COLLOQUIUM MATHEMATICUM

VOL. 71 1996 NO. 1

ON STRONGLY SUM-FREE SUBSETS OF ABELIAN GROUPS

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In his book on unsolved problems in number theory [1] R. K. Guy asks whether for every natural l there exists $n_0 = n_0(l)$ with the following property: for every $n \geq n_0$ and any n elements $a_1, \ldots, a_n$ of a group such that the product of any two of them is different from the unit element of the group, there exist l of the a_i such that $a_{i_j} a_{i_k} \neq a_m$ for $1 \leq j < k \leq l$ and $1 \leq m \leq n$. In this note we answer this question in the affirmative in the first non-trivial case when $l = 3$ and the group is abelian, proving the following result.

THEOREM. *Any finite subset S of an abelian group G with* $\operatorname{card} S \geq 48$ *and the property that $st \neq 1$ for every $s, t \in S$ contains three different elements a, b, c such that $ab, ac, bc \notin S$.*

Let us remark that without the assumption that S is finite the statement is no longer valid: it is enough to consider the set of natural numbers viewed as a subset of $\mathbb{Z}$.

In the proof of the Theorem we use some notions from graph theory. Let G be an abelian group and let S be a finite subset of G with $\operatorname{card} S = n$. If for some $x, y, z \in S$ we have $xz = y$ we connect elements x, y by an arc $\overrightarrow{xy}$ coloured with colour z. We denote the coloured digraph with vertex set S obtained in this way by $\vec{H} = \vec{H}(G, S)$. (Thus, $\vec{H}(G, S)$ is the subgraph induced by S in the Cayley digraph of G based on S.) We denote by $N_-(x)$ and $N_+(x)$ the in- and out-neighbourhoods of a vertex x, i.e.

$$N_-(x) = \{y \in S : \overrightarrow{yx} \text{ is an arc of } \vec{H}\},$$
$$N_+(x) = \{y \in S : \overrightarrow{xy} \text{ is an arc of } \vec{H}\},$$

and set $d_-(x) = |N_-(x)|$, $d_+(x) = |N_+(x)|$ and $\delta_+ = \min_x d_+(x)$.

1991 *Mathematics Subject Classification*: 05E15, 05B10, 20K99.

Both authors were partially supported by US-Polish Research Grant #93113003 and KBN grant 2 P03A 023 09.

If for every $s,t\in S$ we have $st\neq 1$, then $\vec{H}$ contains no directed cycles of length two, i.e. for no pair $x,y\in S$ both arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$ belong to $\vec{H}$. We call a directed graph with this property a *proper* directed graph. Note that, in particular, each proper directed graph on n vertices contains at most $\binom{n}{2}$ arcs.

We deduce the Theorem from the following two facts, corresponding to the cases when $\vec{H}$ is sparse and dense respectively.

CLAIM 1. *If S is such that $\vec{H}=\vec{H}(G,S)$ is a proper directed graph on n vertices with $\delta_+<(n-\sqrt{n}-2)/2$, then S contains three different elements a,b,c such that $ab,ac,bc\notin S$.*

Proof. Choose $a\in A$ in such a way that $d_+(a)=\delta_+$ and let X denote the set of all colours of arcs $\overrightarrow{ax}$ which belong to $\vec{H}$. Consider the set $Y=S\setminus(\{a\}\cup X)$. Since for every $y\in Y$ we have $ay\notin S$, it is enough to find $b,c\in Y$ such that $bc\notin S$.

