \le -1/8],$$

where

$$\tau = (8N)^{-1} \sum_{s \in S} s \sum_{j=1}^{8N^2} \delta_{t_{s,j}}.$$

Proof. Let $p = (1, 0) \in \mathbb{Z}_2 \times G$. Then, for all $\gamma \in E$, $\widehat{\delta_0}(\gamma) = 1$ while $\widehat{\delta_p}(\gamma) = -1$. Thus for $\gamma \in E$, $\widehat{\delta_0}(\gamma) + \widehat{\delta_p}(\gamma) = 0$.

Let $F \subset E$ and μ be a measure provided for F by Lemma 10. Rearrange μ as follows:

$$\mu = (8N)^{-1} \sum_{j=1}^{N} s_j \sum_{q=1}^{r_j} \delta_{t_{j,q}},$$

where $t_{j,q} = t_j$ for all $q \in [1, r_j]$. Set

$$W_j = \begin{cases} 2^{-1}(8N - r_j)(\delta_0 + \delta_p) & \text{for } r_j \text{ even}, \\ \delta_0 + 2^{-1}(8N - r_j - 1)(\delta_0 + \delta_p) & \text{for } r_j \text{ odd}. \end{cases}$$

Let $\phi = \mu + (8N)^{-1}\sum_{j=1}^{N} s_j W_j$. Then one may write ϕ as

$$\phi = (8N)^{-1}\sum_{j=1}^{N} s_j \sum_{q=1}^{8N} \delta_{t_{j,q}}.$$

Note that $\widehat{W_j}(\gamma) \in \{0,1\}$ for $\gamma \in E$ and therefore

$$|\widehat{\phi}(\gamma) - \widehat{\mu}(\gamma)| \le (8N)^{-1}\sum_{j=1}^{N} |\widehat{W_j}(x)| \le 1/8.$$

Thus, for $\gamma \in F$,

$$\Im(\widehat{\phi}(\gamma)) = \Im\{\widehat{\mu}(\gamma) - (\widehat{\mu}(\gamma) - \widehat{\phi}(\gamma))\} \ge 1/4 - |\widehat{\mu}(\gamma) - \widehat{\phi}(\gamma)| \ge 1/8.$$

Likewise, for $\gamma \in (E\backslash F)$,

$$\Im(\widehat{\phi}(\gamma)) = \Im\{\widehat{\mu}(\gamma) - (\widehat{\mu}(\gamma) - \widehat{\phi}(\gamma))\} \le -1/4 + |\widehat{\mu}(\gamma) - \widehat{\phi}(\gamma)| \le -1/8.$$

Next, rewrite ϕ as follows:

$$\phi = (8N)^{-1}\sum_{s\in S} s \sum_{\substack{j\in[1,N]\\ \&\, s_j = s}} \sum_{q=1}^{8N} \delta_{t_{j,q}} = (8N)^{-1}\sum_{s\in S} sV_s.$$

The number of point masses in V_s is $8Nf_s$ for some integer $f_s \in [0,N]$ (f_s is the number of j's such that $s_j = s$). Let

$$Z_s = (N - f_s)(4N)(\delta_0 + \delta_p)$$

and set

$$\tau = \phi + (8N)^{-1}\sum_{s\in S} sZ_s.$$

Note that $\widehat{Z_s}(x) = 0$ for all $x \in E$, $\widehat{\tau}|_E = \widehat{\phi}|_E$, and τ may be written as

$$(8N)^{-1}\sum_{s\in S} s \sum_{q=1}^{8N^2} \delta_{t_{s,q}}. \ \blacksquare$$

Proof of Theorem 8. Set $\phi(\infty) = \infty$ and let $\phi(N) = \sup\{N(E) \mid S(E) \le N\}$. If $\phi(N) < \infty$ for all N, the theorem is proved. Suppose that $\phi(N) = \infty$ for a particular N. That is, there is a sequence of discrete abelian groups Ω_i (with dual group H_i) and subsets $W_i \subset \Omega_i$ such that $S(W_i) \le N$ and $N(W_i) > i$. Let $E_i = \{1\} \times W_i \subset \Gamma_i$, where $\Gamma_i = \mathbb{Z}_2 \times \Omega_i$ and $G_i = \mathbb{Z}_2 \times H_i$ is the group dual to Γ_i. By Lemma 9, $S(E_i) = S(W_i) \le N$ and $N(E_i) = N(W_i)$. Let Γ be the direct sum of the Γ_i, which is the set of all sequences $\{\gamma_i\}_i$ with $\gamma_i \in \Gamma_i$ and at most finitely many $\gamma_i \ne 0$ [assume that the Γ_i's are presented additively]. The dual group of Γ is the following direct product:

$$G = \prod_i G_i.$$

If $\gamma = \{\gamma_i\}_i \in \Gamma$ and $g = \{g_i\}_i \in G$, then $\langle \gamma, g\rangle = \prod_i \langle \gamma_i, g_i\rangle$, where the latter infinite product has at most finitely many factors that differ from 1. Γ_i may be viewed as a subset of Γ in the natural way, as the set of $\gamma \in \Gamma$ such that $\gamma_j = 0$ for $j \neq i$. Denote this canonical copy of Γ_i by Γ_i^*. For $\gamma \in \Gamma_i^* \subset \Gamma$ and $g \in G$,

$$\widehat{\delta_g}(\gamma) = \langle \gamma_i, -g_i\rangle = \widehat{\delta_{g_i}}(\gamma_i),$$

where g_i and γ_i are the respective ith components of g and γ. Thus, $N(E_i) = N(E_i^*)$ and $S(E_i) = S(E_i^*)$ for each $E_i \subset \Gamma_i$ and its canonical image E_i^* in Γ_i^*.

It will be proved that $E^* = \bigcup_i E_i^*$ is an I_0 set and thus $N(E^*) < \infty$ by Theorem 7 of the Appendix. That will contradict equation (3I), which says that $N(E^*) \geq N(E_i^*)$, and thus

$$N(E^*) \geq N(E_i^*) = N(E_i) = N(W_i) > i \quad \text{for all } i.$$

This contradiction will prove that $\phi(N) < \infty$ for all N.

To see that E^* is I_0, let S be a finite set which is $1/(8N)$ dense in $\mathbb{T}$ of cardinality M. It will be shown that E^* is $J(8MN^2, 1/8)$ and hence an I_0 set by Proposition 6.

Let $F^* \subset E^*$, and set $F_i^* = F^* \cap E_i^*$. Let F_i be the pre-image of F_i^* under the canonical embedding of Γ_i into Γ. Because $S(E_i) \leq N$ and $F_i \subset E_i$, Lemma 11 provides a discrete measure μ_i on G_i of the form

$$\mu_i = (8N)^{-1} \sum_{s\in S} s \sum_{j=1}^{8N^2} \delta_{t^i_{s,j}}$$

such that

$$(\forall \gamma \in F_i)[\Im(\widehat{\mu_i}(\gamma)) \geq 1/8] \quad \text{and} \quad (\forall \gamma \in E_i \backslash F_i)[\Im(\widehat{\mu_i}(\gamma)) \leq -1/8].$$

Let $t_{s,j} \in G$ be defined to be $t^i_{s,j}$ in the ith coordinate, and set

$$\mu = (8N)^{-1} \sum_{s\in S} s \sum_{j=1}^{8N^2} \delta_{t_{s,j}}.$$

Because any $\gamma \in E_i^*$ has coordinates equal to 0 apart from the ith coordinate, and $\gamma_i \in E_i$, one has

$$\widehat{\delta_{t_{s,j}}}(\gamma) = \langle -t_{s,j}, \gamma\rangle = \langle -t^i_{s,j}, \gamma_i\rangle = \widehat{\delta_{t^i_{s,j}}}(\gamma_i).$$

For $\gamma \in E_i^*$, it follows that $\widehat{\mu}(\gamma) = \widehat{\mu}_i(\gamma_i)$ with $\gamma_i \in E_i$. Note that $\gamma_i \in F_i$ if and only if $\gamma \in F_i^*$. Thus, for all i,

$$(\forall\gamma \in F_i^*)[\Im(\widehat{\mu}(\gamma)) \geq 1/8] \quad \text{while} \quad (\forall\gamma \in (E_i^*\backslash F_i^*))[\Im(\widehat{\mu}(\gamma)) \leq -1/8].$$

Since $F^* = \bigcup_i F_i^*$, the imaginary part of $\widehat{\mu}$ is at least $1/8$ on F^* and at most $-1/8$ on $E^*\backslash F^*$. This holds for an arbitrary $F^* \subset E^*$, with a measure in $D(8MN^2)$. Thus E^* is $J(8MN^2, 1/8)$. ∎

