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On the type constants with respect to systems of characters of a compact abelian group

by

AICKE HINRICHS (Jena)

Abstract. We prove that there exists an absolute constant c such that for any positive integer n and any system Φ of 2^n characters of a compact abelian group,

$$2^{-n/2} t_\Phi(T) \le c n^{-1/2} t_n(T),$$

where T is an arbitrary operator between Banach spaces, $t_\Phi(T)$ is the type norm of T with respect to Φ and $t_n(T)$ is the usual Rademacher type-2 norm computed with n vectors. For the system of the first 2^n Walsh functions this is even true with $c = 1$. This result combined with known properties of such type norms provides easy access to quantitative versions of the fact that a nontrivial type of a Banach space implies finite cotype and nontrivial type with respect to the Walsh system or the trigonometric system.

1. Introduction. Let $r_1, r_2, \ldots$ be the orthonormal system of the Rademacher functions. The *Rademacher type*-2 *norm* of a bounded linear operator T between Banach spaces X and Y is the infimum over all $c > 0$ such that

$$\Big(\int_0^1 \Big\| \sum_{i=1}^n r_i(t) T x_i \Big\|^2 dt\Big)^{1/2} \le c \Big(\sum_{i=1}^n \|x_i\|^2\Big)^{1/2} \tag{1}$$

for all $x_1, \ldots, x_n \in X$. If n is fixed we get the Rademacher type-2 norm computed with n vectors which we denote by $t_n(T)$. These notions play an important role in the local theory of Banach spaces.

We study relations of type norms defined with systems of characters of a compact abelian group with the quantities $t_n(T)$.

Let $\Phi = (\phi_1, \ldots, \phi_n)$ be an orthonormal system in $L_2(M, \mu)$ for some measure space (M, μ). In analogy to (1) we define the type norm $t_\Phi(T)$ with respect to Φ as the infimum over all $c > 0$ such that

$$\Big(\int \Big\| \sum_{i=1}^n \phi_i(t) T x_i \Big\|^2 d\mu(t)\Big)^{1/2} \le c \Big(\sum_{i=1}^n \|x_i\|^2\Big)^{1/2}$$

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for all $x_1,\dots,x_n \in X$. If T is the identity of a Banach space X we simply write $t_\Phi(X)$ and $t_n(X)$.

The main result of this paper is that there exists a constant c such that

$$2^{-n/2} t_\Phi(T) \le c n^{-1/2} t_n(T) \tag{2}$$

whenever n is a positive integer, Φ is a system of 2^n characters of a compact abelian group and T is a bounded linear operator between Banach spaces.

Some of the statements of our results become more pleasant looking if we use instead of the type norms $t_\Phi(T)$ the normalized quantities

$$\lambda_\Phi(T) = |\Phi|^{-1/2} t_\Phi(T).$$

This corresponds to Pisier's notation $\lambda_n(T) = n^{-1/2} t_n(T)$ from [14]. Then inequality (2) can be restated as follows.

THEOREM 1.1. *There exists a constant c such that for any positive integer n, any system Φ of 2^n characters of a compact abelian group and any bounded linear operator T between Banach spaces*,

$$\lambda_\Phi(T) \le c\lambda_n(T).$$

Section 3 deals with the system of the first 2^n Walsh functions. Because of its special structure we may then even take $c = 1$ in (2). Since the Walsh type norms dominate the Rademacher cotype-2 norms we also get estimates of the cotype-2 norms in terms of type-2 norms, in fact

$$\kappa_{2^n}(T) \le \lambda_n(T).$$

Here $\kappa_m(T) = m^{-1/2} c_m(T)$ and $c_m(T)$ is defined as the infimum of $c > 0$ such that

$$\Big(\sum_{i=1}^m \|Tx_i\|^2\Big)^{1/2} \le c\Big(\int_0^1 \Big\|\sum_{i=1}^m r_i(t)x_i\Big\|^2\,dt\Big)^{1/2}$$

for all $x_1,\dots,x_m \in X$.

In Section 4 we prove Theorem 1.1.

