STUDIA MATHEMATICA

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STUDIA MATHEMATICA

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997

Subscription information (1996): Vols. 117(2,3)–121 (14 issues); $30 per issue.

Correspondence concerning subscription, exchange and back numbers should be addressed to

Institute of Mathematics, Polish Academy of Sciences
Publications Department

Śniadeckich 8, P.O. Box 137, 00-950 Warszawa, Poland, fax 48-22-6293997

Published by the Institute of Mathematics, Polish Academy of Sciences
Typeset in TeX at the Institute
Printed and bound by

drukarnia
herman & herman

PRINTED IN POLAND

ISSN 0039-3223

Holomorphic motions commuting with semigroups

by

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Abstract. A holomorphic family f_z, $|z| < 1$, of injections of a compact set E into the Riemann sphere can be extended to a holomorphic family of homeomorphisms F_z, $|z| < 1$, of the Riemann sphere. (An earlier result of the author.) It is shown below that there exist extensions F_z which, in addition, commute with some holomorphic families of holomorphic endomorphisms of $\overline{\mathbb{C}} \setminus f_z(E)$, $|z| < 1$ (under suitable assumptions). The classes of covering maps and maps with the path lifting property are discussed.

0. Introduction. Holomorphic motions were introduced by Mañé *et al.* [MSS].

DEFINITION 0.1 [MSS]. Let D denote the unit disc and $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Let E be an arbitrary subset of $\overline{\mathbb{C}}$. A map $(v, z) \to f_v(z) : D \times E \to \overline{\mathbb{C}}$ is called a *holomorphic motion* of E in $\overline{\mathbb{C}}$ over D if:

(i) $f_0 = \mathrm{id}_E$;
(ii) f_v is an injection for every $v \in D$, and
(iii) the function $v \to f_v(z) : D \to \overline{\mathbb{C}}$ is holomorphic for every $z \in E$.

The following extension theorem due to the author [Sł1] (cf. also [Sł3] for a simplified proof) is the basis of the results of this paper.

THEOREM 0.2 [Sł1]. *Let $(v, z) \to f_v(z) : D \times E \to \overline{\mathbb{C}}$ be a holomorphic motion. Then there is a holomorphic motion $(v, z) \to F_v(z) : D \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $F_v|E = f_v$ for $v \in D$.*

Partial extension results were earlier obtained by Sullivan and Thurston [ST], who posed the problem answered by Theorem 0.2, and by Bers and Royden [BR].

The reader is referred to [MSS, ST, Su, BR, EKK] and [Sł1,3] for further results on holomorphic motions, their background and applications.

Problems concerning the "natural" extensions of holomorphic motions, in particular of finding equivariant analogues of Theorem 0.2 were formulated

1991 *Mathematics Subject Classification*: Primary 30E25; Secondary 30D45, 30C60.

This research was supported in part by an NSF grant.

by Sullivan and Thurston [ST], Bers and Royden [BR] and Curt McMullen [Mu]. In [Mu], Curt McMullen formulated two questions, which we reproduce here in a somewhat re-digested form (which means that C. McMullen is not responsible for possible shortcomings of this formulation).

Let $(v,z) \to f_v(z) : D \times E \to \overline{\mathbb{C}}$, with E closed, be a holomorphic motion. Define $U_v = \overline{\mathbb{C}} \setminus f_v(E)$, $v \in D$.

Suppose that there is a group of holomorphic families of conformal maps $g_v : U_v \to U_v$, $v \in D$ (with $g_v(z)$ holomorphic in (v,z)). Must there exist a holomorphic motion $(v,z) \to F_v(z) : D \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ extending f and commuting with every g, i.e.,

$$g_v F_v(z) = F_v g_0(z) \quad \text{for } v \in D,\ z \in U_0.$$

The second question is whether the same conclusion is true if the group $\{g\}$ is replaced by a semigroup of holomorphic families of endomorphisms $g_v : U_v \to U_v$.

The following theorem, proved by the author in [Sł3, Sec. 3] (cf. also [Sł2]) provides a complete answer to the first question.

THEOREM 0.3 [Sł2, Sł3]. *Let E_0 be a compact subset of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ having at least 3 distinct points. Let $(v,z) \to f_v^0(z) : D \times E_0 \to \overline{\mathbb{C}}$ be a holomorphic motion. Define $U_v = \overline{\mathbb{C}} \setminus E_v$, where $E_v = f_v^0(E_0)$, $v \in D$, and let G be the group of all fiber-preserving biholomorphic maps g of $U = \bigcup_{v \in D} \{v\} \times U_v \subset D \times \overline{\mathbb{C}}$ of the form $(v,z) \to g(v,z) = (v, g_v(z)) : U \to U$. Then there is a holomorphic motion $(v,z) \to F_v(z) : D \times U_0 \to \overline{\mathbb{C}}$ such that*

(i) $F_v(U_0) = U_v$, $v \in D$;
(ii) $F_v \circ g_0 = g_v \circ F_v$.

A somewhat less general version of this result has been independently obtained by Earle *et al.* [EKK].

The aim of the present paper is to study the second question of McMullen, i.e. the case when g_v's are not necessarily automorphisms. In the next section we point out the intrinsic difficulties of the problem and formulate the results of the paper.

Acknowledgements. The author is grateful to Curt McMullen for communicating him the above questions. Most of the paper was conceived in Spring 1993, during the author's visit at the I.H.E.S. The author would like to thank the Institute and, in particular, Professors M. Berger and D. Sullivan, for the invitation and support.

1. Results

NOTATIONS AND TERMINOLOGY 1.1. U_v, $v \in D$, will be a holomorphic family of domains in $\overline{\mathbb{C}}$, by which we mean that there is a holomorphic motion $(v,z) \to f_v(z) : U_0 \to \overline{\mathbb{C}}$ such that $f_v(U_0) = U_v$. We then say that f_v *traces* U_v. (By Theorem 0.2 this is equivalent to the condition that there is a holomorphic motion f_v^0, $v \in D$, tracing $E_v = \overline{\mathbb{C}} \setminus U_v$. We will assume that $\#(\overline{\mathbb{C}} \setminus U_v) \ge 3$. We will consider a semigroup G of analytic, fiber-preserving maps of the form

$$g : (v,z) \to (v, g_v(z)) : U \to U,$$

where $U = \bigcup_{v \in D} \{v\} \times U_v$, or a single g of this kind. Additional assumptions on G/g will be formulated as needed.