A more direct proof of Theorem 8 can be adapted from [9], in which the following theorem is proved. Consider a Banach algebra B of continuous functions on a compact Hausdorff space $\mathfrak{M}$. Assume that for every closed subset F of $\mathfrak{M}$, there exists a positive number $\varepsilon = \varepsilon(F)$ such that whenever N is both open and closed in F, B contains an element h of norm one satisfying $\Re(h(M)) < 0$ for $M \in N$, $\Re(h(M)) > \varepsilon$ for $M \in F\backslash N$. Then $B = C(\mathfrak{M})$. In [9] a polynomial P is fixed, depending only on ε and some $\varepsilon' > 0$, such that for F, N and the corresponding h of the hypotheses, $P(h)$ satisfies $|P(h)(M)| < \varepsilon'$ for $M \in F\backslash N$ while $|P(h)(M) - 1| < \varepsilon'$ for $M \in N$. Thus χ_N is approximated by $P(h)$ within ε' in $\ell^\infty(F)$. With appropriate scalings ($\varepsilon = 1/(2S(E))$), this could be applied to $h = \widehat{\nu}$ where $\nu = -i\mu$, $\mu \in D(S(E))$ with $\Im(\widehat{\mu}) \geq 1/2$ on some $F \subset E$ while $\Im(\widehat{\mu}) \leq -1/2$ on $E\backslash F$. It is clear that $P(\nu)$ is in $D(n)$ for some n which is determined by $S(E)$ and ε' (and P, which is in turn specified to depend only on $\varepsilon = 1/(2S(E))$ and ε'). If ε' is set equal to $1/144$, one can proceed as in the next paragraphs to get $N(E) \leq 36n$.

Following [12], one could define another degree for I_0 sets. For $\xi = (g_1, \dots, g_n) \in G^n$ and $\gamma \in \Gamma$, let $\xi(\gamma) = (\gamma(g_1), \dots, \gamma(g_n))$. For $\xi \in G^n$ and real $\varepsilon > 0$, let $U(\xi, \varepsilon) = \{\lambda \in \Gamma \mid \sup_i |\lambda(g_i) - 1| < \varepsilon\}$. A basis for the topology of $b\Gamma$ consists of $\gamma + U(\xi, \varepsilon)$, where γ ranges over Γ, ξ ranges over $\bigcup_n G^n$ and ε ranges over $\mathbb{R}^+$. By [6] and [12, Theorem 1, p. 172], $E \subset \Gamma$ is I_0 if and only if there are some k and real $\varepsilon > 0$ such that, for all $F \subset E$, there is some $\xi \in G^k$ for which $F + U(\xi, \varepsilon)$ and $(E\backslash F) + U(\xi, \varepsilon)$ are disjoint. Such sets are said to have order k (regardless of ε) [12]. Define $M(E)$ as the least k for which this result holds for k and $\varepsilon = 1/k$. By following the proof in [12, pp. 175–176], one can prove that $N(E) \leq \psi(M(E))$ for some non-decreasing function ψ such that $\psi(\mathbb{Z}^+) \subset \mathbb{Z}^+$. Also, $M(E) \leq 4N(E)$.

Here's how one could specify ψ. Given f in the unit ball of $\ell_\infty(E)$ and $M(E) \leq k$, one can approximate f within $1/4$ with a linear sum of characteristic functions:

$$\sum_{j=1}^{36} c_j \chi_{F_j} \quad \text{with } |c_j| \leq 1.$$

Each χ_{F_j} can be approximated within $1/144$ by the transform of a measure in $D(n)$ where n is chosen as follows. In [12, p. 175] there is a function $\chi \in A(T^k)$ chosen in a manner which depends only on k. Based upon it, choose N so that

$$\sum_{\substack{(n_1,\ldots,n_k)\in\mathbb{Z}^k\\ \&\, |n_1|+\ldots+|n_k|>N}} |\widehat{\chi}(n_1,\ldots,n_k)| \le 1/144.$$

Set

$$n = \sum_{\substack{(n_1,\ldots,n_k)\in\mathbb{Z}^k\\ \&\, |n_1|+\ldots+|n_k|\le N}} \lceil|\widehat{\chi}(n_1,\ldots,n_k)|\rceil.$$

In [12, p. 175], given an idempotent $e \in \ell_\infty(E)$ and a particular $\xi = (g_1,\ldots,g_k)$ which separates the support of e from its complement with $U(\xi,1/k)$, there is some Φ_e such that $e = \Phi_e \circ \xi|_E$ and $|\widehat{\Phi_e}(n_1,\ldots,n_k)| \le |\widehat{\chi}(n_1,\ldots,n_k)|$. Then, if

$$\mu = \sum_{\substack{(n_1,\ldots,n_k)\in\mathbb{Z}^k\\ \&\, |n_1|+\ldots+|n_k|\le N}} \widehat{\Phi_e}(n_1,\ldots,n_k)\delta_{-n_1g_1-\ldots-n_kg_k},$$

$\mu \in D(n)$ and $\widehat{\mu}$ interpolates e within $1/144$. By doing this to each F_j for f, one interpolates f within $1/2$ by the transform of a measure in $D(36n)$ and hence $N(E) \le 36n$. If $\psi(k) = \sup\{N(E) \mid M(E) \le k\}$, then $\psi(k) < \infty$, ψ is non-decreasing and $N(E) \le \psi(M(E))$.

To see that $M(E) \le 4N(E)$, let $n = N(E) < \infty$ and $F \subset E$. Let $f = 1$ on F and -1 on $E\backslash F$. Let $\mu \in D(n)$ interpolate f within $1/2$. If $\mu = \sum_{j=1}^n c_j\delta_{g_j}$, let $\xi = (g_1,\ldots,g_n)$. If $\lambda \in U(\xi,1/(4n))$, then for all γ,

$$|\widehat{\mu}(\gamma+\lambda) - \widehat{\mu}(\gamma)| \le 1/4.$$

Thus for $\gamma \in F$,

$$\Re(\widehat{\mu}(\gamma+\lambda)) \ge 1/2 - 1/4 = 1/4,$$

while for $\gamma \in E\backslash F$,

$$\Re(\widehat{\mu}(\gamma+\lambda)) \le -1/2 + 1/4 = -1/4.$$

It is evident that $F + U(\xi,1/(4n))$ and $(E\backslash F) + U(\xi,1/(4n))$ are disjoint. Thus $M(E) \le 4n$.

The proof of Theorem 8 provides an analog for I_0 sets of "sup-norm partitions" used among Sidon sets [4, p. 370]. What is different about this construction is the "DC-offset" (an electrical engineering term): shifting the W_i's into "odd" cosets before unioning them. This is not required in the usual sup-norm partition constructions.

PROPOSITION 12. *Let W_i be a sequence of I_0 sets, with W_i a subset of an abelian group Ω_i and $S(W_i) \le N$ for some N. If $\Gamma_i = \mathbb{Z}_2 \times \Omega_i$ and $E_i = \{1\} \times W_i$, then $E = \bigcup_i E_i$ is an I_0 set in the direct sum of the Γ_i's* *with* $S(E) \leq 32MN^2$ (*where* M *is the cardinality of a finite set which is* $1/(8N)$ *dense in* $\mathbb{T}$).

P r o o f. In the proof of Theorem 8, E is $J(8MN^2, 1/8)$. By repeating the interpolating measures 4 times, one sees that E is $J(32MN^2, 1/2)$ and hence $S(E) \leq 32MN^2$. ∎

Proposition 12 is proved in the category of discrete abelian groups, where there is plenty of room to fit diverse groups together. The analog of Proposition 12 is proved within $\mathbb{Z}$ in the next proposition. Some care must be taken with this new construction of I_0 sets, but its basic ideas are simple: rapidly dilate successive sets of the given sequence of I_0 sets and provide a "DC-offset".

PROPOSITION 13. *Let* $\{W_n\}_n$ *be a sequence of finite* I_0 *subsets of* $\mathbb{Z}$ *with* $S(W_n) \leq N$ *for all* n. *There is a sequence of integers* $\{k_n\}$ *with* $k_n \neq 0$ *for all* n *such that*

$$E = \bigcup_n (2k_n W_n + k_n)$$

is an I_0 *set with* $(2k_n W_n + k_n) \cap (2k_j W_j + k_j) = \emptyset$ *for* $n \neq j$.