Recall that a Banach space is *B-convex* if it does not contain the spaces l_1^n uniformly. In Section 5 we sketch how Theorem 1.1 leads via submultiplicativity arguments to easy proofs of the fact that B-convexity implies the existence of $c > 0$ and $r > 2$ such that $t_{\Phi_n}(X) \le cn^{1/r}$ for any n, where Φ_n is the system of the first n trigonometric or Walsh functions. This provides a new approach to the cotype estimates of B-convex Banach spaces in [6]. Actually, we obtain the same estimates already for the a priori bigger Walsh type.

The last section deals with extremal cases in the following sense. It is easy to prove that $t_\Phi(X) \le n^{1/2}$ for any orthonormal system Φ containing n functions. It is well known that $t_n(X) = n^{1/2}$ implies that X contains an arbitrary good copy of l_1^n (see Proposition 6.1 below). Our results now imply

that the same is true if $t_\Phi(X) = 2^{n/2}$, where Φ is the system of the first 2^n Walsh functions. Moreover, we give an elementary proof that $c_n(X) = n^{1/2}$ implies that X contains an arbitrary good copy of l_∞^n.

We use standard Banach space notation. In particular, for Banach spaces X and Y, the set of bounded linear operators from X into Y is denoted by $\mathcal{L}(X,Y)$.

2. A lemma on type constants.

Let us first prove a simple lemma on type norms with respect to orthonormal systems which is nevertheless the basis for all subsequent results.

LEMMA 2.1. *Fix a finite orthonormal system Φ, positive integers m and k and a k-covering $\Phi = \bigcup_{i=1}^m \Phi_i$, i.e. every $\phi \in \Phi$ belongs to exactly k different Φ_i. Then, for $T \in \mathcal{L}(X,Y)$*,

$$t_\Phi(T) \le (m/k)^{1/2} \max_{1\le i\le m} t_{\Phi_i}(T).$$

Proof. Choose $(x_\phi)_{\phi\in\Phi} \subset X$ and estimate

$$\begin{aligned}\Big(\int\Big\|\sum_{\phi\in\Phi}\phi(t)Tx_\phi\Big\|^2 d\mu(t)\Big)^{1/2} &= \Big(\int\Big\|\frac1k\sum_{i=1}^m\sum_{\phi\in\Phi_i}\phi(t)Tx_\phi\Big\|^2 d\mu(t)\Big)^{1/2}\\ &\le \frac1k\sum_{i=1}^m\Big(\int\Big\|\sum_{\phi\in\Phi_i}\phi(t)Tx_\phi\Big\|^2 d\mu(t)\Big)^{1/2}\\ &\le \frac1k\sum_{i=1}^m t_{\Phi_i}(T)\Big(\sum_{\phi\in\Phi_i}\|x_\phi\|^2\Big)^{1/2}\\ &\le \frac1k\Big(\sum_{i=1}^m t_{\Phi_i}(T)^2\Big)^{1/2}\Big(\sum_{i=1}^m\sum_{\phi\in\Phi_i}\|x_\phi\|^2\Big)^{1/2}\\ &\le \Big(\frac mk\Big)^{1/2}\max_{1\le i\le m} t_{\Phi_i}(T)\Big(\sum_{\phi\in\Phi}\|x_\phi\|^2\Big)^{1/2}.\end{aligned}$$

By definition of $t_\Phi(T)$, this implies the claim.

The following corollary is an immediate consequence of the preceding lemma.

COROLLARY 2.2. *Let Φ and Φ_i be as in Lemma 2.1. Assume that all Φ_i have equal cardinality. Then, for $T \in \mathcal{L}(X,Y)$*,

$$\lambda_\Phi(T) \le \max_{1\le i\le m}\lambda_{\Phi_i}(T).$$

3. The Walsh system.

In the next section we deal with type constants with respect to arbitrary sets of characters of a compact abelian group.

Because of the special structure of the Walsh functions it is possible to get sharper estimates by a separate treatment.

