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CHARACTERIZATIONS OF COMPLEX SPACE FORMS BY MEANS OF GEODESIC SPHERES AND TUBES

BY

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We prove that a connected complex space form (M^n, g, J) with $n \geq 4$ can be characterized by the Ricci-semi-symmetry condition $\widetilde{R}_{XY} \cdot \widetilde{\varrho} = 0$ and by the semi-parallel condition $\widetilde{R}_{XY} \cdot \sigma = 0$, considering special choices of tangent vectors X, Y to small geodesic spheres or geodesic tubes (that is, tubes about geodesics), where $\widetilde{R}$, $\widetilde{\varrho}$ and σ denote the Riemann curvature tensor, the corresponding Ricci tensor of type $(0, 2)$ and the second fundamental form of the spheres or tubes and where $\widetilde{R}_{XY}$ acts as a derivation.

1. Introduction. In a previous article [1] the following question was stated: which are the Riemannian manifolds all of whose small geodesic spheres or geodesic tubes are semi-symmetric? In fact, one investigated the weaker *Ricci-semi-symmetry* condition $\widetilde{R}_{XY} \cdot \widetilde{\varrho} = 0$ and also the *semi-parallel* condition $\widetilde{R}_{XY} \cdot \sigma = 0$ for these hypersurfaces, in view of the strong similarities shown in [2], [4] between the intrinsic geometry determined by the Ricci tensor $\widetilde{\varrho}$ and the extrinsic properties related to the second fundamental form σ of the geodesic sphere or tube. The main result was that a connected Riemannian manifold (M^n, g) with $n \geq 4$ is a real space form if and only if its small geodesic spheres are Ricci-semi-symmetric or semi-parallel, where for small geodesic tubes it was sufficient that these conditions are satisfied for the so-called horizontal tangent vectors X, Y to the tube. As a consequence, these properties cannot hold for complex space forms, except when they are flat.

In this paper we look for a special class of tangent vectors X, Y to the tubes or spheres which makes each of the two conditions $\widetilde{R}_{XY} \cdot \widetilde{\varrho} = 0$ and $\widetilde{R}_{XY} \cdot \sigma = 0$ characteristic for complex space forms. It will turn out that the appropriate tangent vectors are the horizontal ones (in the sense of Section 3 and 4), where in the case of geodesic tubes one has additionally to restrict to special points (see Section 2).

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2. Preliminaries. Let (M, g) be an n-dimensional, connected, smooth Riemannian manifold, with $n \geq 4$. Denote by ∇ the Levi-Civita connection and by R and ϱ the corresponding Riemannian curvature tensor and Ricci tensor, respectively. We use the sign convention

$$R_{XY} = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

for tangent vector fields X, Y on M.

Next, we treat some general aspects of complex space forms. Suppose that (M, g, J) is a *Kähler manifold*, that is, J is a $(1,1)$-tensor field on M such that

$$J^2 = -I, \quad g(JX, JY) = g(X, Y), \quad \nabla J = 0 \tag{1}$$

for all tangent vector fields X, Y on M. The *holomorphic sectional curvature* $H(u)$ for a unit tangent vector $u \in T_xM, x \in M$ is the sectional curvature of the plane spanned by $\{u, Ju\}$. So, $H(u) = R_{uJuuJu} = g(R_{uJu}u, Ju)$. If $H(u)$ is independent of u then it is independent of x, i.e., $H(u) = c$, $c \in \mathbb{R}$ and then (M, g, J) is called a space of constant holomorphic sectional curvature c or a *complex space form*. Further, a Kähler manifold of constant holomorphic sectional curvature c is characterized by the following curvature tensor:

$$\begin{aligned} R_{XY}Z = \frac{c}{4}\{&g(X,Z)Y - g(Y,Z)X \\ &+ g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ\}. \end{aligned} \tag{2}$$

(See for example [11].) We also have another useful characterization:

THEOREM 2.1 [8]. *Let (M^n, g, J) be a connected Kähler manifold with dimension $n \geq 4$. Then M is a complex space form if and only if $R_{XJX}X$ is proportional to JX for any vector X tangent to M.*

Now, let m be a point in an arbitrary Riemannian manifold M and γ a geodesic parametrized by arc length such that $\gamma(0) = m$. Denote $u = \gamma'(0)$. Next, let $\{E_1, \ldots, E_n\}$ be the parallel orthonormal frame field along γ with $E_1(0) = u$. Let $G_m(r)$ denote the *geodesic sphere* centered at m and with radius $r < i(m)$, the injectivity radius at m. For a point $p = \gamma(r) = \exp_m(ru) \in G_m(r)$ we have the following expansions for the curvature tensor $\widetilde{R}$, the Ricci tensor $\widetilde{\varrho}$ and the second fundamental form σ of $G_m(r)$ with respect to $\{E_1, \ldots, E_n\}$:

$$(3)\quad \widetilde{R}_{abcd}(p) = \frac{1}{r^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + \left\{R_{abcd} - \frac{1}{3}(R_{ubud}\delta_{ac} + R_{uauc}\delta_{bd} - R_{ubuc}\delta_{ad} - R_{uaud}\delta_{bc})\right\}(m) + O(r),$$

$$\begin{aligned}(4)\quad \widetilde{\varrho}_{ab}(p) = {} & \frac{n-2}{r^2}\delta_{ab} + \left(\varrho_{ab} - \frac{1}{3}\varrho_{uu}\delta_{ab} - \frac{n}{3}R_{uaub}\right)(m) \\ & + r\left(\nabla_u\varrho_{ab} - \frac{1}{4}\nabla_u\varrho_{uu}\delta_{ab} - \frac{n+1}{4}\nabla_u R_{uaub}\right)(m) \\ & + r^2\left(\frac{1}{2}\nabla^2_{uu}\varrho_{ab} - \frac{1}{10}\nabla^2_{uu}\varrho_{uu}\delta_{ab} - \frac{n+2}{10}\nabla^2_{uu}R_{uaub}\right. \\ & + \frac{1}{9}R_{uaub}\varrho_{uu} - \frac{1}{45}\sum_{\lambda,\mu=2}^{n} R^2_{u\lambda u\mu}\delta_{ab} \\ & \left. - \frac{n+2}{45}\sum_{\lambda=2}^{n} R_{uau\lambda}R_{ubu\lambda}\right)(m) + O(r^3),\end{aligned}$$

$$(5)\quad \sigma_{ab}(p) = \frac{1}{r}\delta_{ab} - \frac{r}{3}R_{uaub}(m) + O(r^2)$$

for $a, b, c, d = 2, \ldots, n$, where $R_{abcd} = g(R_{E_aE_b}E_c, E_d)$ and similarly for the other tensors. We refer to [2], [5], [6], [9] for more details.

Since we are working in a Kähler manifold we can make a specific choice for E_2 by means of the initial condition $E_2(0) = Ju = J\gamma'(0)$. Hence, $E_2 = JE_1 = J\gamma'$. When (M^n, g, J) is a space of constant holomorphic sectional curvature c, we can write down complete formulas for $\widetilde{R}$, $\widetilde{\varrho}$ and σ. Using the technique of Jacobi vector fields [9] we find

$$(6)\qquad \sigma = \lambda g + \mu\eta \otimes \eta.$$

This together with (2) and the Gauss equation yields

$$\begin{aligned}(7)\quad \widetilde{R}_{XYZW} = {} & \left(\frac{c}{4} + \lambda^2\right)\{g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\} \\ & + \frac{c}{4}\{g(JX,Z)g(JY,W) - g(JY,Z)g(JX,W) \\ & + 2g(JX,Y)g(JZ,W)\} \\ & + \mu\lambda\{g(X,Z)\eta(Y)\eta(W) + g(Y,W)\eta(X)\eta(Z) \\ & - g(X,W)\eta(Y)\eta(Z) - g(Y,Z)\eta(X)\eta(W)\}.\end{aligned}$$