One observes quickly that the existence of a holomorphic motion F_v tracing U_v and commuting with the action of G implies that

$$g_v = F_v \circ g_0 \circ F_v^{-1}, \quad v \in D,\ g \in G. \tag{1.1}$$

This implies that $g \in G$ is uniquely determined by g_0, and that, for each $g \in G$, all the g_v's are topologically conjugate. The uniqueness property implies that G cannot be too large, in particular it cannot contain too many g's such that $\overline{g_v(U_v)}$ is compact in U_v, $v \in D$. The conjugation property implies that for each $g \in G$, the topological or combinatorial pattern of critical points of g_v should not change with $v \in D$.

The next result, to be proven in Section 2, describes the simplest case in which difficulties of both kinds are avoided.

THEOREM 1.2. *Let U_v, $v \in D$, U and $G = \{g\}$ satisfy Notation and Terminology 1.1. If, in addition, $g_v : U_v \to U_v$ is a covering map for every $v \in D$ and $g \in G$, then there is a holomorphic motion F_v, $v \in D$, tracing U_v and commuting with $g \in G$, i.e. $F_v \circ g_0 = g_v \circ F_v$.*

Proper maps are fairly close to covering maps. The next example shows, however, that some assumptions on critical points have to be introduced.

EXAMPLE 1.3. Consider the maps $g_v(z) = z^2 + v$. As is well known (cf. Douady and Hubbard [DH], Sullivan [Su]) for $|v| < c$, $c > 0$, the Julia set J_v of g_v is a quasicircle. Denote by U_v the unique bounded component of $\mathbb{C} \setminus J_v$. Then J_v moves by holomorphic motion over $|v| < c$ (cf. [DH], [Su, Section 7]) and so does U_v by Theorem 0.2. If there were a holomorphic motion $F : D(0,c) \times U_0 \to \mathbb{C}$ with $F_v(U_0) = U_v$, commuting with the action of g_v, then g_v would be topologically conjugate, via F_v, to g_0. However, g_0 has a hyperattractive fixed point at 0, while the g_v have no hyperattractive fixed points for $0 < |v| < c$. The reason for this is that for $g_0|U_0$ its critical point and fixed point coincide, while for $0 < |v| < c$ they differ.

This kind of phenomena has motivated Mañé *et al.* [MSS] and Sullivan [Su] to introduce the condition of permanence of critical orbit relations [Su, Section 1] for the description of the conjugacy classes of rational maps of $\mathbb{C}$. We now introduce a close analogue of their condition. (The only difference between our version and theirs is that they had, or have chosen, to treat differently the case when (g_v, U_v) is equivalent to a Siegel disc or Herman ring, or has a superattractive fixed point).

DEFINITION 1.4. Let $g_v : U_v \to U_v$, $v \in D$. Assume that $(v,z) \to g_v(z)$ is analytic. We say that the family g satisfies the condition of *permanence of critical orbit relations* if

(i) the critical variety
$$\{(v,z) : v \in D,\ z \in U_v,\ (g_v)'(z) = 0\}$$
is the union of mutually disjoint graphs of analytic functions $c_1, c_2, \ldots : D \to \mathbb{C}$,

(ii) for every n, m, i, j the functions $z \to g_v^n(c_i(z))$ and $z \to g_v^m(c_n(z))$ are either identical or have disjoint graphs.

THEOREM 1.5. *Let $g_v : U_v \to U_v$, $v \in D$, be an analytic family of proper maps satisfying the condition of permanence of critical orbit relations. Let U_v be a holomorphic family of domains with* $\operatorname{card}(\overline{\mathbb{C}} \setminus U_v) \geq 3$. *Then there is a holomorphic motion F_v, $v \in D$, tracing U_v and commuting with g_v, $v \in D$.*

This theorem generalizes [MSS, Theorem D] (cf. also [Su]). Our proof is different from theirs. We will actually prove a more general result, Theorem 1.7, which simultaneously generalizes (or nearly generalizes) Theorems 1.2 and 1.5. (Theorem 1.5 follows immediately from Theorem 1.7 and Proposition 3.5.)

DEFINITION 1.6. Let $g : X \to Y$ be a continuous map. We say that g has the *path lifting property* if for every arc $\gamma : [0,1] \to Y$ and for every $x_0 \in g^{-1}(\gamma(0))$ there is an arc $\sigma : [0,1] \to X$, not necessarily unique, such that $g(\sigma(t)) = \gamma(t)$ for $t \in [0,1]$ and $\sigma(0) = x_0$.

Examples of maps with the path lifting property include covering maps and proper maps. Further examples and basic properties of such maps are discussed in Sections 3, 4. The next theorem is the main result of this paper.

THEOREM 1.7. *Let U_v, $v \in D$, be a holomorphic family of domains in* $\mathbb{C}$, $\operatorname{card}(\mathbb{C} \setminus U_v) \geq 3$, *and $g_v : U_v \to U_v$, $v \in D$, be an analytic family of maps having the path lifting property. Assume that g satisfies the condition of permanence of critical orbit relations. Then there is a holomorphic motion $F_v : U_0 \to U_v$ tracing U_v and commuting with g.*

This result is proven in Section 5, where we also discuss its analogue for semigroups (cf. Definition 5.1 and Proposition 5.2), too technical to be discussed here. The following Corollary of Proposition 5.2 is, perhaps, more attractive.

COROLLARY 1.8. *Let U_v, $v \in D$, U, $G = \{g\}$ satisfy Notation and Terminology 1.1. Assume, in addition, that there is a topological isotopy ϕ_v, $v \in D$, with $\phi_v(U_0) = U_v$ which conjugates the action of G on U_0 and U_v, $v \in D$ (i.e., $g_v = \phi_v \circ g_0 \circ \phi_v^{-1}$ for $g \in G$ and $v \in D$). Then there is a holomorphic motion F_v, $v \in D$, tracing U_v, which quasiconformally conjugates the actions of G on U_v and U_0.*

2. Holomorphic motions commuting with families of covering maps

Proof of Theorem 1.2. The idea of the proof is to lift the actions of $g : U \to U$, $g \in G$, to the universal covering space $\pi : \widetilde{U} \to U$, obtain an equivariant holomorphic motion in $\widetilde{U}$ by means of Theorem 0.3, and observe that it can be pushed down to U.