So, let us now consider the system of the first 2^n Walsh functions

$$\Phi = \Big\{ \prod_{i=1}^{n} r_i^{\varepsilon_i} : \varepsilon_i \in \{0,1\} \Big\},$$

where r_i are again the Rademacher functions. Let us fix the notations $w_{2^n}(T) = t_\Phi(T)$ and $\omega_{2^n}(T) = \lambda_\Phi(T)$ for $T \in \mathcal{L}(X,Y)$. The following fact is taken from [11]. The sequence $(w_{2^n}(T))$ dominates the sequences of type-2 and cotype-2 norms:

$$t_{2^n}(T) \le w_{2^n}(T) \quad \text{and} \quad c_{2^n}(T) \le w_{2^n}(T). \tag{3}$$

The type inequality is easily verified. For the convenience of the reader we include the argument for the cotype inequality.

The following duality relation between cotype and type norms is well known (see [16], Prop. 3.2):

$$c_n(T) \le t_n(T'). \tag{4}$$

Moreover, w_{2^n} is a self-dual quantity:

$$w_{2^n}(T) = w_{2^n}(T'). \tag{5}$$

To show this let us use the discrete version of the Walsh functions given by the Hadamard–Walsh matrices $W_{2^n} = (w_{ij})_{i,j=1}^{2^n}$ defined inductively by

$$W_1 = (1), \qquad W_{2^{n+1}} = 2^{-1/2} \begin{pmatrix} W_{2^n} & W_{2^n} \\ W_{2^n} & -W_{2^n} \end{pmatrix}.$$

Then we have

$$\int_0^1 \Big\| \sum_{i=1}^{2^n} \phi_i(t) x_i \Big\|^2 dt = \sum_{j=1}^{2^n} \Big\| \sum_{i=1}^{2^n} w_{ij} x_{\pi(i)} \Big\|^2,$$

where $\Phi = \{\phi_1, \ldots, \phi_{2^n}\}$ is an enumeration of the first 2^n Walsh functions and π is a suitable permutation of $\{1, \ldots, 2^n\}$.

Therefore, $w_{2^n}(T)$ is the infimum of $c > 0$ such that

$$\Big(\sum_{j=1}^{2^n} \Big\| \sum_{i=1}^{2^n} w_{ij} T x_i \Big\|^2 \Big)^{1/2} \le c \Big(\sum_{i=1}^{2^n} \|x_i\|^2 \Big)^{1/2}$$

for all $x_1, \ldots, x_{2^n} \in X$. Then (5) can be shown using the proof of Proposition 3.4 in [12] together with the symmetry of the Hadamard–Walsh matrices, i.e. $W_{2^n} = W_{2^n}^t$.

The claimed inequality is now an immediate consequence of (4), the first inequality of (3) and (5).

THEOREM 3.1. *For $T \in \mathcal{L}(X,Y)$ and $n \ge 1$,*

$$\omega_{2^n}(T) \le \lambda_n(T).$$

In particular,

$$\kappa_{2^n}(T) \le \lambda_n(T).$$

Proof. Look at the sets $\{\phi r_i : i = 1, \ldots, n\}$, $\phi \in \Phi$. These are subsets of Φ and form an n-covering, so that the theorem follows immediately from Corollary 2.2.

Remark. These estimates are tight in the following sense. If for some function f and some constant $c > 0$,

$$\kappa_{f(n)}(T) \le c\lambda_n(T)$$

for any T and n, then there exist $K > 1$ and $\alpha > 0$ such that

$$f(n) \ge \alpha K^n.$$

This follows by considering the identity I_n of $l_\infty^{f(n)}$ for which $\kappa_{f(n)}(I_n) = 1$ and $\lambda_n(I_n) \le c'((1 + \log f(n))/n)^{1/2}$ for some constant $c' > 0$ independent of n.

4. Characters of a compact abelian group. The abstract reason why we can find n-coverings of the first 2^n Walsh functions is that they can be viewed as the elements of the dual group of a finite Cantor group $\{1,-1\}^n$. This enables us to extend the method to arbitrary systems of characters of a compact abelian group.