By contraction we then obtain

$$(8)\quad \widetilde{\varrho} = \left\{(n-2)\lambda^2 + (n+1)\frac{c}{4} + \mu\lambda\right\}g + \left\{(n-3)\mu\lambda - \frac{3c}{4}\right\}\eta \otimes \eta,$$

where g denotes the induced metric and $\lambda = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}r$, $\mu+\lambda = \sqrt{c}\cot\sqrt{c}\,r$ for $c > 0$, $\eta(X) = g(X, E_2(r))$ and X, Y, Z, W are tangent vectors to $G_m(r)$. When $c < 0$ one has to replace cot by coth and the formulas for $c = 0$ are obtained by taking the limit as $c \to 0$.

Now, we will consider *geodesic tubes*, that is, tubes about a geodesic curve. We refer to [4], [5], [7], [9], [10] for more details. Let $\sigma : [a, b] \to M$ be a smooth embedded geodesic curve and let P_r denote the tube of radius r about σ, where we suppose r to be smaller than the distance from σ to its nearest focal point. In that case, P_r is a hypersurface of M. Let σ be parametrized by the arc length and denote by $\{e_1, \ldots, e_n\}$ an orthonormal basis of $T_{\sigma(a)}M$ such that $e_1 = \dot\sigma(a)$. Further, let $E_1, \ldots, E_n$ be the vector fields along σ obtained by parallel translation of $e_1, \ldots, e_n$. Then $E_1 = \dot\sigma$ and $\{E_1, \ldots, E_n\}$ is a parallel orthonormal frame field along the geodesic σ. Next, let $p \in P_r$ and denote by γ the geodesic through p which cuts σ orthogonally at $m = \sigma(t)$. We parametrize γ by arc length such that $\gamma(0) = m$ and take $(E_2, \ldots, E_n)$ such that $E_2(t) = \gamma'(0) = u$. Finally, let $\{F_1, \ldots, F_n\}$ be the orthonormal frame field along γ obtained by parallel translation of $\{E_1(t), \ldots, E_n(t)\}$ along γ.

For the hypersurface P_r one then has the following expansions with respect to this parallel frame field [4], [10]:

$$\begin{aligned}
(9)\qquad \widetilde{R}_{1abc}(p) = {} & \Big(R_{1abc} - \frac{1}{2}R_{1ubu}\delta_{ac} + \frac{1}{2}R_{1ucu}\delta_{ab}\Big)(m) \\
& + r\Big(\nabla_u R_{1abc} - \frac{1}{3}\nabla_u R_{1ubu}\delta_{ac} + \frac{1}{3}\nabla_u R_{1ucu}\delta_{ab}\Big)(m) \\
& + r^2\Big(\frac{1}{2}\nabla^2_{uu}R_{1abc} + \frac{1}{6}R_{1ubu}R_{aucu} - \frac{1}{6}R_{1ucu}R_{aubu} \\
& - \frac{1}{8}\nabla^2_{uu}R_{1ubu}\delta_{ac} + \frac{1}{8}\nabla^2_{uu}R_{1ucu}\delta_{ab} \\
& - \frac{1}{8}R_{1u1u}R_{1ubu}\delta_{ac} + \frac{1}{8}R_{1u1u}R_{1ucu}\delta_{ab} \\
& - \frac{1}{24}\sum_{\lambda=3}^{n} R_{1u\lambda u}R_{bu\lambda u}\delta_{ac} \\
& + \frac{1}{24}\sum_{\lambda=3}^{n} R_{1u\lambda u}R_{cu\lambda u}\delta_{ab}\Big)(m) + O(r^3),
\end{aligned}$$