We now recall the description of the universal covering space of U, denoted here by $\pi : \widetilde{U} \to U$, given in [Sł3, Proposition 3.4]. Define $\widetilde{U}_v = \pi^{-1}(\{u\} \times U_v)$ and let $\pi_v : \widetilde{U}_v \to U_v$ be the map defined by $\{v\} \times \pi_v(x) = \pi(x)$, $x \in \widetilde{U}_v$, and let Γ be the covering group. Then the $\widetilde{U}_v$ are topological discs, $\pi_v : \widetilde{U}_v \to U_v$, $v \in D$, are universal covering maps and Γ is isomorphic to the fundamental group $\pi^1(U_v), v \in D$. With the unique complex structure in $\widetilde{U}$ making π an analytic map the fibers $\widetilde{U}_v$ are conformally equivalent to the unit disc and the deck transformations $\gamma : \widetilde{U} \to \widetilde{U}$, $\gamma \in \Gamma$, are biholomorphic. For $g \in G$, let $\widetilde{g} : \widetilde{U} \to \widetilde{U}$ be any of its lifts to $\widetilde{U}$. Denote by $\widetilde{G}$ the set of all such lifts. Then $\widetilde{g}(\widetilde{U}_v) = \widetilde{U}_v$ and the map $\widetilde{g}_v = \widetilde{g}|\widetilde{U}_v$ is a lift of g_v to the universal covering space $\pi_v : \widetilde{U}_v \to U_v$. As is well known, such a lifting must be a homeomorphism when g_v is a covering map. We need the following two observations.

ASSERTION 1. *Any holomorphic motion tracing U_v can be lifted to a holomorphic motion tracing $\widetilde{U}$, i.e. there is a homeomorphism*
$$(v,z) \to \widetilde{f}_v(z) : D \times \widetilde{U}_0 \to \widetilde{U}$$
such that for every $z \in \widetilde{U}_0$, $v \to \widetilde{f}_v(z) : D \to \widetilde{U}$ is an analytic mapping and $\widetilde{f}_v(\{v\} \times \widetilde{U}_0) = \widetilde{U}_v$, $v \in D$.

ASSERTION 2. *There is a biholomorphic, fiber-preserving embedding of $\widetilde{U}$ onto an open subset of $D \times \mathbb{C}$.*

We first conclude the proof of the theorem, assuming the above facts. Taken together, Assertions 1 and 2 allow us to assume, without loss of generality, that $\widetilde{U} \subset D \times \mathbb{C}$, with $\widetilde{U}_v \subset \{v\} \times \mathbb{C}$, $v \in D$, and that the family $\widetilde{U}_v$ varies holomorphically. Denote by $\mathcal{H}$ the group of fiber-preserving biholomorphic maps $h : \widetilde{U} \to \widetilde{U}$. In particular, $h_v = h|\widetilde{U}_v : \widetilde{U}_v \to \widetilde{U}_v$ are conformal maps. Note that

$$\widetilde{G}, \Gamma \subset \mathcal{H}.$$

By Theorem 0.3 there is a holomorphic motion $\widetilde{F}(v,z) = \widetilde{F}_v(z)$, $\widetilde{F} : D \times U_0 \to \widetilde{U}$, $\widetilde{F}_v(\widetilde{U}_0) = \widetilde{U}_v$, $v \in D$, such that

$$h_v \circ \widetilde{F}_v = \widetilde{F}_v \circ h_0 \quad \text{for } h \in \mathcal{H},\ v \in D.$$

We now define $F_v : U_0 \to U_v$, $v \in D$. For $z_0 \in U_0$, let $F_v(z_0) = \pi_v \widetilde{F}_v(w_0)$ if $w_0 \in \pi_0^{-1}(z_0)$. The definition is independent of the choice of w_0, for if $w_1, w_0 \in \pi_0^{-1}(z_0)$, then there is $\gamma \in \Gamma$ such that $w_1 = \gamma_0(w_0)$ and so $\pi_v \widetilde{F}_v(w_0) = \pi_v \widetilde{F}_v(w_1)$.

To check that the F_v have the required properties it is actually more convenient to look at the trajectories of $\widetilde{F}_v$ and F_v. The former define a holomorphic foliation of $\widetilde{U}$ that is preserved by the action of Γ, i.e., for every $\gamma \in \Gamma$, γ maps one foliation leaf onto another foliation leaf. Thus the projection π maps the foliation of $\widetilde{U}$ onto a foliation of U, whose leaves are trajectories of F_v. This gives immediately that F_v is a holomorphic motion. Since the foliation of $\widetilde{U}$ is left invariant by $\mathcal{H} \supset \{\widetilde{g} : g \in G\} \cup \Gamma$, it is clear that the projected foliation in U is preserved by G, which is equivalent to $g_v F_v = F_v g_0$, $g \in G$.

It remains to prove the two assertions.

Observe first that Assertion 2 follows from Assertion 1. The statement of the latter means that the family of Riemann surfaces $\widetilde{U}_v$, $v \in D$, is a simple analytic family in the sense of Earle and Fowler [EF], i.e., it satisfies condition (a) of Proposition 4.1 in [Sł3]. By this result the family $\widetilde{U}_v$, $v \in D$, is equivalent to a Bers model, which, in particular, implies Assertion 2.

As for Assertion 1 observe that since $\pi : \widetilde{U} \to U$ and $\mathrm{id} \times \pi_0 : D \times \widetilde{U} \to D \times U_0$ are both universal coverings, the homeomorphism $(v,z) \to (v, f_v(z)) : D \times U_0 \to U$ can be lifted to a homeomorphism $D \times \widetilde{U}_0 \to \widetilde{U}$ which makes the diagram

$$\begin{array}{ccc} D \times \widetilde{U}_0 & \longrightarrow & \widetilde{U} \\ \downarrow {\scriptstyle \mathrm{id} \times \pi_0} & & \downarrow {\scriptstyle \pi} \\ D \times U_0 & \longrightarrow & U \end{array}$$

commutative. Since it moves fibers to fibers, it must be of the form $(v,z) \to (v, \widetilde{f}_v(z))$, where $\widetilde{f}_v : \widetilde{U}_0 \to \widetilde{U}_v$, $v \in D$, are homeomorphisms. If we specify one point $(0, w_0) \in D \times \widetilde{U}_0$ and require that it is mapped onto $w_0 \in \widetilde{U}$, then the lifting $\widetilde{f}$ is unique, $\widetilde{f}_0(w_0) = w_0$, $\mathrm{id}_{U_0} \circ \pi_0 = \pi_0 \circ \widetilde{f}_0$ and so $\widetilde{f}_0 = \mathrm{id}_{\widetilde{U}_0}$. Since $\pi(\widetilde{f}_v(z)) = f_v(\pi_0(z))$, and π is locally biholomorphic, $\widetilde{f}_v(z)$ must be holomorphic in z for every fixed $z \in \widetilde{U}_0$. This completes the proof of Assertion 1. ∎

3. Mappings with the path lifting property. We restrict our discussion of mappings with the path lifting property to the case of open Riemann surfaces or planar domains because that is what we need for our applications.