To this end, let Γ be the dual group of a finite abelian group G. If we fix a subset $\Lambda \subset \Gamma$, then the sets $\gamma\Lambda$, $\gamma \in \Gamma$, produce a $|\Lambda|$-covering of Γ. So we can formulate the next consequence of Lemma 2.1.

PROPOSITION 4.1. *Let Γ be the dual group of a finite abelian group and let $\Lambda \subset \Gamma$. Then for $T \in \mathcal{L}(X,Y)$,*

$$\lambda_\Gamma(T) \le \lambda_\Lambda(T).$$

To get an estimate in terms of type-2 constants we have to look for large subsets Λ of Γ that behave—at least for the involved quantities—like Rademacher functions. Such subsets are provided by subsets with small Sidon constants.

Let us recall the definition. The *Sidon constant* $S(\Lambda)$ is defined as the smallest constant c such that

$$\sum_{\gamma \in \Lambda} |\alpha_\gamma| \le c \sup_{t \in G} \Big| \sum_{\gamma \in \Lambda} \alpha_\gamma \gamma(t) \Big|$$

for all complex numbers α_γ. We need the following result of [15]:

THEOREM 4.2. *There exists a constant $c > 0$ such that for any compact abelian group G with Haar measure μ and any system $\Lambda = (\gamma_1, \ldots, \gamma_n)$ of characters of G,*

$$\frac{1}{cS(\Lambda)}\Big(\int_0^1 \Big\|\sum_{i=1}^n r_i(t)x_i\Big\|^2\,dt\Big)^{1/2} \le \Big(\int_G \Big\|\sum_{i=1}^n \gamma_i(t)x_i\Big\|^2\,d\mu(t)\Big)^{1/2}$$
$$\le cS(\Lambda)\Big(\int_0^1 \Big\|\sum_{i=1}^n r_i(t)x_i\Big\|^2\,dt\Big)^{1/2}$$

for all $x_1, \ldots, x_n \in X$.

COROLLARY 4.3. *For $T \in \mathcal{L}(X, Y)$,*

$$t_\Lambda(T) \le cS(\Lambda)t_{|\Lambda|}(T).$$

We want to find large subsets with small Sidon constants. This can be done with the help of a special class of character sets. A subset Λ of Γ is called *dissociate* if an equation

$$\prod_{\gamma\in\Lambda} \gamma^{\varepsilon_\gamma} = 1$$

with $\varepsilon_\gamma \in \{-1, 0, 1\}$ implies $\varepsilon_\gamma = 0$. Observe that we implicitly assume that Λ does not contain the trivial character. The next proposition on the Sidon constants of dissociate sets is proved in [4] (Prop. 1, p. 100).

PROPOSITION 4.4. *There exists an absolute constant c such that $S(\Lambda) \le c$ for any dissociate set Λ.*

We can find large dissociate subsets in any set of characters:

PROPOSITION 4.5. *Let Γ_0 be a finite set of characters of a compact abelian group. Then there exists a dissociate subset $\Lambda \subset \Gamma_0$ with*

$$|\Lambda| \ge \log_3 |\Gamma_0|.$$

Proof. Choose a maximal dissociate subset $\Lambda \subset \Gamma_0$. Then Γ_0 must be contained in the set

$$[\Lambda] = \Big\{\prod_{\gamma\in\Lambda} \gamma^{\varepsilon_\gamma} : \varepsilon_\gamma \in \{-1, 0, 1\}\Big\}.$$

Otherwise we could add any $\gamma_0 \in \Gamma_0 \setminus [\Lambda]$ to Λ without changing the dissociation property. Thus

$$|\Gamma_0| \le |[\Lambda]| \le 3^{|\Lambda|},$$

which proves the claim.

Remark. The estimate in Proposition 4.5 is sharp, as the example of the n-fold product of the cyclic group of order 3 shows. In this group, which has cardinality 3^n, a subset is dissociate if and only if it is a linear

independent system viewed as a subset in the n-dimensional vector space over the field defined by the cyclic group of order 3.

I would like to thank J. Seigner for the hint to use dissociate sets.