Suppose that such a pair b,c does not exist. Then, for every $b,c\in Y$, $\vec{H}$ must contain an arc $\overrightarrow{bx}$ coloured with c, in particular, $d_+(b)\geq |Y|-1$. Hence $\vec{H}$ contains δ_+ arcs starting at a, δ_+^2 arcs with tails in X and at least $|Y|(|Y|-1)=(n-\delta_+-1)(n-\delta_+-2)$ starting at vertices from Y. But elementary calculations show that if $\delta_+<(n-\sqrt{n}-2)/2$ then

$$\delta_+ + \delta_+^2 + (n-\delta_+-1)(n-\delta_+-2) > \binom{n}{2},$$

which contradicts the assumption that $\vec{H}$ is proper. ∎

CLAIM 2. *If S is such that $\vec{H}=\vec{H}(G,S)$ is a proper directed graph on $n\geq 48$ vertices with $\delta_+\geq(n-\sqrt{n}-2)/2$, then S contains three different elements a,b,c such that $abc=1$.*

Proof. Assume that $\vec{H}=\vec{H}(G,S)$ is proper and $\delta_+\geq(n-\sqrt{n}-2)/2$. We show that $\vec{H}$ contains a directed cycle of length three with all arcs coloured with different colours.

Let $x\in S$ be chosen in such a way that $d_-(x)\geq\delta_+$ and let $\vec{A}$ be the set of all edges leading from $N_+(x)$ to $N_-(x)$. Then, clearly,

$$|\vec{A}|\geq |N_+(x)|\delta_+ - \binom{|N_+(x)|}{2} - |N_+(x)|(n-|N_+(x)|-|N_-(x)|-1).$$

Now remove from $\vec{A}$ all arcs $\overrightarrow{yz}$ which are such that either $\overrightarrow{xy}$ and $\overrightarrow{yz}$, or $\overrightarrow{yz}$ and $\overrightarrow{zx}$ are of the same colour. Clearly the set $\vec{A}'$ obtained in this way contains at least $|\vec{A}|-|N_+(x)|+|N_-(x)|$ arcs. We claim that for n large enough the size of $\vec{A}'$ is greater than $(|N_+(x)|+|N_-(x)|)/2$. Indeed, from the fact that $\delta_+\leq|N_+(v)|\leq n-\delta_+$ and $\delta_+\geq(n-\sqrt{n}-2)/2$, it follows that $|\vec{A}'|\geq|\vec{A}|-O(n)\geq n^2/8-O(n\sqrt{n})$ and so it is larger than $n/2$ if

$n \geq n_0$ for some sufficiently large n_0 (an elementary but somewhat tedious computation show that it is enough to take $n_0 = 48$).

Thus, $|\vec{A}'| > (|N_+(x)| + |N_-(x)|)/2$ and either two arcs from $\vec{A}'$ have a common tail or two of them have a common head. Consider the former case; the latter can be dealt with in an analogous way. Then there exist $y, z_1, z_2 \in S$ such that the arcs $\overrightarrow{xy}$, $\overrightarrow{yz_1}$, $\overrightarrow{yz_2}$, $\overrightarrow{z_1x}$, $\overrightarrow{z_2x}$ belong to $\vec{H}$ and moreover, for $i = 1, 2$, the arc $\overrightarrow{yz_i}$ is coloured with a colour different from that of $\overrightarrow{xy}$ and $\overrightarrow{z_ix}$. Note that no vertex of $\vec{H}$ is the head of two arcs coloured with the same colour and so at least one of the arcs $\overrightarrow{z_1x}$ and $\overrightarrow{z_2x}$, say $\overrightarrow{z_1x}$, has colour different from that of $\overrightarrow{xy}$. But then all arcs of a directed cycle xyz_1 are coloured with different colours, say, a, b and c, and $abc = 1$. ■

Proof of Theorem. Note that if for some $x_1x_2x_3 \in S$ we have $x_1x_2x_3 = 1$, then none of the products x_1x_2, x_1x_3 and x_2x_3 belongs to S since otherwise we would have $x_i^{-1} \in S$ for some $i = 1, 2, 3$, contradicting the assumption on S. Thus, the assertion follows immediately from Claims 1 and 2. ■

REFERENCES

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Received 3 November 1995;
revised 9 January 1996