$$\begin{aligned}
(10)\qquad \widetilde{R}_{abcd}(p) = {} & \frac{1}{r^2}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + R_{abcd}(m) \\
& - \frac{1}{3}(R_{budu}\delta_{ac} - R_{bucu}\delta_{ad} + R_{aucu}\delta_{bd} - R_{audu}\delta_{bc})(m) \\
& + O(r),
\end{aligned}$$

$$\widetilde{\varrho}_{11}(p) = \varrho_{11}(m) - (n-1)R_{1u1u}(m) + O(r), \tag{11}$$

$$\begin{aligned}\widetilde{\varrho}_{1a}(p) = \varrho_{1a}(m) &- \frac{n-1}{2}R_{1uau}(m) \\ &+ r\left(\nabla_u \varrho_{1a} - \frac{n}{3}\nabla_u R_{1uau}\right)(m) \\ &+ r^2\left(\frac{1}{2}\nabla^2_{uu}\varrho_{1a} - \frac{n+1}{8}\nabla^2_{uu}R_{1uau} + \frac{1}{6}\varrho_{uu}R_{1uau}\right. \\ &\left. - \frac{3n-5}{24}R_{1u1u}R_{1uau} - \frac{n+1}{24}\sum_{\lambda=3}^{n} R_{1u\lambda u}R_{au\lambda u}\right)(m) \\ &+ O(r^3),\end{aligned} \tag{12}$$

$$\begin{aligned}\widetilde{\varrho}_{ab}(p) = \frac{n-3}{r^2}\delta_{ab} &+ \left(\varrho_{ab} - \frac{n-1}{3}R_{aubu}\right. \\ &\left. - \frac{1}{3}\varrho_{uu}\delta_{ab} - \frac{2}{3}R_{1u1u}\delta_{ab}\right)(m) + O(r),\end{aligned} \tag{13}$$

$$\sigma_{11}(p) = O(r), \tag{14}$$

$$\sigma_{1a}(p) = -\frac{r}{2}R_{1uau}(m) + O(r^2), \tag{15}$$

$$\sigma_{ab}(p) = \frac{1}{r}\delta_{ab} + O(r) \tag{16}$$

for $a, b, c, d \in \{3, \dots, n\}$.

Now, suppose that (M^n, g, J) is a Kähler manifold. Then, a point $p = \exp_m(ru)$ on the geodesic tube P_r will be called a *special point* when $u = J\dot{\sigma}(t)$, that is, $F_2 = JF_1$. For complex space forms of holomorphic sectional curvature c, computing the second fundamental form of P_r by means of the technique of Jacobi vector fields at such a special point yields [7]

$$\sigma(p) = \lambda\, g + \mu\, \eta \otimes \eta, \tag{17}$$

where g denotes the induced metric and $\lambda = \frac{\sqrt{c}}{2}\cot\frac{\sqrt{c}}{2}r$, $\mu + \lambda = -\sqrt{c}\tan\sqrt{c}\,r$ for $c > 0$. The values for $c < 0$ are obtained as usual by replacing the trigonometric functions by the corresponding hyperbolic functions and for $c = 0$ one has to take the limit $c \to 0$. The tensor η in this case is determined by $\eta(X)(p) = g(X, F_1(r))$ for tangent vectors X to P_r at the special point p. Since σ has the same form as in (6), proceeding in the same way results in formally the same expressions for $\widetilde{R}$ and $\widetilde{\varrho}$ as in (7) and (8), respectively. One only has to keep in mind that in the case of geodesic tubes, these formulas are only valid for the special points.

3. Horizontally Ricci-semi-symmetric and horizontally semi-parallel geodesic spheres. A vector $X \in T_pG_m(r)$ is called *horizontal* if

X is orthogonal to $J\gamma'_{|p}$, where γ denotes the unit speed geodesic connecting m and p. This means that $\eta(X) = 0$. Moreover, the space of horizontal tangent vectors to $G_m(r)$ at p is spanned by $E_3(r), \ldots, E_n(r)$.