Combining all these observations gives

PROPOSITION 4.6. *There exists an absolute constant c such that for the dual group Γ of any finite abelian group,*

$$\lambda_\Gamma(T) \le c\lambda_{[1+\log_3|\Gamma|]}(T).$$

However, with a little more work we can do better. We want to prove the estimate of the preceding proposition for arbitrary finite sets of characters of a compact abelian group. This is done by exhausting the given set with not too many dissociate subsets.

Let A be a nonempty finite set, $0 < c \le 1$ and consider the following procedure. Set $A_1 = A$. In the ith step we remove a subset $B_i \subset A_i$ of at least $c \log |A_i|$ elements and let A_{i+1} be the remaining set. This procedure eventually stops, possibly leaving over a singleton. If this is the case we take it in the last step. Let m be the number of steps. Then we have got a disjoint partition

$$A = \bigcup_{i=1}^m B_i.$$

LEMMA 4.7. *The number of steps in the described procedure satisfies*

$$m \le \frac{3|A|}{c(1 + \log |A|)}.$$

Proof. Since $m \le |A|$, the assertion is certainly true for $|A| \le e^2$. For $|A| > e^2$ we proceed by induction on $|A|$.

If $m = 1$ we have nothing to prove. Otherwise we apply the induction hypothesis to the set A_2 and get

$$m - 1 \le \frac{3|A_2|}{c(1 + \log |A_2|)}.$$

Now let us consider the function

$$f(x) = \frac{x}{1 + \log x}.$$

Then

$$f'(x) = \frac{\log x}{(1 + \log x)^2} \quad \text{and} \quad f''(x) = \frac{1 - \log x}{x(1 + \log x)^3}.$$

Therefore f is increasing for $x \ge 1$ and f' is decreasing for $x \ge e$.

If $|A_2| \geq e$, we conclude by $\log |A| > 2$ that

$$f(|A|) - f(|A_2|) = f'(\xi)(|A| - |A_2|) \geq f'(|A|)c\log |A| = c\left(\frac{\log |A|}{1+\log |A|}\right)^2 \geq \frac{4c}{9},$$

where ξ is some number in the interval $[|A_2|, |A|]$. Therefore,

$$m \leq \frac{3}{c} f(|A_2|) + 1 \leq \frac{3}{c} f(|A|) = \frac{3|A|}{c(1+\log |A|)}.$$

If $|A_2| < e$, since $|A| > e^2$ we get

$$m \leq 1 + |A_2| \leq 3 \leq \frac{3|A|}{c(1+\log |A|)},$$

which completes the induction and the proof.

THEOREM 4.8. *There exists an absolute constant c such that if Λ is a finite set of characters of a compact abelian group then for any $T \in \mathcal{L}(X,Y)$,*

$$\lambda_\Lambda(T) \leq c\lambda_{[1+\log |\Lambda|]}(T).$$

Proof. We use Proposition 4.5 to apply the preceding procedure to $A = \Lambda$ with $c = 1/\log 3$ such that all the B_i are dissociate, except possibly the last singleton which may be the trivial character. We may assume that no B_i has cardinality greater than $[1 + \log_3 |\Lambda|]$.

Then by Lemma 2.1 with $k = 1$, Proposition 4.4 and Corollary 4.3 we get

$$t_\Lambda(T) \leq cm^{1/2} t_{[1+\log_3 |\Lambda|]}(T) \leq c' m^{1/2} t_{[1+\log |\Lambda|]}(T).$$

The estimate for m in Lemma 4.7 proves the theorem.

Now Theorem 1.1 is an immediate consequence of the preceding theorem applied to $|\Phi| = 2^n$ and the monotonicity of $t_m(T)$.

5. Applications. In this section we indicate how estimates of the form

$$\alpha_{2^n}(T) \leq c\lambda_n(T)$$

may be used in presence of submultiplicativity to derive power type estimates of $\alpha_n(X)$ for a B-convex Banach space X. Since the methods and results are essentially known (see the remarks following Theorem 5.2) we do not go into details.