PROPOSITION 3.1. *Let W, U be open Riemann surfaces and $g : W \to U$ be an analytic endomorphism with the path lifting property. Let C denote the set of critical points of g and $V = U \setminus \overline{g(C)}$. Then $g|g^{-1}(V) : g^{-1}(V) \to V$ is a covering map.*

Proof. Since g is analytic, $g'(z) \neq 0$ on $g^{-1}(V)$ and so $g|g^{-1}(V)$ is a local homeomorphism. Because g has the path lifting property, g is onto, hence $g|g^{-1}(V)$ is onto V, and the latter map has the unique path lifting property (being a local homeomorphism). By the topological monodromy theorem, $g|g^{-1}(V)$ is a covering map onto V. ∎

Remark 3.2. It is an obvious observation that all the analytic self-maps $g : W \to W$ with the path lifting property form a semigroup with respect to composition.

LEMMA 3.3. *Let W, U be open Riemann surfaces and let $g : W \to U$ have the path lifting property. Let $\gamma : [0,1] \to U$ be an arc. Assume that for every $0 \le t < 1$ the point $\gamma(t)$ is not a critical value (but $\gamma(1)$ can be a critical value). Let $g^{-1}(\gamma(0)) = \{z_n : n = 1,2,\ldots\}$ and let $\sigma_n : [0,1] \to W$ denote a lifting of γ such that $\sigma_n(0) = z_n$, $n = 1,2,\ldots$ Then*

(a) *For each n, σ_n is unique.*
(b) *$\{\sigma_n(1) : n = 1,2,\ldots\} = g^{-1}(\gamma(1))$.*

Proof. (a) Fix n. Suppose there are two liftings σ_n, τ_n with $\sigma_n(0) = z_n = \tau_n(0)$. Let $t^* \in [0,1]$ be the largest t such that $\sigma_n(s) = \tau_n(s)$ for all $0 \le s \le t$. Suppose $t^* < 1$. Then $g(\sigma_n(t^*)) = \gamma(t^*) \notin g(C)$ and so $\sigma_n(t^*) \notin C$. There is a neighbourhood W_0 of $\sigma_n(t^*) = \tau_n(t^*)$ such that $g|W_0 : W_0 \to g(W_0)$ is a local homeomorphism. Choose $\varepsilon > 0$ such that $\sigma_n([t^*-\varepsilon, t^*+\varepsilon]) \subset W_0$, $\tau_n([t^*-\varepsilon, t^*+\varepsilon]) \subset W_0$ and $\gamma([t^*-\varepsilon, t^*+\varepsilon]) \subset g(W_0)$. Since $g|W_0$ is a local homeomorphism, $\sigma_n|[t^*-\varepsilon, t^*+\varepsilon] = \tau_n|[t^*-\varepsilon, t^*+\varepsilon]$, in contradiction to the maximality of t^*. Thus $t^* = 1$.

(b) Let $\{w_m\} = g^{-1}(\gamma(1))$. By the path lifting property for each m there is an arc $\mu_m : [0,1] \to W$, perhaps not unique (if $\gamma(1)$ is a critical value), such that $\mu_m(1) = w_n$ and $g\mu_n = \gamma$. But then μ_m must be equal to one of the arcs σ_n, namely the unique one for which $\sigma_n(0) = \mu_m(0)$. Thus $g^{-1}(\gamma(1)) \subset \{\sigma_n(1) : n = 1,2,\ldots\}$. The reverse inclusion is obvious. ∎

We now consider some natural examples of analytic endomorphisms having the path lifting property. The class of almost proper maps was introduced by E. Bishop [Bi].

DEFINITION 3.4 [Bi, GR, Ch. VII, Sec. C.1]. Let $g : X \to Y$ be a continuous map. We say that g is *almost proper* if, for every compact subset $K \subset Y$, every connected component of $g^{-1}(K)$ is compact. We say that g is *locally almost proper* if for every $y_0 \in Y$ there is a compact neighbourhood N of y_0 such that $g|g^{-1}(N) : g^{-1}(N) \to N$ is almost proper.

It is clear that the class of locally almost proper maps contains covering maps, proper maps and branched regular coverings (cf. [Ma, III.F.2]).

PROPOSITION 3.5. *Let W, U be open Riemann surfaces and $g : W \to U$ be an analytic map. Assume $g : W \to U$ is almost proper. Then g has the path lifting property.*

For the proof we need the following result due to Bishop.

PROPOSITION 3.6. *Let $g : W \to U$ be a locally almost proper map between open Riemann surfaces. Then for every point $z_0 \in U$ there is a compact disc neighbourhood $\overline{D}_{z_0}$ with interior D_{z_0} such that $g^{-1}(D_{z_0}) = \bigcup X_n$, X_n are connected open sets with mutually disjoint closures $\overline{X}_n$ and the restricted maps*

$$g|X_n : X_n \to D_{z_0}, \quad g|\overline{X}_n : \overline{X}_n \to \overline{D}_{z_0}$$

are proper and onto.

For the proof observe that we can choose $\overline{D}_{z_0}$ as a subset of some open neighbourhood V of z_0 such that $g|g^{-1}(V) : g^{-1}(V) \to V$ is almost proper, in which case the statement was proved by Bishop [Bi] (cf. also [GR, Lemma VII.C.3]).

Proof of Proposition 3.5 (sketch). Choosing a suitable finite covering of the arc $\gamma : [0,1] \to U$ by neighbourhoods D_{z_0} with the properties required in Proposition 3.6 we can assume that there is a subdivision $0 = t_0 < t_1 < \ldots < t_m = 1$ and neighbourhoods $D_0, D_1, \ldots, D_m$ as in Proposition 3.6 such that

$$\gamma_j = \gamma([t_{j-1}, t_{j+1}]) \subset D_j, \quad j = 1, \ldots, m-1,$$

and none of $\gamma(t_0), \gamma(t_1), \ldots, \gamma(t_m)$ is a critical value. It is clearly enough to find, for every $\xi_0 \in g^{-1}(\gamma(t_j))$, a lifting $\sigma_j : [t_j, t_{j+1}] \to W$ such that $\sigma_j(t_j) = \xi_0$ and $g\sigma_j = \gamma_j$. Let $g^{-1}(D) = \bigcup X_n$, where the X_n have properties as in Proposition 3.6, in particular are connected and mutually disjoint. Let X_{n_0} be the unique component containing ξ_0.