LEMMA 14. *Let* $E \subset \mathbb{Z}$. *For any* N, *let* S *be a finite set which is* $1/(8N)$ *dense in* $\mathbb{T}$. *Assume that* $S(E) \leq N$ *and that* $E \subset k + 2k\mathbb{Z}$ *for some non-zero integer* k. *Let* $F \subset E$. *Then for each* $s \in S$ *there are* $8N^2$ *points of* $\mathbb{T}$, *here labeled as* $t_{s,j}$, *such that*

$$(\forall \gamma \in F)[\Im(\widehat{\tau}(\gamma)) \geq 1/8] \quad \textit{and} \quad (\forall \gamma \in (E' \backslash F))[\Im(\widehat{\tau}(\gamma)) \leq -1/8],$$

where

$$\tau = (8N)^{-1} \sum_{s \in S} s \sum_{j=1}^{8N^2} \delta_{t_{s,j}}.$$

P r o o f. Let $\mathbb{T}$, the dual group of $\mathbb{Z}$, be presented as the interval $(-\pi, \pi]$ with operations modulo 2π. An integer n acts on $t \in \mathbb{T}$ as follows:

$$n(t) = \langle n, t \rangle = e^{int}.$$

For all $x \in E$, $\widehat{\delta_0}(x) = 1$ while

$$\widehat{\delta_{\pi/k}}(x) = e^{ix\pi/k} = e^{i(k+2kj)\pi/k} = e^{i\pi} = -1.$$

Thus, for $x \in E$, $\widehat{\delta_0}(x) + \widehat{\delta_{\pi/k}}(x) = 0$. From this point, the proof is identical to that of Lemma 11, with $\delta_{\pi/k}$ replacing δ_p in that proof. ∎

P r o o f o f P r o p o s i t i o n 13. Without loss of generality, we may assume that $W_n \neq \emptyset$ for all n. The integers k_n shall be chosen inductively. Let $k_1 = 1$; given k_j for $j \leq n$, let D_n be the maximum absolute value of any element of $\bigcup_{j \leq n} (2k_j W_j + k_j)$. Fix some finite subset S which is $1/(8N)$ dense in $\mathbb{T}$ and of cardinality Q. For $n > 1$ choose $k_n \geq 32NQD_{n-1}$ and let $E_n = k_n + 2k_nW_n$. Since every element of E_n is an odd multiple of k_n, $|x| \geq k_n$ for all $x \in E_n$; since $E_n \neq \emptyset$, $D_n \geq k_n$. Since $F_1 \neq \emptyset$, $D_n \geq k_1 > 0$. Thus, for $n > 1$, $k_n \geq 32NQD_{n-1} > D_{n-1}$, which guarantees that E_n is disjoint from E_j for $j < n$. Finally, for $j < n$ and $x \in E_j$,

$$k_n \geq (32NQ)^{n-j} D_j \geq (32NQ)^{n-j}|x|.$$

In particular, $k_n \geq (32NQ)^{n-1}D_1 \geq (32NQ)^{n-1}$ for $n > 1$. [Of course, $k_1 = 1 \geq (32NQ)^0$ as well.]

Let $F \subset E$ and $F_i = F \cap E_i$. Lemma 14 provides a discrete measure μ_1 on $\mathbb{T}$ of the form

$$\mu_1 = (8N)^{-1} \sum_{s \in S} s \sum_{j=1}^{8N^2} \delta_{t^1_{s,j}}$$

such that

$$(\forall \gamma \in F_1)[\Im(\widehat{\mu_1}(\gamma)) \geq 1/8] \quad \text{and} \quad (\forall \gamma \in E_1 \backslash F_1)[\Im(\widehat{\mu_1}(\gamma)) \leq -1/8].$$

Proceed inductively. Suppose that for $j < n$ one has μ_j such that

$$(\forall \gamma \in F_j)[\Im(\widehat{\mu_j}(\gamma)) \geq 1/8] \quad \text{and} \quad (\forall \gamma \in E_j \backslash F_j)[\Im(\widehat{\mu_j}(\gamma)) \leq -1/8],$$

where

$$\mu_j = (8N)^{-1} \sum_{s \in S} s \sum_{q=1}^{8N^2} \delta_{t^j_{s,q}}$$

and $|t^j_{s,q} - t^{j-1}_{s,q}| \leq \pi/k_j$ for $j \in (1, n)$, $s \in S$, and $q \in [1, 8N^2]$. Because $E_n = k_n + 2k_nW_n$ with $k_n \neq 0$, one has $S(E_n) = S(W_n) \leq N$. By Lemma 14, there is some μ such that

$$(\forall \gamma \in F_n)[\Im(\widehat{\mu}(\gamma)) \geq 1/8] \quad \text{and} \quad (\forall \gamma \in E_n \backslash F_n)[\Im(\widehat{\mu}(\gamma)) \leq -1/8],$$

where

$$\mu = (8N)^{-1} \sum_{s \in S} s \sum_{q=1}^{8N^2} \delta_{z^n_{s,q}}.$$

However, since every $x \in E_n$ is a multiple of k_n, for any integers $p_{q,s}$,

$$\widehat{\delta_{w+z^n_{s,q}}}(x) = \widehat{\delta_{z^n_{s,q}}}(x) \quad \text{for } w = 2\pi p_{q,s}/k_n.$$

Thus $\widehat{\mu}|_{E_n} = \widehat{\lambda}|_{E_n}$ when

$$\lambda = (8N)^{-1} \sum_{s \in S} s \sum_{q=1}^{8N^2} \delta_{z^n_{s,q} + p_{q,s}2\pi/k_n}.$$

Choose $p_{q,s}$ so that

$$|z^n_{s,q} + p_{q,s}2\pi/k_n - t^{n-1}_{s,q}| \leq \pi/k_n.$$

Let $\mu_n = \lambda$ with this choice of the $p_{q,s}$. That is, $t^n_{s,q} = z^n_{s,q} + p_{q,s}2\pi/k_n$.

It follows that, for each $s \in S$ and $1 \le q \le 8N^2$, $t_{s,q} = \lim_{j\to\infty} t^j_{s,q}$ exists because

$$\sum_{j=2}^{\infty} |t^j_{s,q} - t^{j-1}_{s,q}| \le \sum_{j=2}^{\infty} \pi/k_j \le \pi \sum_{j=2}^{\infty} (32NQ)^{-j+1} < \infty.$$

Moreover, for $x \in E_j$ and $n > j$,

$$\begin{aligned}
|\widehat{\delta_{t^n_{s,q}}}(x) - \widehat{\delta_{t^j_{s,q}}}(x)| &= |e^{-ixt^n_{s,q}} - e^{-ixt^j_{s,q}}| \\
&= \Big| \sum_{w=j+1}^{n} e^{-ixt^w_{s,q}} - e^{-ixt^{w-1}_{s,q}} \Big| \\
&\le \sum_{w=j+1}^{n} |e^{-ixt^w_{s,q}} - e^{-ixt^{w-1}_{s,q}}| \\
&\le \sum_{w=j+1}^{n} |x(t^w_{s,q} - t^{w-1}_{s,q})| \le |x| \sum_{w=j+1}^{n} (\pi/k_w) \\
&\le \pi |x| \sum_{w=j+1}^{n} |x|^{-1} (32NQ)^{-(w-j)} \\
&< (\pi/(32NQ))(1 - 1/(32NQ))^{-1} \\
&= \pi/(32NQ - 1) < \pi/(31NQ).
\end{aligned}$$

If one fixes j and lets $n \to \infty$, then for $x \in E_j$,

$$|\widehat{\delta_{t_{s,q}}}(x) - \widehat{\delta_{t^j_{s,q}}}(x)| \le \pi/(31NQ).$$

Set

$$\varrho = (8N)^{-1} \sum_{s\in S} s \sum_{q=1}^{8N^2} \delta_{t_{s,q}}.$$

Then, for all $x \in E_j$,

$$\begin{aligned}
|\widehat{\mu_j}(x) - \widehat{\varrho}(x)| &= \Big| (8N)^{-1} \sum_{s\in S} s \sum_{q=1}^{8N^2} (\widehat{\delta_{t^j_{s,q}}}(x) - \widehat{\delta_{t_{s,q}}}(x)) \Big| \\
&\le (8N)^{-1} \sum_{s\in S} |s| \sum_{q=1}^{8N^2} (\pi/(31NQ)) = \pi/31.
\end{aligned}$$

Thus for all i,

$$\begin{aligned}
&(\forall \gamma \in F_i)[\Im(\widehat{\varrho}(\gamma)) \ge 1/8 - \pi/31] \quad \text{and} \\
&(\forall \gamma \in (E_i \backslash F_i))[\Im(\widehat{\varrho}(\gamma)) \le -1/8 + \pi/31].
\end{aligned}$$

Since $F = \bigcup_i F_i$, the imaginary part of $\widehat{\varrho}$ is at least .02 on F and at most $-.02$ on $E \backslash F$. Because this holds for any $F \subset E$ with a measure in $D(8QN^2)$, E is $J(8QN^2, .02)$ and hence I_0. ■

Proportions of Sidon sets are I_0 sets. The following theorem originated in conversations with Gilles Pisier.

THEOREM 15. *Let Γ be a discrete abelian group. Then $E \subset \Gamma$ is Sidon if and only if there are N and some real $r > 0$ such that, for all finite $F \subset E$, there is some $H \subset F$ for which $|H| \geq r|F|$ and $S(E) \leq N$.*

A key ingredient of the proof of Theorem 15 is a theorem of Pisier's [14, p. 941]. Other critical ingredients are recycled from [3, 13].