Applications of submultiplicativity rest on the next lemma.

LEMMA 5.1. *Let $(\alpha_n)_{n\geq 1}$ be a sequence of nonnegative real numbers satisfying the following conditions:*

(i) *there exist $c, s \geq 1$ such that $\alpha_{2^n} \leq cn^{-1/s}$ for any n,*
(ii) *$(n^{1/2}\alpha_n)_{n\geq 1}$ is nondecreasing,*

(iii) *the subsequence $(\alpha_{2^n})_{n\geq 1}$ is submultiplicative, i.e. $\alpha_{2^{m+n}} \leq \alpha_{2^m}\alpha_{2^n}$ for any m, n.*

Then $(n^{1/r}\alpha_n)_{n\geq 1}$ is bounded, where r is given by $r = (1+ec^s)s\log 2$.

Proof. Choose n_0 such that $ec^s \leq n_0 < ec^s + 1$. Then by the assumptions on α_n we get

$$2^{n_0/r}\alpha_{2^{n_0}} \leq 2^{n_0/r} c n_0^{-1/s} < 1.$$

For given n choose k such that $2^{n_0 k} \leq n < 2^{n_0(k+1)}$. Since the assumptions on c and s imply $r \geq 2$, we can estimate

$$n^{1/r}\alpha_n \leq 2^{n_0/2} 2^{n_0 k/r} \alpha_{2^{n_0(k+1)}} \leq 2^{n_0/2} 2^{n_0 k/r} (\alpha_{2^{n_0}})^{k+1} = 2^{n_0(1/2-1/r)} (2^{n_0/r}\alpha_{2^{n_0}})^{k+1} \leq 2^{n_0(1/2-1/r)}.$$

This finishes the proof.

If X is a fixed Banach space, condition (iii) is in particular satisfied for the cotype numbers $\kappa_n(X)$ (see [10], Lemma 13.4) and for $\lambda_{\Phi_n}(X)$, where Φ_n is the system of the first n Walsh functions in the natural order. Moreover, for the system $\Phi_n = \{e_k : k = 1, \ldots, n\} \subset L_2[0,1]$ of the first n trigonometric functions

$$e_k(t) = e^{2\pi ikt},$$

there exists an equivalent submultiplicative sequence $(\alpha_n(X))_{n\geq 1}$, in fact $\frac{1}{3}\alpha_n(X) \leq \lambda_{\Phi_n}(X) \leq \alpha_n(X)$ for any $n \geq 1$. This is proved in [13]. Conditions (i) and (ii) are also satisfied in all these cases.

Then using Theorems 3.1 and 4.8 the following can be shown.

THEOREM 5.2. *Let X be a B-convex Banach space so that for some $q \geq 2$ and $K > 0$,*

$$\lambda_n(X) \leq Kn^{-1/q} \quad \textit{for any } n. \tag{6}$$

Let $(\alpha_n(X))_{n\geq 1}$ be the sequence $(\kappa_n(X))_{n\geq 1}$ or $(\lambda_{\Phi_n}(X))_{n\geq 1}$ with Φ_n the system of the first n Walsh or trigonometric functions. Then there exist $L > 0$ and $r \geq 2$ depending only on q and K such that

$$\alpha_n(X) \leq Ln^{-1/r} \quad \textit{for any } n. \tag{7}$$

Remarks. (i) Condition (6) means that X is of weak type q' (if $q > 2$) or of type 2 (if $q = 2$). Here q' is the conjugate number of q given by $1/q + 1/q' = 1$. Then K may be taken as the weak type q' constant and type 2 constant, respectively. For $\alpha_n(X) = \kappa_n(X)$, (7) says that X is of weak cotype r. For the concepts of weak type and weak cotype see [8].

(ii) For the cotype and the Walsh functions it follows from Theorem 3.1 that we can take $r = (1 + eK^q)q\log 2$. The cotype case is already contained in [6] with exactly this r. Power type estimates in the Walsh and cotype case may also be proved with Propositions 6.2 and 6.3 using again submultiplicativity (e.g. [10], Lemma 13.5). But this does not give quantitative estimates of r. According to a remark of Bourgain in [2] the Walsh function part was proved by Pisier.