Since $g|X_{n_0} : X_{n_0} \to D_j$ has only finitely many critical points and since ξ_0 is not a critical point there is a homotopy $\gamma(s,t)$ such that

$$\gamma : [t_j, t_{j+1}] \times [0,1] \to D_j, \quad \gamma(t_j, s) = \gamma(t_j), \quad 0 \le s \le 1,$$
$$\gamma(t,1) = \gamma(t), \quad t_j \le t \le t_{j+1},$$

and $\gamma(t,s)$ is not a critical value of $g|X_{n_0}$ for $(t,s) \in [t_j, t_{j+1}] \times [0,1)$. Since $g|X_{n_0} \setminus g^{-1}g(C) : X_{n_0} \setminus g^{-1}g(C) \to D_j \setminus g(C)$ is obviously a covering, and since the arc $\gamma^s(t) = \gamma(t,s)$, $t \in [t_j, t_{j+1}]$, omits $\gamma(C)$, it has a unique lifting starting with ξ_0. Thus there is a continuous function

$$\sigma_0 : [t_j, t_{j+1}] \times [0,1) \to X_{n_0} \setminus g^{-1}g(C)$$

such that $g \circ \sigma_0 = \gamma|[t_j, t_{j+1}] \times [0,1)$. Since $\overline{X}_{n_0}$ is compact, $g|\overline{X}_{n_0} : \overline{X}_{n_0} \to D_j$ is a proper map with discrete fibers, and one can show easily that σ_0 has a continuous extension $\sigma : [t_j, t_{j+1}] \times [0,1] \to \overline{X}_{n_0}$ which lifts the homotopy γ. (Note that cluster sets of σ_0, at the points $(t,1)$, must be connected subsets of fibers of g, hence have to consist of single points). Then $\sigma_j(t) = \gamma(t,1)$, $t_j \le t \le t_{j+1}$, is the required lifting of γ_j. (Note that σ_j might be nonunique because it depends on the choice of the homotopy $\gamma(t,s)$.) ∎

4. Analytic families of endomorphisms with the path lifting property

THEOREM 4.1. *Let U_v, $v \in D$, and $g_v : U_v \to U_v$, $v \in D$, satisfy all the assumptions of Theorem 1.5. Let $a : D \to \mathbb{C}$ be a holomorphic function such that, for every $v \in D$, $a(v)$ is not a critical value of g_v. Then there is a sequence (finite or infinite) of analytic functions $h_j : D \to \mathbb{C}$ with mutually disjoint graphs such that $g_v(h_j(v)) = a(v)$, $v \in D$, $j = 1, 2, \ldots$*

The proof of this theorem is based on the next lemma, presumably well known. Its proof is an easy exercise and is omitted.

LEMMA 4.2. *Let $\pi : W \to D$ be a local homeomorphism onto D. Assume that for every $v_0 \in D$, $z_0, z_1 \in \pi^{-1}(v_0)$, $r > 0$ such that $\overline{D}(v_0, r) \subset D$ and analytic function $h_0 : D(v_0, r) \to W$ satisfying $h_0(v_0) = z_0$ and $\pi h_0 = \mathrm{id}_{D(v_0,r)}$, there exists an analytic function $h_1 : D(v_0, r) \to W$ such that $h_1(v_0) = z_1$ and $\pi h_1 = \mathrm{id}_{D(v_0,r)}$. Then $\pi : W \to D$ is a covering map.*

Proof of Theorem 4.1. Let

$$W = \{(v,z) \in D \times \mathbb{C} : g_v(z) = a(v)\}$$

and $\pi(v,z) = v$. Then $\pi : W \to D$ is a local homeomorphism onto D, by the assumptions of the theorem. Let $v_0 \in D$, and let (v_0, z_0), (v_0, z_1) be two arbitrary points of the fiber $\pi^{-1}(v_0)$. Suppose π has an analytic section over $D(v_0, r)$, $r > 0$. Then it must be of the form $v \to (v, h_0(v))$, where $g_v h_0(v) = a(v)$ for $|v - v_0| < r$. Our proof of the theorem is based on the following assertion.

ASSERTION. *Whenever $g_{v_0}(z_0) = g_{v_0}(z_1) = a(v_0)$, $v_0 \in D$, $r > 0$ and $h_0 : D(v_0, r) \to \mathbb{C}$ is an analytic function such that $h_0(v) \in U_v$, $h_0(v_0) = z_0$ and $g_v(h_0(v)) = a(v)$, then there is an analytic function $h_1 : D(v_0, r) \to \mathbb{C}$ such that $h_1(v_0) = z_1$, $h_1(v) \in U_v$ and $g_v(h_1(v)) = a(v)$ in $D(v_0, r)$.*

Assuming the assertion, the function $v \to (v, h_1(v))$ is an analytic section of π passing through (v_0, z_1). Thus $\pi : W \to D$ satisfies the assumptions of Lemma 4.2 and so is a covering map. By the monodromy theorem $W = \{(v,z) : g_v(z) = a(v)\}$ is the union of countably many mutually disjoint graphs of analytic functions $h_j : D \to \mathbb{C}$.

It remains to prove the assertion.

Choose a real analytic arc $\sigma : [0,1] \to U_{v_0}$ joining z_0 to z_1, i.e., $\sigma(0) = z_0$, $\sigma(1) = z_1$, in such a way that it omits the critical points of g_{v_0}.

Denote by C_v the critical points of g_v. The assumption of the permanence of critical orbit relations and the assumption that $a(v) \notin g_v(C_v)$, $v \in D$, imply that the sets $g_v(C_v) \cup \{a(v)\}$ move holomorphically. Extending this motion to a motion tracing U_v, $v \in D$, by means of Theorem 0.2 we conclude that there is a holomorphic motion $(v,z) \to \phi_v(z) : D(v_0, r) \times U_{v_0} \to \mathbb{C}$ (it is more convenient to have the origin of the motion at v_0 now) such that

$$\phi_{v_0} = \mathrm{id}_{U_0}, \quad \phi_0(U_{v_0}) = U_v, \quad \phi_v(U_{v_0} \setminus g_{v_0}(C_{v_0})) = U_v \setminus g_v(C_v)$$

and

$$\phi_v(a(v_0)) = a(v).$$

Let $\gamma^v(t) = \phi_v(\gamma(t))$, $0 \le t \le 1$, $|v - v_0| < r$. Then γ^v is a closed path in U_v, omitting all the critical values of g_v and joining $a(v)$ to itself.

By the path lifting property there are unique arcs $\sigma^v : [0,1] \to U_v \setminus C_v$, $|v - v_0| < r$, such that $\sigma^v(0) = h_0(v)$ and $g_v \sigma^v(t) = \gamma^v(t)$, $t \in [0,1]$. Our plan is to show that $h_1(v) := \sigma^v(1)$, $|v - v_0| < r$, satisfies all the required conditions.