Then a small geodesic sphere $G_m(r)$ is said to be *horizontally Ricci-semi-symmetric* if $\widetilde{R}_{XY} \cdot \widetilde{\varrho} = 0$ for all horizontal tangent vectors on $G_m(r)$.

The notion of *horizontally semi-parallel* geodesic spheres is defined in a similar way by means of the condition $\widetilde{R}_{XY} \cdot \sigma = 0$.

First, we prove the following result for complex space forms.

THEOREM 3.1. *Let* (M^n, g, J), $n \geq 4$, *be a complex space form. Then the small geodesic spheres in* M *are horizontally Ricci-semi-symmetric and horizontally semi-parallel.*

Proof. Using (8) it is easy to see that

$$-(\widetilde{R}_{XY} \cdot \widetilde{\varrho})(W, W) = 2\mu_2 \eta(\widetilde{R}_{XY} W)\eta(W),$$

where $\mu_2 = (n-3)\mu\lambda - 3c/4$. But $\eta(\widetilde{R}_{XY}W) = -g(\widetilde{R}_{XY}E_2, W)$. So, we have to show that

$$\widetilde{R}_{XY}E_2 = 0 \tag{18}$$

for horizontal tangent vectors to $G_m(r)$.

Using (6) we see in the same way that (18) implies $\widetilde{R}_{XY} \cdot \sigma = 0$.

By means of (7) it is easy to verify that (18) is indeed satisfied for horizontal tangent vectors. ■

Next, we prove the converse theorems.

THEOREM 3.2. *Let* (M^n, g, J), $n \geq 4$, *be a Kähler manifold such that its small geodesic spheres are horizontally semi-parallel. Then* (M, g, J) *is a complex space form.*

Proof. Using (3) and (5) and considering the coefficient of r^{-1} in the power series expansion of

$$(\widetilde{R}_{ab} \cdot \sigma)_{cd} = 0$$

for $a, b = 3, \ldots, n$ and $c, d = 2, \ldots, n$ yields

$$-\delta_{ac}R_{dubu} + \delta_{bc}R_{duau} - \delta_{ad}R_{cubu} + \delta_{bd}R_{cuau} = 0.$$

Next, take $a = d \neq b$ and $c = Ju$ (that is, $c = 2$). Then we also have $a \neq c, b \neq c$ since $a, b \geq 3$, and we get $R_{Juubu} = 0$ for $b \geq 3$. This implies that $R_{uJuux} = 0$ for x orthogonal to Ju. Hence, Theorem 2.1 yields that (M, g, J) is a complex space form. ■

THEOREM 3.3. *Let* (M^n, g, J), $n \geq 4$, *be a Kähler manifold such that its small geodesic spheres are horizontally Ricci-semi-symmetric. Then* (M, g, J) *is a complex space form.*

P r o o f. The assumption in the theorem yields $(\widetilde{R}_{ab} \cdot \widetilde{\varrho})_{cd} = 0$ for $a, b = 3, \ldots, n$ and $c, d = 2, \ldots, n$. Using the power series expansions (3) and (4) and considering the coefficient of r^{-2}, r^{-1} and r^0 gives three conditions in which we make the choice $b = d \neq a$ and $c = Ju$ (that is, $c = 2$). This leads to the following conditions:

$$\varrho_{aJu} = \frac{n}{3} R_{auJuu}, \tag{19}$$

$$(\nabla_u \varrho)_{aJu} = \frac{n+1}{4} (\nabla_u R)_{auJuu}, \tag{20}$$

$$\begin{aligned} 0 = {} & \frac{1}{2} (\nabla^2_{uu} \varrho)_{aJu} - \frac{n+2}{10} (\nabla^2_{uu} R)_{auJuu} \\ & + \frac{1}{9} R_{auJuu}\, \varrho_{uu} - \frac{n+2}{45} \sum_{\lambda=2}^{n} R_{\lambda uJuu} R_{\lambda uau} \end{aligned} \tag{21}$$

for a orthogonal to $\operatorname{span}\{u, Ju\}$.