(iii) The results for the Walsh and trigonometric functions are particular cases of Bourgain's result [3] up to the constants.

(iv) For a Banach space X, let

$$h_n(X) = \sup\{d(X_n, l_2^{\dim X_n}) : X_n \subset X,\ \dim X_n \le n\}$$

and

$$t_n^{\max}(X) = \sup\{t_\Phi(X) : \Phi \text{ orthonormal system of } n \text{ functions}\},$$

where d denotes the Banach–Mazur distance. These sequences are equivalent, more precisely we have

$$t_n^{\max}(X) \le h_n(X) \le 2t_n^{\max}(X).$$

For a proof see [12], Theorem 5.6, and apply an obvious discretization procedure. The basic remaining question in this context was asked by Pisier [17]. If X is B-convex, does there exist $c > 0$ and $r > 2$ such that $h_n(X) \le cn^{1/r}$ for any n?

(v) It follows from Theorem 27.7 of [18] that there exists a $c > 0$ such that for any positive integer n and any orthonormal system Φ of cardinality 2^n,

$$\lambda_\Phi(T) \le c(\lambda_n(T)\|T\|)^{1/2}$$

for all bounded linear operators T. Actually, in [18] only the case of identities of Banach spaces is treated but the extension to arbitrary operators is straightforward.

6. Extremal cases. For Banach spaces X and Y, let $d(X,Y)$ be the Banach–Mazur distance of X and Y. The starting point of this section is the following result of Pisier.

PROPOSITION 6.1. *Let X be a real Banach space. If $t_n(X) = n^{1/2}$ then, for any $\varepsilon > 0$, X contains an n-dimensional subspace X_n with $d(X_n, l_1^n) < 1+\varepsilon$.*

This can be found in [14] or [1] and is used there together with submultiplicativity of the sequence t_n to show equivalence of B-convexity and nontrivial type.

Then Theorem 3.1 implies immediately

PROPOSITION 6.2. *Let X be a real Banach space and let Φ be the system of the first 2^n Walsh functions. If $t_\Phi(X) = 2^{n/2}$ then, for any $\varepsilon > 0$, X contains an n-dimensional subspace X_n with $d(X_n, l_1^n) < 1+\varepsilon$.*

It is also well known that the equivalence of uniform containment of l_1^n and nontrivial type has a counterpart for uniform containment of l_∞^n and finite cotype. But to the best of my knowledge the published proofs (see e.g. [5], [7], [9], [10]) are by no means easy. In fact, they show the more general result that

$$p_X := \sup\{p : X \text{ has type } p\} = \inf\{r : X \text{ contains } l_r^n \text{ uniformly}\},$$
$$q_X := \inf\{q : X \text{ has cotype } q\} = \sup\{r : X \text{ contains } l_r^n \text{ uniformly}\}.$$

We give an elementary proof of the special case $q = \infty$ by demonstrating the following statement.

PROPOSITION 6.3. *Let X be a real Banach space. If $c_n(X) = n^{1/2}$ then, for any $\varepsilon > 0$, X contains an n-dimensional subspace X_n with $d(X_n, l_\infty^n) < 1+\varepsilon$.*

Proof. We have to show that, given $\varepsilon > 0$, there exist $x_1, \ldots, x_n$ in X such that

$$(1-\varepsilon)\max_{1\le i\le n}|a_i| \le \Big\|\sum_{i=1}^n a_i x_i\Big\| \le (1+\varepsilon)\max_{1\le i\le n}|a_i|.$$

By assumption, we can find $x_1, \ldots, x_n$ such that

$$\sum_{i=1}^n \|x_i\|^2 = n \quad\text{and}\quad \frac{1}{2^n}\sum_{\varepsilon_i=\pm1}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^2 \le (1+\delta)^2.$$