Define $\sigma_t(v) = \sigma^v(t)$. We first show that $\sigma_t(v)$ is jointly continuous in $(t,v) \in [0,1] \times D(v_0, r)$ and analytic in v. Fix $r_0 < r$. Consider any $t_0 \in [0,1]$ such that σ_{t_0} is a continuous function on $\overline{D}(v_0, r_0)$. Since the map $(v,z) \to (v, g_v(z))$, if restricted to a small enough neighbourhood of the graph $\{(v, \sigma_{t_0}(v)) : |v - v_0| \le r_0\}$, is a biholomorphic map onto some neighbourhood of the graph

$$\{(v, \gamma^v(t_0)) : |v - v_0| \le r_0\},$$

there is $\varepsilon > 0$ such that $\sigma_t(v)$ is continuous on

$$\{t_0 \le t \le \min(t_0 + \varepsilon, 1)\} \times \overline{D}(v_0, r_0).$$

Since $v \to \gamma^v(t)$ are analytic functions on $D(v_0, r)$, all σ_t's, $t_0 \le t \le \min(t_0 + \varepsilon, 1)$, are analytic on $\overline{D}(v_0, r_0)$. Observe that $t_0 = 0$ satisfies the above conditions. Hence there is a largest $t^* \in (0,1]$ such that $\sigma_t(v)$ is jointly continuous on $[0, t^*) \times \overline{D}(v_0, r)$. We want to show that this function is uniformly continuous on this set, which holds if and only if the cluster sets

$$\mathrm{Cl}(v) = \{z = \lim_{n \to \infty} \sigma_{t_n}(v_n) : t_n \nearrow t^*,\ v_n \to v\},$$

for $|v - v_0| \le r_0$, are singletons. Since $g_v(\sigma_t(v)) = \gamma^v(t)$ is a jointly continuous function,

$$\mathrm{Cl}(v) \setminus (\partial U_v \cup \{\infty\}) \subset g_v^{-1}(\gamma^v(t^*)), \quad |v - v_0| \le r_0.$$

On the other hand, the cluster sets are connected compact subsets of the Riemann sphere, for

$$\mathrm{Cl}(v) = \bigcap_{\varepsilon > 0} \mathrm{Closure}\,\{z = \sigma(t, w) : t \in [t^* - \varepsilon, t^*),\ w \in \overline{D}(v, \varepsilon)\}.$$

Finally, $\sigma_{t^*}(v) \in \mathrm{Cl}(v)$, since σ^v is a continuous arc. Since the fibers $g_v^{-1}(\gamma^v(t^*))$ are discrete subsets of U_v, we conclude that $\mathrm{Cl}(v) = \{\sigma_{t^*}(v)\}$ for $|v - v_0| \le r_0$. Thus $\sigma_t(v)$ is jointly continuous on $[0, t^*] \times \overline{D}(v_0, r_0)$, and, by the earlier argument, on $[0, \min(t^* + \varepsilon, 1)] \times \overline{D}(v_0, r_0)$ for some $\varepsilon > 0$, which contradicts the maximality of t^*, unless $t^* = 1$. Since $r^* < r$ is arbitrary,

$$(t, v) \to \sigma_t(v) : [0,1] \times D(v_0, r) \to \mathbb{C}$$

is continuous. As we have observed above, being continuous, σ_t must also be analytic on $D(v_0, r)$.

Let $h_1(v) = \sigma_1(v)$, $|v - v_0| < r$. Then $g_v(h_1(v)) = \gamma^v(1) = a(v)$, and $h_1(v_0) = \sigma(1) = z_1$, and $h_1(v) \in U_v$. This completes the proof of the assertion. ■

5. Proof of Theorem 1.7

Proof of Theorem 1.7. For brevity, for any function $f \in H(D, \mathbb{C})$ denote by $g \circ f$ or $g^n \circ f$, $n = 1, 2, \ldots$, the functions $g \circ f(v) = g_v(f(v))$, $g^n \circ f(v) = g_v^n(f(v))$, $v \in U_v$, $v \in D$. Let $\mathcal{C} = \{c_1, c_2, \ldots\}$, where the c_n are as in Definition 1.4, and let

$$\mathcal{F} = \bigcup_{n \ge 0} g^n(\mathcal{C}) = \{g^n \circ c_j : n \ge 0,\ j = 1, 2, \ldots\}.$$

Define inductively $\mathcal{F}_k$, $k = 0, 1, 2, \ldots$, by $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_{k+1} = g^{-1}(\mathcal{F}_k) \cup \mathcal{F}_k$, $k = 0, 1, \ldots$, where $g^{-1}(\mathcal{F}_k) = \{f \in H(D, \mathbb{C}) : g \circ f \in \mathcal{F}_k\}$.

ASSERTION 1. *Let $a(v)$ be holomorphic in D with $a(v) \in U_v$ for $v \in D$. Assume that either $a(v) \notin g_v(C_v)$ for every $v \in D$ or $a \in g(\mathcal{C})$. Then*

$$\{(v, z) \in D \times \mathbb{C} : g_v(z) = a(v)\}$$

is the union of mutually disjoint graphs of functions analytic in D.

ASSERTION 2. *For every $k = 0, 1, 2, \ldots$ functions in $\mathcal{F}_k$ have mutually disjoint graphs.*

Concerning the proof of Assertion 1, the case when all $a(v)$ are noncritical points was established in Theorem 4.1.

When $a \in g(\mathcal{C})$, consider a holomorphic motion $\phi_v : U_0 \to U_v$, $v \in D$, like at the beginning of the proof of Theorem 4.1, such that $\phi_v(g_0(C_0)) = g_v(C_v)$, $\phi_v(U_0 \setminus g_0(C_0)) = U_v \setminus g_v(C_v)$, in particular $\phi_v(a(0)) = a(v)$. Since C_0 is a countable set, there is a closed arc $\gamma : [0,1] \to U_0$ such that $\gamma(1) = a(0)$ and for every $0 \le t < 1$, $\gamma(t)$ is not a critical point of g_0. As in the proof of Theorem 4.1 define $\gamma^v : [0,1] \to U_v$ by $\gamma^v(t) = \phi_v(\gamma(t))$. Let $a_t(v) = \gamma^v(t)$. Then a_t's are analytic functions with $a_1 = a$.