Proof of Theorem 15. To prove sufficiency, suppose that $E \subset \Gamma$ has some N and real $r > 0$ such that, for every finite subset $F \subset E$,

$$(\exists H \subset F)(|H| \geq r|F| \text{ and } S(H) \leq N).$$

Then H is $I(\phi(N), 1/2)$ by Theorem 8. By the proof of Theorem 7 of the Appendix, condition (3) of that theorem holds with $M = 2$ and $\delta = (1/2)^{1/\phi(N)}$. It follows that, for every f in the unit ball of $\ell_\infty(H)$, there is some $\mu \in M_d(G)$ such that $\widehat{\mu}|_H = f$ and $\|\mu\|_{M_d(G)} \leq L = 2\sum_{j=1}^\infty 2^{-j/\phi(N)} < \infty$. Thus, there is a constant L which depends only on N and satisfies $\|f\|_{B_d(H)} \leq L\|f\|_{\ell_\infty(H)}$ for all $f \in \ell_\infty(H)$. Since $\|f\|_{B(H)} \leq \|f\|_{B_d(H)}$, one has $\|f\|_{B(H)} \leq L\|f\|_{\ell_\infty(H)}$. Thus H is a Sidon set with Sidon constant at most L. That suffices to make E be Sidon by Corollary 2.3 of [14, p. 924].

Now suppose that E is Sidon. By [14, p. 941] there is some $\delta > 0$ such that, for all finite $F \subset E$, there are at least $2^{\delta|F|}$ points g_j of G such that, for $i \neq j$,

$$\sup_{\gamma \in F} |\gamma(g_j) - \gamma(g_i)| \geq \delta. \tag{5}$$

Necessarily, $\delta \leq 2$.

Let $F \subset E$ with $|F| = n$. Enumerate F as $\gamma_1, \ldots, \gamma_n$. Choose p so that $\tau = 2\pi/p < \delta/2$ (e.g., let $p = 1 + \lceil 4\pi/\delta \rceil$). Partition $\mathbb{T}$ into disjoint arcs, T_k, $0 \leq k < p$, of the form

$$T_k = \{e^{i\theta} \mid k\tau \leq \theta < (k+1)\tau\}.$$

Let $Q = \lceil (1 - 2^{-\delta/2})^{-1} \rceil$ and set $\tau' = \tau/Q$. Partition each T_k into Q arcs $U_{k,m}$ of the form

$$U_{k,m} = \{e^{i\theta} \mid k\tau + m\tau' \leq \theta < k\tau + (m+1)\tau'\},$$

for $0 \leq m < Q$. Finally, let $\mathcal{S}_0$ denote a set of at least $2^{\delta|F|}$ points of G which satisfy inequality (5).

Define $\mathcal{S}_i$ inductively. Let

$$\mathcal{S}^i_k = \{g \in \mathcal{S}_{i-1} \mid \gamma_i(g) \in T_k\} \quad \text{and} \quad \mathcal{S}^i_{k,m} = \{g \in \mathcal{S}_{i-1} \mid \gamma_i(g) \in U_{k,m}\}.$$

Then $\mathcal{S}_{i-1} = \bigcup_{k=0}^{p-1} \mathcal{S}^i_k$ and $\mathcal{S}^i_k = \bigcup_{m=0}^{Q-1} \mathcal{S}^i_{k,m}$. There is some $m(i,k)$ such that

$$|\mathcal{S}^i_{k,m(i,k)}| \le Q^{-1}|\mathcal{S}^i_k|.$$

So,

$$\Big|\bigcup_{k=0}^{p-1} \mathcal{S}^i_{k,m(i,k)}\Big| \le Q^{-1}|\mathcal{S}_{i-1}|.$$

Let

$$\mathcal{S}_i = \mathcal{S}_{i-1} \setminus \bigcup_{k=0}^{p-1} \mathcal{S}^i_{k,m(i,k)}.$$

Then $|\mathcal{S}_i| \ge (1-Q^{-1})|\mathcal{S}_{i-1}|$. By induction one has $|\mathcal{S}_n| \ge (1-Q^{-1})^n|\mathcal{S}_0|$. Note that $Q \ge (1-2^{-\delta/2})^{-1}$; consequently, $(1-Q^{-1}) \ge 2^{-\delta/2}$. Therefore,

$$|\mathcal{S}_n| \ge (1-Q^{-1})^n|\mathcal{S}_0| \ge (2^{-\delta/2})^n 2^{\delta n} = 2^{n\delta/2}.$$

For $1 \le i \le n$ and $1 \le k < p$, let $I_{i,k}$ be the arc between $U_{k-1,m(i,k-1)}$ and $U_{k,m(i,k)}$. For $k = 0$, let $I_{i,0}$ be the arc between $U_{p-1,m(i,p-1)}$ and $U_{0,m(i,0)}$. Necessarily,

$$I_{i,k} \subset \{e^{i\theta} \mid (k-1)\tau + \tau' \le \theta < (k+1)\tau - \tau'\}. \tag{6}$$

The length (and hence the diameter) of each of these arcs is at most $(2 - 2/Q)\tau < 2\cdot(\delta/2) = \delta$. For $j \ne k$ there are arcs of length τ' separating $I_{i,k}$ from $e^{ij\tau}$ within $\mathbb{T}$, namely $U_{k-1,m(i,k-1)}$ and $U_{k,m(i,k)}$ when $1 \le k < p$, and $U_{p-1,m(i,p-1)}$ and $U_{0,m(i,0)}$ for $k = 0$.

Each sequence $\{k_i\}_{i=1}^n$, with $0 \le k_i < p$, defines a cylinder in $\ell_\infty(F)$ of the following form:

$$W[\{k_i\}_{i=1}^n] = \{f \in \ell_\infty(F) \mid f(\gamma_i) \in I_{i,k_i}\}.$$

For $g \in G$, let $f_g(\gamma) = \gamma(g)$ for $\gamma \in F$. Because these cylinders are disjoint, each f_g is in at most one of them. $g \in \mathcal{S}_n$ was specified to guarantee that f_g would be in at least one of these cylinders. For $g \in \mathcal{S}_n$, define $h(g) \in \ell_\infty(F)$ by $h(g)(\gamma_i) = k_i$ where $f_g(\gamma_i) \in I_{i,k_i}$ and thus $f_g \in W[\{k_i\}_{i=1}^n]$. Because each cylinder has diameter less than δ, each cylinder contains at most one f_g for $g \in \mathcal{S}_n$. Hence $|h(\mathcal{S}_n)| = |\mathcal{S}_n| \ge 2^{n\delta/2}$. For any subset $H \subset F$, let Π^H be this projection: for $f \in \ell_\infty(F)$, $\Pi^H(f) = f|_H$. By Corollary 2 of [13, p. 742], there is a constant $c'' > 0$ which depends only on $\delta/2$ and p (which themselves depend only on δ) such that there are some $H \subset F$ and integers $a < b$ from $[1,p]$ such that

$$|H| \ge c''|F| \quad \text{and} \quad \{a,b\}^H \subset \Pi^H(h(\mathcal{S}_n)).$$

If $b-a \le p/2$, let $a' = a$ and $b' = b$. If $b-a > p/2$, let $a' = b$ and $b' = a+p$. In either case, let $a'' = a' \bmod p$ and $b'' = b' \bmod p$. Then $\{a'', b''\} = \{a, b\}$ with $a' < b'$ and $b' - a' \le p/2$.

Case 1: $b' - a' \ge 2$. Let $c = (a' + b')/2$. Then $b' - c \ge 1$, $c - a' \ge 1$, $b' - c \le p/4$ and $c - a' \le p/4$. If $z_2 \in I_{i,b''}$, then $z_2 = e^{i\theta}$ with

$$c\tau + \tau' \le (b'-1)\tau + \tau' \le \theta < (b'+1)\tau - \tau' < c\tau + p\tau/4 + 1,$$

because $\tau = 2\pi/p < \delta/2$ and $\delta \le 2$ (see condition (6)). Hence

$$e^{-ic\tau} z_2 = e^{i(\theta - c\tau)} \quad \text{with } \tau' \le \theta - c\tau < \pi/2 + 1.$$

Thus $e^{-ic\tau} z_2$ is in the upper half-plane, with

$$\Im(e^{-ic\tau} z_2) \ge \tau'' = \min\{\sin(\tau'), \sin(\pi/2+1)\} > 0.$$

Likewise, if $z_1 \in I_{i,a''}$, then then $z_1 = e^{i\theta}$ with

$$c\tau - p\tau/4 - 1 < (a'-1)\tau + \tau' \le \theta < (a'+1)\tau - \tau' < c\tau - \tau'.$$

Hence

$$e^{-ic\tau} z_1 = e^{i(\theta - c\tau)} \quad \text{with } -\pi/2 - 1 < \theta - c\tau < -\tau'.$$

Thus $e^{-ic\tau} z_1$ is in the lower half-plane, with

$$\Im(e^{-ic\tau} z_1) < -\tau'' < 0.$$

Because $\{a,b\}^H \subset \Pi^H(h(\mathcal{S}_n))$ and $\{a,b\} = \{a'', b''\}$, for any $A \subset H$ there is some $g \in \mathcal{S}_n$ such that $h(g)(\gamma) = b''$ for $\gamma \in A$ and $h(g)(\gamma) = a''$ for $\gamma \in H \backslash A$. Let $\mu = e^{-ic\tau}\delta_{-g}$; $\mu \in D(1)$. For $\gamma \in A$ we have

$$\Im(\widehat{e^{-ic\tau}\delta_{-g}}(\gamma)) = \Im(e^{-ic\tau}\gamma(g)) \ge \tau''.$$

Likewise, for $\gamma_i \in H\backslash A$,

$$\Im(\widehat{e^{-ic\tau}\delta_{-g}}(\gamma)) = \Im(e^{-ic\tau}\gamma(g)) < -\tau''.$$

This proves that H is $J(1, \tau'')$.