Here $\delta > 0$ will be chosen later. We may assume $\|x_1\| \ge \ldots \ge \|x_n\|$. We conclude from

$$x_1 = \frac{1}{2^n}\sum_{\varepsilon_i=\pm1}\varepsilon_1\sum_{i=1}^n \varepsilon_i x_i$$

that

$$1 \le \|x_1\| \le \frac{1}{2^n}\sum_{\varepsilon_i=\pm1}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\| \le \Big(\frac{1}{2^n}\sum_{\varepsilon_i=\pm1}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^2\Big)^{1/2} \le 1+\delta.$$

Since for arbitrary $\xi_1, \ldots, \xi_m \in \mathbb{R}$,

$$\sum_{i=1}^m\sum_{j=1}^m (\xi_i-\xi_j)^2 = 2m\sum_{i=1}^m \xi_i^2 - 2\Big(\sum_{i=1}^m \xi_i\Big)^2,$$

we have furthermore

$$\begin{aligned}\sum_{\varepsilon_i=\pm1}\sum_{\varepsilon_i^0=\pm1}&\Big(\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\| - \Big\|\sum_{i=1}^n \varepsilon_i^0 x_i\Big\|\Big)^2\\ &= 2^{n+1}\sum_{\varepsilon_i=\pm1}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^2 - 2\Big(\sum_{\varepsilon_i=\pm1}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|\Big)^2\\ &\le 2^{2n+1}[(1+\delta)^2-1] \le 2^{2n+3}\delta,\end{aligned}$$

if we assume $\delta < 1$. Choose $\varepsilon_i^0 \in \{+1, -1\}$, $i = 1, \ldots, n$, such that

$$\Big\|\sum_{i=1}^{n} \varepsilon_i^0 x_i\Big\| = \min\Big\{\Big\|\sum_{i=1}^{n} \varepsilon_i x_i\Big\| : \varepsilon_i = \pm 1\Big\} \le 1 + \delta.$$

Combining the above observations gives

$$\Big\|\sum_{i=1}^{n} \varepsilon_i x_i\Big\| \le \Big\|\sum_{i=1}^{n} \varepsilon_i^0 x_i\Big\| + 2^{(n+3)/2}\delta^{1/2} \le 1 + 2^{n+2}\delta^{1/2}.$$

If we let $\delta' \ge 2^{n+2}\delta^{1/2}$, then

$$\Big\|\sum_{i=1}^{n} \varepsilon_i x_i\Big\| \le 1 + \delta'$$

for all $\varepsilon_i = \pm 1$. Now we conclude from $\|x_i\| \le \|x_1\| \le 1 + \delta$ that

$$\|x_i\|^2 = n - \sum_{j \ne i} \|x_j\|^2 \ge n - (n-1)(1+\delta)^2 \ge 1 - 3n\delta,$$

therefore also

$$\|x_i\| \ge 1 - 3n\delta \quad \text{for } i = 1, \ldots, n.$$

Finally, we let $\xi_1, \ldots, \xi_m \in \mathbb{R}$ and $\sup_k |\xi_k| = 1$. An extreme point argument shows

$$\Big\|\sum_{i=1}^{n} \xi_i x_i\Big\| \le 1 + \delta'.$$

On the other hand, choosing $1 \le i \le n$ such that $|\xi_i| = 1$, we get

$$\begin{aligned} 2 - 6n\delta \le 2\|\xi_i x_i\| &\le \Big\|\xi_i x_i + \sum_{j \ne i} \xi_j x_j\Big\| + \Big\|\xi_i x_i - \sum_{j \ne i} \xi_j x_j\Big\| \\ &\le \Big\|\sum_{i=1}^{n} \xi_i x_i\Big\| + (1 + \delta'). \end{aligned}$$

Hence

$$1 - 6n\delta - \delta' \le \Big\|\sum_{i=1}^{n} \xi_i x_i\Big\|,$$

which completes the proof with δ, δ' such that $\delta < 1$, $6n\delta + \delta' < \varepsilon$ and $2^{n+2}\delta^{1/2} < \delta' < \varepsilon$.

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FSU Jena
Mathematisches Institut
07740 Jena, Germany
E-mail: nah@hpux.rz.uni-jena.de

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