Since all the points $a_0(v)$ are noncritical, there are analytic functions $h_1, h_2, \ldots$ in D with mutually disjoint graphs such that

$$g_v^{-1}(a_0(v)) = \{h_n(v) : n = 1, 2, \ldots\},$$

and since all points of γ^v except for the terminal point $\gamma^v(1)$ are noncritical values of g_v, the arc γ^v has, by Lemma 3.3(a), a unique lifting σ_n^v for each initial point $h_n(v)$. That is, there are $\sigma_n^v : [0,1] \to U_v$, $n = 1, 2, \ldots$, $v \in D$, such that

$$\sigma_n^v(0) = h_n(v), \quad g_v(\sigma_n^v(t)) = \gamma^v(t), \quad 0 \le t \le 1.$$

Since the arc γ^v contains critical values, the results established in the proof of Theorem 4.1 do not apply directly to γ^v and σ_n^v. They apply, however, for every $0 < \varepsilon < 1$, to the shorter arcs $\gamma^v|[0, 1-\varepsilon]$ and $\sigma_n^v|[0, 1-\varepsilon]$. They can be summarized as follows:

Assertion 3. *For every* $n = 1, 2, \ldots,$

$$(t, v) \to \sigma_n^v(t) : [0,1) \times D \to \mathbb{C}$$

is a continuous function, and $v \to \sigma_n^v(t) : D \to \mathbb{C}$ *is analytic for* $0 \le t < 1$, $n = 1, 2, \ldots$

Since the half-open arcs $\sigma_n^v|[0,1)$ are mutually disjoint for different n, the analytic functions $v \to \sigma_n^v(t) : D \to \mathbb{C}$, $n = 1, 2, \ldots$, define a holomorphic motion which, by the lambda lemma [MSS, Section 1.2] is jointly continuous and has a continuous extension to a holomorphic motion ψ_v of the closure of the set $E = \{\sigma_n^0(t) : 0 \le t < 1,\ n = 1, 2, \ldots\}$ such that

$$\psi_v(\overline{E}) = \mathrm{Cl}\{\sigma_n^v(t) : 0 \le t < 1,\ n = 1, 2, \ldots\},$$
$$\psi_v(\sigma_n^0(t)) = \sigma_n^v(t), \quad t \in [0,1),\ v \in D,\ n = 1, 2, \ldots$$

On the other hand$g(c_1) \in \mathcal{F}_0 \subset \mathcal{F}_k$,, by Lemma 3.3(a), $\lim_{t\to 1} \sigma_n^v(t) = \sigma_n^v(1)$, pointwise, for $v \in D$, $n = 1, 2, \ldots$ This convergence must be uniform on compact subsets of D, by the property of ψ, and so $\sigma_n^v(1) = \psi_v(\sigma_n^0(1))$. Thus the functions $v \to \sigma_n^v(1) : D \to \mathbb{C}$, $n = 1, 2, \ldots$, are analytic and their graphs are either mutually disjoint or identical. Since $\{\sigma_n^v(1) : n = 1, 2, \ldots\} = g_v^{-1}(\gamma(1)) = g_v^{-1}(a(v))$, by Lemma 3.3(b), Assertion 1 follows.

We now show Assertion 2 by induction on k. The case $k = 0$ is just the content of Definition 1.4. Assume the statement for $\mathcal{F}_k$ and consider any $a \in \mathcal{F}_k$. Suppose that for some $v_0 \in D$, $a(v_0)$ is a critical value of g_{v_0}. Then there is c_i such that $g_{v_0}(c_i(v_0)) = a(v_0)$. On the other hand, $g(c_i) \in \mathcal{F}_0 \subset \mathcal{F}_k$. Hence $g(c_i)$ and a both belong to $\mathcal{F}_k$, and by inductive assumption they must be identical since they are equal over v_0. Thus all values $a(v)$ are critical. We conclude that Assertion 1 applies to every $a \in \mathcal{F}_k$ and so for every $a \in \mathcal{F}_k$ the set

$$\{(v, z) : g_v(z) = a(v)\} \tag{5.1}$$

is the union of mutually disjoint graphs of analytic functions in $g^{-1}(a)$. The varieties (5.1) corresponding to distinct a's in $\mathcal{F}_k$ are mutually disjoint, since the graphs of a's are mutually disjoint, and so all the functions in $g^{-1}(\mathcal{F}_k)$ have mutually disjoint graphs. It remains to observe that if $a \in g^{-1}(\mathcal{F}_k)$, $b \in \mathcal{F}_k$ and $a(v_0) = b(v_0)$ for some v_0, then $a \equiv b$. It is easy to show by induction on k that

$$g(\mathcal{F}_k) \subset \mathcal{F}_k \qquad \text{for } k \ge 0,$$
$$g(\mathcal{F}_k) \subset \mathcal{F}_{k-1} \qquad \text{for } k \ge 1.$$

Thus $g(a) = g(b) \in \mathcal{F}_{k-1}$. Let $a_1 = g(a)$. Applying the statement (5.1) to $a_1 \in \mathcal{F}_{k-1}$ we infer that $a \equiv b$, since their graphs intersect. This completes the proof of Assertion 2.

Denote by $\theta_v(z)$ the "large orbits" of g_v, that is,

$$\theta_v(z) = \bigcup_{n,m\ge 0} g_v^{-n} g_v^m(z)$$

(cf. Sullivan [Su]). Let

$$Z_v = \bigcup\{\theta_v(z) : z \in C_v\}, \qquad v \in D.$$

It follows from the definition of $\mathcal{F}_k$'s, Assertion 2 and (5.1) that the set

$$Z = \bigcup_{v\in D} \{v\} \times Z_v$$

is the union of the graphs of all the functions in $\bigcup_{k\ge 0} \mathcal{F}_k$, which are mutually disjoint. This defines a foliation of Z which is unique (the fibers Z_v are countable sets) and invariant with respect to g. Hence the formula

$$f_v^Z(z_0) = f(v),$$

where f is the unique function in $\bigcup_{k\ge 0} \mathcal{F}_k$ such that $f(0) = z_0$, defines a holomorphic motion $(v, z) \to f_v^Z(z) : D \times Z_0 \to \mathbb{C}$ such that $f_v^Z(Z_0) = Z_v$ and $g_v \circ f_v^Z = f_v^Z \circ (g_0|Z_0)$.

By the lambda lemma [MSS, Section 1.2], f_v^Z extends to a unique holomorphic motion $\widetilde{f}_v^Z$ of the closure of Z_0, i.e., $\widetilde{f}_v^Z : \overline{Z}_0 \to \overline{Z}_v$, $|v| < 1$.

By the extension theorem (Theorem 0.2) the domains $V_v := U_v \setminus \overline{Z}_v$, $|v| < 1$, move holomorphically. Hence $g_v|V_v : V_v \to V_v$ are covering maps for

$|v| < 1$ (note that $g_v^{-1}(V_v) = (V_v)$). Applying Theorem 1.2 to the family of domains V_v, $v \in D$, and the cyclic semigroup $\{g^n : n = 1, 2, \ldots\}$ we see that there is a holomorphic motion $\psi_v : V_0 \to V_v$ commuting with g. It is easy to see that letting

$$F_v(z) = \begin{cases} \widetilde{f}_v(z) & \text{for } z \in \overline{z}_0 \text{ and} \\ \psi_v(z) & \text{for } z \in V_0 = U_0 \setminus \overline{z}_0,\ v \in D, \end{cases} \tag{5.2}$$

we obtain a holomorphic motion

$$F_v : U_0 \to U_v, \quad v \in D, \quad F_v(U_0) = U_v,$$

which commutes with the action of g. ∎

We now consider the semigroup case. First we define the analogue of the condition of permanence of critical orbits relations for the semigroup case.