Case 2: $b' = a' + 1$. Because $\{a,b\}^H \subset \Pi^H(h(\mathcal{S}_n))$ and $\{a,b\} = \{a'', b''\}$, for every $A \subset H$ there are g_1 and g_2 such that

$$(\forall \gamma \in A)(h(g_1)(\gamma) = b'' \text{ and } h(g_2)(\gamma) = a''),$$

while

$$(\forall \gamma \in H\backslash A)(h(g_2)(\gamma) = a'' \text{ and } h(g_2)(\gamma) = b'').$$

The arc $U_{i,m(i,a'')}$ equals $\{e^{i\theta} \mid x \le \theta < x + \tau'\}$ with $a'\tau \le x < x + \tau' \le b'\tau$. If $z_2 \in I_{i,b''}$, then $z_2 = e^{i\theta}$ with $x + \tau' \le \theta < (b'+1)\tau - \tau'$. If $z_1 \in I_{i,a''}$, then $z_1 = e^{i\theta}$ with $(a'-1)\tau + \tau' \le \theta < x$. Thus, for $\gamma_i \in A$, $\gamma_i(g_1 - g_2) = \gamma_i(g_1)/\gamma_i(g_2) = e^{i\theta}$ with

$$\tau' < \theta < (b-a)\tau + 2\tau - 2\tau' = (3 - 2/Q)\tau < 3.$$

Thus, when $\gamma \in A$, $\gamma(g_1 - g_2)$ is in the upper half-plane and

$$\Im(\gamma(g_1 - g_2)) \geq \tau''' = \min\{\sin(\tau'), \sin(3)\}.$$

For $\gamma_i \in H\backslash A$,

$$\gamma_i(g_1 - g_2) = \gamma_i(g_1)/\gamma_i(g_2) = e^{i\theta}$$
$$\text{with } -3 < (-3 + 2/Q)\tau < \theta < a' - b' = -\tau'.$$

Thus, when $\gamma \in H\backslash A$, $\gamma_i(g_1 - g_2)$ in the lower half-plane with

$$\Im(\gamma(g_1 - g_2)) \leq -\tau'''.$$

This makes H a $J(1, \tau''')$ set. ■

The proof of Theorem 15 produces "proportional" subsets of Sidon sets (and therefore I_0 sets) which are of order 1 according to [12, pp. 182–186]. In [12] this unresolved question was posed: must I_0 sets be finite unions of order 1 sets?

Are Sidon sets finite unions of I_0 sets? David Grow asked in [5] whether Sidon sets had to be finite unions of I_0 sets. Theorem 15 provides some evidence that they could be, but that question is not resolved here. The next two theorems provide a necessary condition: one for $\mathbb{Z}$ and one for the category of abelian groups.

Definition. For discrete abelian groups Γ and $E \subset \Gamma$, let $\nu(E, m)$ be the minimum number of I_0 sets of degree at most m of which E is the union and let $\nu(E, m) = \infty$ when no such finite union exists.

Theorem 16. *If every Sidon subset of $\mathbb{Z}$ is a finite union of I_0 sets, then there is some $m \in \mathbb{Z}^+$ and a non-decreasing function $\phi : [1, \infty) \to \mathbb{Z}^+$ such that*

$$\nu(E, \phi(r)) \leq \phi(r) \quad \text{if } \alpha(E) \leq r.$$

Theorem 17. *Suppose that, for all abelian groups Γ and Sidon subsets E of Γ, E is the finite union of I_0 sets. Then there is a non-decreasing function $\phi : [0, \infty) \to \mathbb{Z}^+$ such that*

$$\alpha(E) \leq r \quad \text{implies} \quad \nu(E, \phi(r)) \leq \phi(r).$$

These lemmas will be helpful. Their proofs are close to the definitions.

Lemma 18. *For discrete abelian groups Γ and subsets E and F of Γ, if $E \subset F$ then $\nu(E, m) \leq \nu(F, m)$. If $m \leq n$, then $\nu(E, m) \geq \nu(E, n)$.*

Lemma 19. *For $E \subset \mathbb{Z}$ and integers $k \neq 0$ and q, $\alpha(kE + q) = \alpha(E)$, $N(kE + q) = N(E)$, and $\nu(kE + q, m) = \nu(E, m)$.*

Lemma 20. *For discrete abelian groups Γ and $E \subset \Gamma$,*

$$\nu(E, m) = \sup\{\nu(F, m) \mid F \subset E \;\&\; F \text{ is finite}\}. \tag{4F}$$

The proof of Lemma 20 is postponed until after the proof of Theorem 16.

Proof of Theorem 16. Suppose that, for all real $r \geq 1$, there is some m such that

$$\alpha(E) \leq r \quad \text{implies} \quad \nu(E, m) \leq m. \tag{7}$$

If $\phi(r)$ is defined to be the minimum m such that condition (7) holds, then ϕ is non-decreasing with r and meets the requirements of the theorem.

So, for some real $r \geq 1$, suppose that for all m there is some $E_m \subset \mathbb{Z}$ for which $\alpha(E_m) \leq r$ and $\nu(E_m, m) > m$. By Lemma 20, there is a finite subset F_m of E_m with $\alpha(F_m) \leq r$ and $\nu(F_m, m) > m$. Let

$$F = \bigcup_m k_m F_m.$$

By Lemmas 18 and 19, $\nu(F, m) \geq \nu(k_m F_m, m) = \nu(F_m, m) > m$ for all m. Thus F is not a finite union of I_0 sets. If we choose k_m to increase rapidly, F will be a Sidon set; this will contradict the hypotheses.

To make F be Sidon let $k_1 = 1$ and, for $m > 1$, let $k_m > \pi^2 2^m M_{m-1}$, where M_t is the maximum absolute value of an element of $\bigcup_{s<t} k_s F_s$. Then, just as in the proof of Proposition 12.2.4, pages 371–372 of [4], $\{k_m F_m\}_m$ is a sup-norm partition for F: if p_m is a $k_m F_m$-polynomial (on $\mathbb{T}$) and is non-zero for at most finitely many m, then

$$\sum_{m=1}^{\infty} \|p_m\|_\infty \leq 2\pi \Big\| \sum_{m=1}^{\infty} p_m \Big\|_\infty.$$

Recall that $B(F)$ (the restrictions to F of Fourier transforms of bounded Borel measures on $\mathbb{T}$) is the Banach space dual of $\mathrm{Trig}_F(\mathbb{T})$ (the trigonometric polynomials with spectrum in F). For $p \in \mathrm{Trig}_F(\mathbb{T})$, let p_m denote its summand in $\mathrm{Trig}_{k_m F_m}(\mathbb{T})$ under the natural decomposition. Then $f \in B(F)$, and

$$\begin{aligned}
|\langle f, p\rangle| &= \Big| \sum_{m=1}^{\infty} \langle f, p_m\rangle \Big| \leq \sum_{m=1}^{\infty} |\langle f, p_m\rangle| \\
&\leq \sum_{m=1}^{\infty} \|f|_{k_m F_m}\|_{B(k_m F_m)} \|p_m\|_\infty \\
&\leq \Big(\sup_{m \in \mathbb{Z}^+} \|f|_{k_m F_m}\|_{B(k_m F_m)} \Big) \sum_{m=1}^{\infty} \|p_m\|_\infty \\
&\leq (r \sup_{m \in \mathbb{Z}^+} \|f|_{k_m F_m}\|_\infty)(2\pi \|p\|_\infty) \leq (2\pi r \|f\|_\infty) \|p\|_\infty.
\end{aligned}$$

Thus, $\|f\|_{B(F)} \leq 2\pi r \|f\|_\infty$. By the definition of Sidon constant, $\alpha(F) \leq 2\pi r$ and thus F is Sidon. ■

Proof of Theorem 17. As in the proof of Theorem 8, suppose that there is some $r \in [1, \infty)$ such that, for all m, there is an abelian group Γ_m and $F_m \subset \Gamma_m$ for which $\alpha(F_m) \le r$ and $\mu(F_m, m) > m$. Let Γ be the direct sum of the Γ_m's. Embed Γ_m into Γ canonically: $x \mapsto \gamma_x$, where $\gamma_x(m) = x$ and $\gamma_x(j) = 0$ for $j \ne m$. Under this embedding, neither $\alpha(F_m)$ nor $\nu(F_m, m)$ changes. Let

$$F = \bigcup_{m=1}^{\infty} F_m.$$

Then for all m, $\nu(F, m) \ge \nu(F_m, m) > m$. Evidently, F is not the finite union of I_0 sets.