These three conditions are exactly those needed in the proof of Theorem 12 of [3, pp. 198–201]. Applying the same method (polarization and summation procedures) therefore leads to the required result. ■

4. Horizontally Ricci-semi-symmetric and horizontally semi-parallel geodesic tubes. In [1] a tangent vector X to a small geodesic tube P_r is said to be *horizontal* if X is orthogonal to F_1, the parallel translate of $\dot{\sigma}$ along γ.

Now, if (M^n, g, J) is a Kähler manifold, for special points $p \in P_r$ we see that $X \in T_p P_r$ is horizontal if X is orthogonal to $J\gamma'_{|p}$. Hence, a horizontal vector X at a special point p is determined by the condition $\eta(X) = 0$ and the spaces of horizontal vectors at p are spanned by $F_3, \ldots, F_n$ at p.

Next, a small geodesic tube P_r will be called *horizontally Ricci-semi-symmetric for special points* if $\widetilde{R}_{XY} \cdot \widetilde{\varrho} = 0$ for all horizontal tangent vectors X, Y at special points, and similarly P_r is said to be *horizontally semi-parallel for special points* if $\widetilde{R}_{XY} \cdot \sigma = 0$ for the same choice of vectors X, Y.

We then have

THEOREM 4.1. *Let (M^n, g, J), $n \geq 4$, be a complex space form. Then the small geodesic tubes in M are horizontally Ricci-semi-symmetric and horizontally semi-parallel for special points.*

P r o o f. In the same way as in Theorem 3.1 we find that $\widetilde{R}_{XY} F_1 = 0$ implies $\widetilde{R}_{XY} \cdot \widetilde{\varrho} = 0$ and $\widetilde{R}_{XY} \cdot \sigma = 0$ for X, Y tangent to P_r. So, we have to show that

$$\widetilde{R}_{XY} F_1 = 0 \tag{22}$$

for horizontal tangent vectors at special points. But at special points $\widetilde{R}$ has the same form as in (7). Using the horizontality of X, Y, it is easy to see that (22) holds. ■

Finally, we consider the converse theorems.

THEOREM 4.2. *Let (M^n, g, J), $n \geq 4$, be a Kähler manifold all of whose geodesic tubes are horizontally semi-parallel for special points. Then (M, g, J) is a complex space form.*

Proof. The assumption yields $(\widetilde{R}_{ab} \cdot \sigma)_{1c} = 0$ for $a, b, c = 3, \ldots, n$. Using the power series expansions (9), (10), (14)–(16) and considering the coefficient of r^{-1} yields $R_{1cab} = 0$. Now, take $b = c = Ja$. Then, since $F_1(0) = -Ju$, we get $R_{JuJaaJa} = 0$ and hence $R_{uaJaa} = 0$, for a orthogonal to the plane (u, Ju). Since this must hold for all tubes, the result follows from Theorem 2.1. ■

THEOREM 4.3. *Let (M^n, g, J), $n \geq 4$, be a Kähler manifold all of whose geodesic tubes are horizontally Ricci-semi-symmetric for special points. Then (M, g, J) is a complex space form.*

Proof. Using (9)–(13) we can write down the power series expansion for $(\widetilde{R}_{ab} \cdot \widetilde{\varrho})_{1a} = 0$, $a, b = 3, \ldots, n$.

Considering the coefficient of r^{-2} and taking $b = Ja$ results in $\varrho(u, a) = R_{aJuuJu} + (n-3)R_{uJaaJa}$ for any unit tangent vectors a, u on M, with a orthogonal to u and Ju. Switching a and u and subtracting the equations obtained yields, for $n \neq 4$ and a, u as above, that $\varrho(u, a) = (n-4)R_{uJaaJa}$ and hence $\varrho(a, Ju) = (n-4)R_{auJuu}$. Although the coefficient of R_{auJuu} in this expression is different from the one in (19), using a similar polarization and summation procedure as in the first part of the proof of Theorem 12 in [3, p. 198] gives the result for $n \neq 4$. (We omit the details.)