DEFINITION 5.1. Let U_v, $v \in D$, be a holomorphic family of domains and $G = \{g\}$ a semigroup of holomorphic families of maps $g_v : U_v \to U_v$, $v \in D$. Assume that every $g \in G$ satisfies Definition 1.4 and let $\mathcal{C}^g$ denote the set of holomorphic functions $c_j : D \to \mathbb{C}$, $j = 1, 2, \ldots$, such that the critical variety of g is the union of the graphs of c_j's. Define $\mathcal{C} = \bigcup_{g \in G} \mathcal{C}^g$. We will say that the semigroup G satisfies the condition of *permanence of critical orbit relations* if the set

$$\mathcal{F} = \mathcal{C} \cup \bigcup_{n=1}^{\infty} \{g_{2n+1} g_{2n}^{-1} g_{2n-1} \ldots g_2^{-1} g_1(\mathcal{C}) : g_j \in G\} \tag{5.3}$$

consists of functions whose graphs are mutually disjoint (except when identical).

The above definition is much less satisfactory than Definition 1.4 because even in the case of a cyclic semigroup $\{g^n : n = 1, 2, \ldots\}$ it assumes most of what had to be demonstrated within the proof of Theorem 1.7.

PROPOSITION 5.2. *Let G and U_v, $v \in D$, satisfy the notation and conditions of Definition 5.1. Assume in addition that for every $g \in G$, all the endomorphisms $g_v : U_v \to U_v$, $v \in D$, have the path lifting property. Then there is a holomorphic motion $F_v : U_0 \to U_v$, $v \in D$, commuting with the action of G.*

Proof (sketch). We proceed as at the end of the last proof. Let $Z_v = \{a(v) : a \in \mathcal{F}\}$. It is a direct consequence of the (unfortunately very strong) Definition 5.1 that the set $Z = \bigcup_{v \in D} \{v\} \times Z_v$ is the union of mutually disjoint graphs of analytic functions. It is clear that Z_v is forward and backward invariant for $g_v \in G$. Similarly to the proof of Theorem 1.7, there exist holomorphic motions $f_v^Z : Z_0 \to Z_v$ and $\overline{f}_v^Z : \overline{Z}_0 \to \overline{Z}_v$, the sets $V_v = U_v \setminus \overline{Z}_v$ are forward and backward invariant with respect to $g \in G$, and all the $g_v | V_v : V_v \to V_v$, $g \in G$, are covering maps. Applying the theorem to $g | \bigcup_v \{v\} \times V_v$ we obtain a holomorphic motion $\psi_v : V_0 \to V_v$ commuting with G. Defining $F_v : U_0 \to U_v$ by (5.2) we obtain a holomorphic motion with all the required properties. ∎

Proof of Corollary 1.8. Let $\phi_v : U_0 \to U_v$, $v \in D$, denote the isotopy conjugating the action of G on U_0 and U_v, that is, for every $g \in G$ and $v \in D$,

$$g_v = \phi_v \circ g_0 \circ \phi_v^{-1}.$$

Let, for $v \in D$,

$$Z_v = \bigcup \Big\{ (g_v^{2n+1})(g_v^{2n})^{-1}(g_v^{2n-1}) \ \ldots (g_v^2)^{-1} g_v^1(z) : z \in \bigcup_{g \in G} C_g, g^1, \ldots, g^{2n+1} \in G \Big\}.$$

It is clear that Z_v is backward and forward invariant with respect to g_v, $g \in G$. Furthermore, $Z_v = \phi_v(Z_0)$, that is, the set $Z = \bigcup_{v \in D} \{v\} \times Z_v$ is the union of the graphs of continuous functions of the form

$$v \to \phi_v(z^*) : D \to \mathbb{C}, \qquad z^* \in Z_0. \tag{5.4}$$

If $z^* \in (g_0^{2n+1})(g_0^{2n})^{-1} \ldots g_0^{-1} g_0^1(z)$ for fixed $z \in \bigcup C_0^g$ and fixed $g^1, \ldots, g^{2n+1}$, then the graph (5.4) is contained in the variety

$$\bigcup_{|v|<1} \{v\} \times (g_v^{2n+1})(g_v^{2n})^{-1} \ldots (g_v^2)^{-1}(g_v^1)(z)$$

and so $v \to \phi_v(z^*)$ is an analytic function. Let $f_v = \phi_v | Z_0$. Then $v \to f_v : Z_0 \to Z_v$ is a holomorphic motion and the proof can be finished in the same way as the proof of Proposition 5.2. ∎

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Received July 4, 1994 (3304)
Revised version January 19, 1995 and November 22, 1995

Note added in proof (April 1996). For the notion of the permanence of critical orbit relations and its application to rational functions, see also the new preprint by C. McMullen and D. Sullivan, *Quasiconformal homeomorphism and dynamics III: Teichmüller space of the conformal dynamical system*, preprint, October 1995, in particular Section 7.

On asymptotic density and uniformly distributed sequences

by

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Abstract. Assuming Martin's axiom we show that if X is a dyadic space of weight at most continuum then every Radon measure on X admits a uniformly distributed sequence. This answers a problem posed by Mercourakis [10]. Our proof is based on an auxiliary result concerning finitely additive measures on ω and asymptotic density.

1. Introduction. Let K be a compact Hausdorff space. We denote by $P(K)$ the set of all probability Radon measures on K. If $x \in K$ then $\delta_x \in P(K)$ denotes the usual Dirac measure.

Given $\lambda \in P(K)$, a sequence $(x_n) \subseteq K$ is said to be λ-*uniformly distributed* (λ-u.d.) if

$$\frac{1}{n}\sum_{i\le n}\delta_{x_i} \to \lambda$$

in the weak* topology, that is, for every real-valued continuous function f defined on K one has

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} f(x_i) = \int_K f\,d\lambda.$$

The theory of uniformly distributed sequences originated in the classical notion of a sequence in the unit interval which is uniformly distributed (with respect to the Lebesgue measure). For many years the case of a compact metric space K was mainly studied. The uniform distribution with respect to the Haar measure of a given compact group also attracted much attention. The book by Kuipers and Neiderreiter [7] surveys these topics.

Recall that every Radon measure defined on a compact metric space has a uniformly distributed sequence. On the other hand, Losert [8] noted that no nonatomic measure on $\beta\omega$ admits such a sequence (since every

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1991 *Mathematics Subject Classification*: Primary 11B05, 28C15; Secondary 03E50.
Partially supported by KBN grant 2 P 301 043 07.