To see that F is a Sidon set, set $E = F\backslash\{0\}$ and $E_m = F_m\backslash\{0\}$. Then $\{E_m\}_{m=1}^{\infty}$ is a sup-norm partition of E. Specifically, let G be the compact group dual to Γ (Γ is given the discrete topology). For $p \in \mathrm{Trig}_E(G)$, if p_j denotes its natural summand in $\mathrm{Trig}_{E_j}(\Gamma)$, then

$$\sum_{j=1}^{\infty} \|p_j\|_\infty \le \pi \|p\|_\infty,$$

by Lemma 12.2.2 of page 370 of [4]. To apply that lemma two things are required. First, no E_j may contain 0, which is true here. Second, in the language of [4], the ranges of $\{p_j\}_{j=1}^{\infty}$ are 0-additive: given $\{g_j\}_{j=1}^{\infty}$ from G, there is some $g \in G$ for which

$$\Big|p(g) - \sum_{j=1}^{\infty} p_j(g_j)\Big| = 0. \tag{8}$$

Here's a proof of equation (8). G is the infinite direct product of $G_m = \widehat{\Gamma_m}$. That is, $g \in G$ if and only if

$$g : \mathbb{Z}^+ \to \bigcup_m G_m, \qquad \text{with } g(m) \in G_m.$$

Let $g \in G$ satisfy $g(j) = g_j(j)$. Note that for any character γ used in p_j, $\langle \gamma, g\rangle$ is determined by $g(j)$ (because γ is 0 in every other coordinate):

$$\langle \gamma, g\rangle = \prod_s \langle \gamma(s), g(s)\rangle = \langle \gamma(j), g(j)\rangle = \langle \gamma(j), g_j(j)\rangle = \langle \gamma, g_j\rangle.$$

Thus $p(g) = \sum_{j=1}^{\infty} p_j(g) = \sum_{j=1}^{\infty} p_j(g_j)$. Once it is known that E is sup-norm partitioned by the E_t's, then just as in the proof of Theorem 16 one has

$$\alpha(E) \le \pi \sup_t \alpha(E_t) \le \pi r.$$

That proves that E is Sidon. Since $\{0\}$ is a Sidon set, and the union of two Sidon sets is Sidon [11], $E \cup \{0\}$ is Sidon. Because $F \subset E \cup \{0\}$, that makes F be Sidon as well. ∎

P r o o f o f L e m m a 20. Let t equal the right-hand side of (4F). By Lemma 18, $t \le \nu(E,m)$. Consider next the reversed inequality. For finite $F \subset E$ there are I_0 sets $I_{q,F}$ (possibly equal to $\emptyset$) with I_0-degree no more than m such that

$$F = \bigcup_{q=1}^{t} I_{q,F}.$$

Without loss of generality, it may be assumed that the $I_{q,F}$'s are disjoint for distinct q's. Hence

$$\chi_F = \sum_{q=1}^{t} \chi_{I_{q,F}}. \tag{9}$$

By using Alaoglu's theorem in $\ell_\infty(\Gamma) = \ell_1(\Gamma)^*$ with successive subnets t times, there is a subnet F_β of the net of all finite subsets of E (ordered by increasing inclusion) such that

$$\lim_{\beta\to\infty} \chi_{I_{q,F_\beta}} = f_q \quad \text{weak-* in } \ell_\infty(\Gamma), \qquad \text{for } 1 \le q \le t.$$

This convergence implies pointwise convergence on Γ.

Necessarily, $f_q = \chi_{I_q}$ for some set $I_q \subset \Gamma$. By equation (9),

$$\sum_{q=1}^{t} \chi_{I_q} = \lim_{\beta\to\infty} \sum_{q=1}^{t} \chi_{I_{q,F_\beta}} = \lim_{\beta\to\infty} \chi_{F_\beta} = \chi_E.$$

Thus, E is the disjoint union of the I_q's. Because each I_q is the limit of I_{q,F_β} with $N(I_{q,F_\beta}) \le m$, we have $N(I_q) \le m$ by Proposition 5. ∎

We conclude this section by observing that the class of finite unions of I_0 sets is F_σ in 2^Γ.

Proposition 21. *The class of subsets of Γ which are finite unions of I_0 sets is F_σ in 2^Γ: they are $\bigcup_i \{E \subset \Gamma \mid \nu(E,i) \le i\}$, where $\{E \subset \Gamma \mid \nu(E,i) \le i\}$ is closed in 2^Γ.*

P r o o f. E is in the class if and only if there are m and n such that $\nu(E,m) \le n$. Since $\nu(E,m) \le n$ implies $\nu(E,i) \le i$ for $i = \max\{m,n\}$, this class is equal to $\bigcup_i \mathcal{U}_i$, where

$$\mathcal{U}_i = \{E \subset \Gamma \mid \nu(E,i) \le i\}.$$

As in the proof of Lemma 2, equation (4F) and Lemma 18 imply that $\mathcal{U}_i$ is closed in 2^Γ. ∎

Appendix

Lemma 1. *For $E \subset \Gamma$,*

$$AP(E) = C(b\Gamma)|_E = C(\overline{E})|_E = AP(\Gamma)|_E.$$

Proof. Let us adopt as the definition of $AP(E)$ that it is the closure in $\ell_\infty(E)$ of $B_d(E)$. First consider $AP(E) = C(b\Gamma)|_E$. Let $g \in C(b\Gamma)$. By [18, p. 32], there is a sequence $\mu_j \in M_d(G)$ such that $\widehat{\mu_j}$ converges uniformly on Γ to g. Necessarily, since $E \subset \Gamma$,

$$\widehat{\mu_j}|_E \in B_d(E) \quad \text{and} \quad \lim_{j\to\infty} \widehat{\mu_j}|_E = g|_E \text{ in } \ell_\infty(E).$$

That puts $g|_E$ in $AP(E)$. Conversely, suppose that $w \in AP(E)$. There is a sequence of $\mu_j \in M_d(G)$ such that $\widehat{\mu_j}|_E$ converges uniformly on E to w. Because E is dense in $\overline{E}$ and this convergence is uniform on E, it follows that

$$\lim_{j\to\infty} \widehat{\mu_j}|_{\bar{E}} = f$$

for some f which is a continuous function on $\overline{E}$ and $f|_E = w$. Because $b\Gamma$ is compact and Hausdorff, it is normal; thus Tietze's extension theorem applies to f and there is some $g \in C(b\Gamma)$ such that $g|_{\bar{E}} = f$ (see [2]). Since $E \subset \overline{E}$,

$$w = f|_E = g|_E.$$

Thus, $w \in C(b\Gamma)|_E$.

Next, consider $C(b\Gamma)|_E = C(\overline{E})|_E$. Let $f \in C(\overline{E})$. As happened in the previous paragraph, Tietze's extension theorem provides some $g \in C(b\Gamma)$ such that $g|_{\bar{E}} = f$. Since $E \subset \overline{E}$, one has $f|_E = g|_E$. Conversely, suppose that $g \in C(b\Gamma)$. Then $g|_{\bar{E}} \in C(\overline{E})$. Necessarily, since $E \subset \overline{E}$,

$$g|_E = (g|_{\bar{E}})|_E.$$

Finally, consider $C(b\Gamma)|_E = AP(\Gamma)|_E$. Let $f \in AP(\Gamma)$. By [18, p. 32], f extends to a continuous function $g \in C(b\Gamma)$. Since $E \subset \Gamma$, $f|_E = g|_E$. Conversely, let $g \in C(b\Gamma)$; by [18, p. 32], $g|_\Gamma \in AP(\Gamma)$. Since $E \subset \Gamma$,

$$g|_E = (g|_\Gamma)|_E. \ \blacksquare$$

Definition. $E \subset \Gamma$ is called *helsonian* if and only if $\overline{E} \subset b\Gamma$ is a Helson set in $b\Gamma$.

Proposition 2. *$E \subset \Gamma$ is helsonian if and only if $B_d(E) = AP(E)$.*

Proof. Suppose that $E \subset \Gamma$ is helsonian. Let $f \in AP(E)$. By Lemma 1, there is some $g \in C(\overline{E})$ such that $g|_E = f$. By hypothesis, $\overline{E} \subset b\Gamma$ is Helson; the definition of Helson is that, for every continuous function g on $\overline{E}$, there is some $\mu \in L_1(G_d) = M_d(G)$ such that $\widehat{\mu}|_{\bar{E}} = g$. Because $E \subset \overline{E}$,

$$\widehat{\mu}|_E = g|_E = f.$$

Thus, $AP(E) \subset B_d(E)$; by condition (1) of the first section, $AP(E) = B_d(E)$.