For $n = 4$ we consider the coefficient of r^0. In this expression we regroup equal terms and use the identity $\nabla^2_{uu}\varrho_{1a} = \nabla^2_{uu}R_{1uau} + \nabla^2_{uu}R_{1bab}$. Finally, taking $b = Ja$ results in

$$\begin{aligned} 0 &= (2R_{uaJaa} + R_{auJuu})(R_{JuuJuu} - R_{JuaJua}) \\ &\quad + R_{auJau}(2R_{uJaaJa} - R_{aJuuJu}) \end{aligned}$$

for a, u unit tangent vectors on M, with a orthogonal to u and Ju.

First, we replace a and u by $a/\|a\|$ and $u/\|u\|$ respectively. Then we obtain a homogeneous expression which is also valid for non-unit vectors a and u.

Next, we polarize this expression, replacing a by $\alpha a + \beta u$, which we may do, since $\alpha a + \beta u$ is orthogonal to u, Ju if a is orthogonal to u, Ju. Writing down the coefficient of $\alpha^3\beta^2$ and β^5 yields

$$AB + DC = 0, \qquad DB - AC = 0, \tag{23}$$

where

$$A = 2R_{uaJaa} + R_{auJuuu}, \qquad C = R_{auJau},$$
$$B = R_{JuuJuu} - R_{auau}, \qquad D = R_{aJuuJu} - 2R_{uJaaJa}.$$

Since (23) is a homogeneous system of linear equations with determinant different from zero if $A \neq 0$, we always get $AB = 0$. Explicitly, this means

$$(2R_{uaJaa} + R_{auJuu})(R_{JuuJuu} - R_{auau}) = 0 \tag{24}$$

for unit tangent vectors a, u on M, with a orthogonal to u, Ju.

Again, we homogenize (24) and polarize, replacing a by $\alpha a + \beta u$ and u by $\beta a - \alpha u$. Writing down the coefficients of the polynomial obtained by this procedure gives

$$\begin{cases} (2H+G)X = 0, \\ 2(2H+G)E + 3KX = 0, \\ (2H+G)Z + 24KE - (G-H)X = 0, \\ 2(2H+G)F + 3K(Z+X) - 2(G-H)E = 0, \\ (2H+G)Y + 24K(F+E) - (G-H)Z - (2G+H)X = 0, \\ 2(2G+H)E - 3K(Z+Y) + 2(G-H)F = 0, \\ (2G+H)Z - 24KF + (G-H)Y = 0, \\ 2(2G+H)F - 3KY = 0, \\ (2G+H)Y = 0, \end{cases} \tag{25}$$

where

$$X = R_{JuuJuu} - R_{auau},$$
$$Y = R_{JaaJaa} - R_{auau},$$
$$Z = 2R_{JuuJaa} + 4R_{JuaJua} - 2R_{auau},$$
$$E = R_{aJuuJu}, \quad F = R_{uJaaJa},$$
$$G = R_{auJuu}, \quad H = R_{uaJaa}, \quad K = R_{auJau}.$$

First we suppose that $2H + G = 0$. The last two equations in (25) then yield that $HF = 0$. On the contrary, if $2H + G \neq 0$, we can use the first four equations to derive that $F = 0$. So, in both cases we obtain $HF = 0$, which means that $R_{uaJaa}R_{uJaaJa} = 0$ for all a, u tangent to M with a orthogonal to u, Ju. Replacing u by $u + Ju$ in this condition eventually leads to $R_{uaJaa} = 0$. Then the result for $n = 4$ follows by Theorem 2.1. ∎

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