Next, suppose that $AP(E) = B_d(E)$ and let $f \in C(\overline{E})$. By Lemma 1, $f|_E \in AP(E)$; since $AP(E) = B_d(E)$,

$$f|_E = \widehat{\mu}|_E \quad \text{for some } \mu \in M_d(G).$$

Since $\widehat{\mu}$ is continuous on $b\Gamma$ and $\overline{E} \subset b\Gamma$, $\widehat{\mu}|_{\bar{E}}$ is continuous on $\overline{E}$. Because both $\widehat{\mu}|_{\bar{E}}$ and f are continuous on $\overline{E}$, E is dense in $\overline{E}$, and $f|_E = \widehat{\mu}|_E$, one has

$$f = \widehat{\mu}|_{\bar{E}}.$$

This makes $\overline{E}$ be a Helson subset of $b\Gamma$ and hence E helsonian. ∎

Proposition 3. *Helsonian implies Sidon.*

Proof. By [18, p. 115, Thm. 5.6.3], $\overline{E} \subset b\Gamma$ is Helson if and only if there is some $K \in \mathbb{R}^+$ such that, for all bounded Borel measures μ supported on $\overline{E}$,

$$\|\mu\| \le K\|\widehat{\mu}\|_{\ell_\infty(G_d)}.$$

This applies to the discrete measures supported on E, $\mu \in M_d(E)$. Because $E \subset \Gamma$, for $\mu \in M_d(E)$ one has $\widehat{\mu}$ continuous on G with respect to the original compact topology on G. Thus, for $\mu \in \ell_1(E) = M_d(E)$,

$$\|\mu\| \le K\|\widehat{\mu}\|_{C(G)}. \tag{A-1}$$

Let $W(G)$ be the space $\widehat{\ell_1(E)}$, with the supremum norm. By (A-1) it is a closed subspace of $C(G)$ and equivalent under $\phi = \widehat{\ }$ to $\ell_1(E)$. Therefore, using Banach space dualities, ϕ^* is an equivalence between $W(G)^*$ and $\ell_\infty(E)$. Since $W(G)$ is a closed subspace of $C(G)$, $W(G)^*$ is a quotient Banach space of $C(G)^* = M(G)$: $w \in W(G)^*$ if and only if there is some $\nu \in M(G)$ such that $w = \nu + W(G)^\perp$, where

$$W(G)^\perp = \{\mu \in M(G) \mid \mu(W(G)) = \{0\}\}.$$

Thus, for $w \in W(G)^*$ and $f \in \ell_1(E)$, if $w = \nu + W(G)^\perp$, then

$$\langle \phi^*(w), f\rangle = \langle w, \phi(f)\rangle = \langle \nu, \widehat{f}\rangle.$$

However, because $f = \sum_{y\in E} c_y\delta_y$ with $\sum_{y\in E} |c_y| < \infty$, we may use Fubini's theorem in the following calculation:

$$\begin{aligned}\langle \nu, \widehat{f}\rangle &= \int_G \widehat{f}(x)\,d\nu(x) = \int_G \Big(\sum_{y\in E}\langle -x, y\rangle c_y\Big)d\nu(x)\\ &= \sum_{y\in E} c_y \int_G \langle -x, y\rangle\,d\nu(x) = \sum_{y\in E} c_y\widehat{\nu}(y) = \langle \widehat{\nu}, f\rangle.\end{aligned}$$

Since this holds for all $f \in \ell_1(E)$, $\phi^*(w) = \widehat{\nu}|_E$ in $\ell_\infty(E)$. Thus, since ϕ^* is onto $\ell_\infty(E)$, $B(E) = \ell_\infty(E)$ and hence E is Sidon. ∎

Proposition 4. *$B(E) = AP(E)$ implies that E is I_0.*

P r o o f. Since

$$\|f\|_{B(E)} \geq \|f\|_\infty,$$

the two Banach spaces have equivalent norms: there is some $K \in \mathbb{R}^+$ such that

$$\|f\|_{B(E)} \leq K\|f\|_\infty.$$

As in [11], this is equivalent to the Sidonicity of E: $\ell_\infty(E) = B(E)$. Since $AP(E) = B(E)$, one therefore has $AP(E) = \ell_\infty(E)$ and thus E is an I_0 set. ■

EXAMPLE 5. Helsonian does not imply I_0.

P r o o f. In general, the union of two helsonian sets E and F is helsonian, because the union of two Helson sets is Helson [4, pp. 48–67] and

$$\overline{E \cup F} = \overline{E} \cup \overline{F}.$$

Apply this to the sets $\{2^n\}_n$ and $\{2^n + n\}_n$, which are sufficiently lacunary to be I_0 sets and hence helsonian [19]. However, the two sets have some cluster points in common in $b\mathbb{Z}$ and hence the function which is 1 on one of them and 0 on the other cannot be extended almost periodically to all of $\mathbb{Z}$. To see that they have a cluster point in common, note that there is a net $\{n_\beta\} \subset \mathbb{Z}^+$ such that $n_\beta \to 0$ in $b\mathbb{Z}$. By the compactness of $b\mathbb{Z}$, there is a subnet β_t for which $2^{n_{\beta_t}}$ is convergent in $b\mathbb{Z}$. By the continuity of the group operations in $b\mathbb{Z}$,

$$\lim_t 2^{n_{\beta_t}} = \lim_t (2^{n_{\beta_t}} + n_{\beta_t}). \ ■$$

Kalton's Theorem revisited. This result of Kalton's is close to previous work by Kahane, J.-F. Méla, Ramsey and Wells [7, 12, 17].

DEFINITION. Let $D(N)$ denote the set of discrete measures μ on G for which

$$\mu = \sum_{j=1}^{N} c_j \delta_{t_j},$$

where $|c_j| \leq 1$ and $t_j \in G$ for each j. For $E \subset \Gamma$ and $\delta \in \mathbb{R}^+$, let $AP(E, N, \delta)$ be the set of $f \in \ell_\infty(E)$ for which there exists $\mu \in D(N)$ such that

$$\|f - \widehat{\mu}|_E\|_\infty \leq \delta.$$

E is said to be $I(N, \delta)$ if the unit ball in $\ell_\infty(E)$ is a subset of $AP(E, N, \delta)$.

LEMMA 6. *For $E \subset \Gamma$ and $\delta \in \mathbb{R}^+$, the set $AP(E, N, \delta)$ is closed in $\mathbb{C}^E$ (the space of all complex functions on E with the topology of pointwise convergence).*

Proof. Let f_α be a net of functions from $AP(E,N,\delta)$ which converge to some $f \in \mathbb{C}^E$. Let $\mu_\alpha \in D(N)$ satisfy
$$\|f_\alpha - \widehat{\mu_\alpha}|_E\|_\infty \le \delta.$$
Write μ_α as
$$\mu_\alpha = \sum_{i=1}^N c_{i,\alpha}\delta_{t_{i,\alpha}},$$
with $|c_{i,\alpha}| \le 1$ and $t_i \in G$ for all i. Because G and the unit disc of $\mathbb{C}$ are compact, one may choose successive subnets of the α's so that, if one labels the final net with β, then
$$\lim_\beta c_{i,\beta} = c_i \in \mathbb{C} \quad \text{and} \quad \lim_\beta t_{i,\beta} = t_i \in G, \qquad \text{for all } i.$$
Of course, $|c_i| \le 1$. Let $\mu = \sum_{i=1}^N c_i\delta_{t_i}$. Since the topology on G is that given by uniform convergence on compact subsets of Γ, we have, for all $x \in \Gamma$ and each i,
$$\lim_\beta \widehat{\delta_{t_{i,\beta}}}(x) = \lim_\beta \langle -x, t_{i,\beta}\rangle = \langle -x, t_i\rangle = \widehat{\delta_{t_i}}(x).$$
It follows that, for all $x \in E \subset \Gamma$,
$$\lim_\beta \widehat{\mu_\beta}(x) = \lim_\beta \sum_{i=1}^N c_{i,\beta}\widehat{\delta_{t_{i,\beta}}}(x) = \sum_{i=1}^N c_i\widehat{\delta_{t_i}}(x) = \widehat{\mu}(x).$$
Therefore, for all $x \in E$,
$$|f(x) - \widehat{\mu}(x)| = \lim_\beta |f_\beta(x) - \widehat{\mu_\beta}(x)| \le \delta.$$
Thus $f \in AP(E,N,\delta)$. ∎

THEOREM 7. *For any discrete abelian group Γ and $E \subset \Gamma$, the following are equivalent*:

(1) *E is an I_0 set.*

(2) *There is some $\delta \in (0,1)$ and some N for which E is $I(N,\delta)$.*

(3) *There is some $\delta \in (0,1)$ and some $M \in \mathbb{R}^+$ such that*, *for all f in the unit ball of $\ell_\infty(E)$*, *there are points $g_j \in G$ and complex numbers c_j with $|c_j| \le M\delta^j$ for which*
$$f = \widehat{\mu}|_E, \qquad \textit{where } \mu = \sum_{j=1}^\infty c_j\delta_{g_j}.$$

(4) *For all $\delta \in (0,1)$ there is some N for which E is $I(N,\delta)$.*

(5) *$B_d(E) = \ell_\infty(E)$.*

Proof. (1)⇒(2). Assume (1) above, and consider (2) with $\delta = 1/2$. Let $\mathbb{T}$ denote the complex numbers of modulus 1 and $\mathbb{T}^E$ the set of all functions on E with values in $\mathbb{T}$. Condition (1) implies that

$$\mathbb{T}^E \subset \bigcup_n AP(E, n, 1/5). \tag{A-2}$$

Since $AP(E, n, 1/5)$ is closed in $\mathbb{C}^E$ as is $\mathbb{T}^E$ (under the topology of pointwise convergence), $AP(E, n, 1/5) \cap \mathbb{T}^E$ is a closed subset of $\mathbb{T}^E$ and hence measurable. Because condition (A-2) involves the union of sets which increase with n, there is some N for which the measure of $AP(E, N, 1/5) \cap \mathbb{T}^E$ is at least $1/2$ for the Haar measure on $\mathbb{T}^E$. Since $\mathbb{T}^E$ is a connected topological group, a theorem of Kemperman's implies that $AP(E, N, 1/5) \cdot AP(E, N, 1/5) = \mathbb{T}^E$ (see [10]). So, for any $f \in \mathbb{T}^E$, there are functions f_1 and f_2 in $AP(E, N, 1/5) \cap \mathbb{T}^E$ such that $f = f_1 f_2$. There are discrete measures μ_1 and μ_2 in $D(N)$ such that $\widehat{\mu_1}$ approximates f_1 within $1/5$ on E and $\widehat{\mu_2}$ approximates f_2 within $1/5$ on E. It follows that, for $x \in E$,

$$\begin{aligned}
|f(x) - \widehat{\mu_1 * \mu_2}(x)| &= |(f_1 \cdot f_2)(x) - \widehat{\mu_1}(x)\widehat{\mu_2}(x)| \\
&\le |f_1(x)[f_2(x) - \widehat{\mu_2}(x)]| + |\widehat{\mu_2}(x)[f_1(x) - \widehat{\mu_1}(x)]| \\
&\le 1/5 + (1/5) \cdot (|f_2(x)| + 1/5) = (1/5) \cdot (11/5) < 1/2.
\end{aligned}$$

Note that $\mu_1 * \mu_2$ can be represented as a sum of N^2 point masses with complex coefficients bounded by 1 in absolute value:

$$\mu_1 * \mu_2 = \Big(\sum_{i=1}^N c_i \delta_{x_i}\Big) * \Big(\sum_{j=1}^N d_j \delta_{y_j}\Big) = \sum_{i,j} (c_i d_j) \delta_{x_i + y_j}.$$

Finally, note that g on E with $\|g\|_\infty \le 1$ is an average of two functions in $\mathbb{T}^E$: there exist g_1 and g_2 in $\mathbb{T}^E$ such that $g = (g_1 + g_2)/2$. [In $\mathbb{C}$, project $g(x)$ to two points of modulus one for which the line segment joining them is perpendicular to the radial segment from 0 to $g(x)$. If $g(x) = 0$, let $g_1(x) = 1$ while $g_2(x) = -1$.] If $\mu_i \in D(N^2)$ approximates g_i within $1/2$, then

$$\begin{aligned}
\|g - (1/2)(\widehat{\mu_1 + \mu_2}|_E)\|_\infty &\le (1/2)(\|g_1 - \widehat{\mu_1}|_E\|_\infty + \|g_2 - \widehat{\mu_2}|_E\|_\infty) \\
&\le (1/2)(1/2 + 1/2) = 1/2.
\end{aligned}$$

This puts g in $AP(E, 2N^2, 1/2)$.

(2)$\Rightarrow$(3). Condition (2) will be applied inductively. Let $f \in \ell_\infty(E)$ with $\|f\|_\infty \le 1$. There is some $\mu_1 \in D(N)$ such that

$$\|f - \widehat{\mu_1}|_E\|_\infty \le \delta.$$

Next, suppose $\mu_i \in D(N)$ have been selected for $i \le J$, such that

$$\Big\|f - \sum_{i=1}^J \delta^{i-1} \widehat{\mu}_i|_E\Big\| \le \delta^J.$$

Apply condition (2) to

$$g = \delta^{-J}\Big(f - \sum_{i=1}^{J} \delta^{i-1}\widehat{\mu_i}|_E\Big)$$

to obtain $\mu_{J+1} \in D(N)$ such that

$$\|g - \widehat{\mu_{J+1}}|_E\|_\infty \le \delta.$$

Then

$$\Big\|f - \sum_{i=1}^{J+1} \delta^{i-1}\widehat{\mu_i}\Big\|_\infty = \delta^J \|g - \widehat{\mu_{J+1}}|_E\|_\infty \le \delta^{J+1}.$$

By the induction principle, there is a sequence $\mu_i \in D(N)$ such that

$$f = \sum_{i=1}^{\infty} \delta^{i-1}\widehat{\mu_i}|_E.$$

One may enumerate the point masses used in μ_i consecutively for each i, say as δ_{x_j}, so that the coefficient of δ_{x_j} is bounded by δ^{i-1} for $(i-1)N < j \le iN$. Let c_j be this coefficient. Then, since $\delta \in (0,1)$,

$$|c_j| \le \delta^{i-1} = \delta^{\lceil j/N\rceil - 1} \le \delta^{(j/N)-1} = (1/\delta)(\delta^{1/N})^j.$$

This proves condition (3) with $M = 1/\delta$ and $\delta^{1/N}$ in the role of δ.

(3)$\Rightarrow$(4). Let condition (3) hold with M and some $\delta' \in (0,1)$ and consider any $\delta \in (0,1)$ for condition (4). Since $\delta' \in (0,1)$ there is some N' such that

$$M \sum_{j=N'+1}^{\infty} (\delta')^j = M(\delta')^{N'+1}/(1-\delta') \le \delta.$$

Specifically, one needs

$$(N'+1)\log(\delta') \le \log([\delta(1-\delta')/M])$$

and hence

$$N' \ge \{\log([\delta(1-\delta')/M])/\log(\delta')\} - 1.$$

For $j \le N'$, set $m_j = \lceil M(\delta')^j \rceil$.

Let f be in the unit ball of $\ell_\infty(E)$. By condition (3), there are coefficients c_j and elements t_j of G such that $|c_j| \le M(\delta')^j$ and

$$f = \widehat{\mu}|_E, \qquad \text{where } \mu = \sum_{j=1}^{\infty} c_j \delta_{t_j}.$$

Let $p_j = \lceil |c_j| \rceil$; necessarily, $p_j \le m_j$. Set $c_j = |c_j| e^{i\theta_j}$ for some real θ_j. Then

$$c_j \delta_{t_j} = \sum_{i=1}^{m_j} c_{j,i} \delta_{t_{j,i}},$$

where $t_{j,i} = t_j$ for all i and

$$c_{j,i} = \begin{cases} e^{i\theta_j} & \text{for } 1 \le i < p_j, \\ e^{i\theta_j}(|c_j| - p_j + 1) & \text{for } i = p_j, \\ 0 & \text{for } i > p_j. \end{cases}$$

It follows that

$$\|f - \widehat{\nu}|_E\|_{\ell_\infty(E)} \le \delta,$$

where

$$\nu = \sum_{j=1}^{N'} c_j \delta_{t_j} = \sum_{j=1}^{N'} \sum_{i=1}^{m_j} c_{i,j} \delta_{t_{i,j}}$$

is a sum of $N'' = \sum_{j=1}^{N'} m_j$ point masses with coefficients bounded by 1 in absolute value. Thus $f \in AP(E, N'', \delta)$ and E is an $I(N'', \delta)$ set.

(4)⇒(5). (4) implies (2), which has been shown to imply (3). Let $f \in \ell^\infty(E)$. If $f = 0$, $f \in B_d(E)$ trivially. If $f \neq 0$, apply (3) to $g = f/\|f\|_\infty$ to obtain a discrete measure μ such that $\widehat{\mu}|_E = g$. Clearly,

$$\widehat{\|f\|_\infty \mu}|_E = f.$$

(5)⇒(1). By equation (1) of the introduction, $B_d(E) \subset AP(E) \subset \ell_\infty(E)$. If $B_d(E) = \ell_\infty(E)$, then $AP(E) = \ell_\infty(E)$ and hence E is an I_0 set. ∎

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Reçu par la Rédaction le 21.6.1995