STUDIA MATHEMATICA 119 (1) (1996)

Local Toeplitz operators based on wavelets: phase space patterns for rough wavelets

by

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Abstract. We consider two standard group representations: one acting on functions by translations and dilations, the other by translations and modulations, and we study local Toeplitz operators based on them. Local Toeplitz operators are the averages of projection-valued functions $g \mapsto P_{g,\phi}$, where for a fixed function ϕ, $P_{g,\phi}$ denotes the one-dimensional orthogonal projection on the function $U_g\phi$, U is a group representation and g is an element of the group. They are defined as integrals $\int_W P_{g,\phi}\,dg$, where W is an open, relatively compact subset of a group. Our main result is a characterization of function spaces corresponding to local Toeplitz operators with pth power summable eigenvalues, $0 < p \le \infty$.

0. Introduction. Let U be a representation of a group G acting on functions defined on $\mathbb{R}^d$. Let b be a function defined on G and let ϕ be a function defined on $\mathbb{R}^d$. We define the operator $T_{b,\phi}$ acting on functions on $\mathbb{R}^d$ by

$$T_{b,\phi}(f) = \int_G b(g)\langle f, U_g\phi\rangle U_g\phi\,dg, \tag{0.1}$$

where $\langle\,,\,\rangle$ is the inner product of $L^2(\mathbb{R}^d)$, and dg is a left invariant measure on G. The operator $T_{b,\phi}$ is called a *Toeplitz operator* based on the representation U.

It is an intriguing problem to understand the dependence of $T_{b,\phi}$ on its parameters b, ϕ. As it stands, the definition (0.1) is rather too general for a thorough study of the problem. We restrict attention to some specific representations U. In fact, in this paper we consider only two representations:

(i) the Schrödinger representation of the reduced Heisenberg group

$$\varrho(p,q,t)\phi(x) = e^{2\pi it}e^{-\pi ipq}e^{2\pi ipx}\phi(x-q), \tag{0.2}$$

where $p, q \in \mathbb{R}^d$, $0 \le t < 1$,

1991 *Mathematics Subject Classification*: 45C05, 81R30, 47B10, 46E30.

Key words and phrases: singular values, time-frequency localization.

The main part of this work has been done while the author was holding Lise Meitner fellowship of the Austrian Science Foundation. It has also been supported in part by KBN Grant 2P 301 051 07.

(ii) the natural representation of the "$ax+b$" group

$$\tau(u,s)\psi(x)=s^{-d/2}\psi\Big(\frac{x-u}{s}\Big), \tag{0.3}$$

where $u\in\mathbb{R}^d$, $s>0$.

In both cases (i) and (ii) if $b=1$ and ϕ, ψ satisfy some regularity conditions then the formula (0.1) becomes a resolution of the identity, i.e. $T_{b,\phi}=\mathrm{id}$. The condition on ϕ that is necessary and sufficient for this resolution of the identity to hold is

$$\|\phi\|_{L^2}=1$$

in the case of (i), and

$$\|\phi\|_{L^2}<\infty,\qquad \int_0^\infty|\hat\phi(s\xi)|^2\frac{d\xi}{\xi}=1\quad\text{for almost every }\xi$$

in the case (ii) ($\hat\phi$ denotes the Fourier transform of ϕ).

As long as b is a nonnegative, compactly supported function, larger than some $\varepsilon>0$, on some open set W, the spectral properties of $T_{b,\phi}$ do not essentially depend on b. Operators $T_{b,\phi}$ with b's having the above properties are called *local Toeplitz operators*. We drop the symbol b and write T_ϕ for local Toeplitz operators based on ϱ and $\mathcal{T}_\psi$ for local Toeplitz operators based on τ.

In the paper [R3] Rochberg considered the operator $\mathcal{T}_\psi$ with ψ the Haar function. He assumed that b is a bounded, nonnegative, compactly supported function. He showed that the nth eigenvalue of $\mathcal{T}_\psi$ is at least cn^{-2} and that $\mathcal{T}_\psi$ belongs to the Schatten ideal S^p for all $p>1/2$. Two-sided eigenvalue estimates for $\mathcal{T}_\psi$ were obtained later by the present author in [N2]. These improve the result of Rochberg, but do not allow the study of the general case. In the present paper we treat the general case. Our results describe the Schatten ideal norms (quasi-norms for $0<p<1$) of both $\mathcal{T}_\psi$ and T_ϕ in terms of some expressions involving Fourier transforms of ψ and ϕ.

Let us sketch the results. Let m be a smooth bump function defined on $\mathbb{R}^d$ which is concentrated around the point 0 and has suitably large support. Let

$$\phi^{k,n}=(m_k(m_n\phi)^\wedge)^\vee, \tag{0.4}$$

where $m_n(x)=m(x-n)$, k,n belong to the integral lattice in $\mathbb{R}^d$, and $^\wedge$, $^\vee$ denote the Fourier transform and the inverse Fourier transform, respectively. Our main result about T_ϕ shows that for $0<p\le\infty$ the following equivalence of norms holds:

$$\|T_\phi\|_{S^p}^p\cong\sum_{k,n}\|\phi^{k,n}\|_{L^2}^{2p}. \tag{0.5}$$

The constants of equivalence do not depend on ϕ, but do depend on m and p.

The norm that appears on the right hand side of (0.5) is the well-known modulation space norm. It has been extensively studied by Feichtinger and Gröchenig and its behaviour is well understood (see [FG]).

We sketch a one-dimensional result for $\mathcal{T}_\psi$. The higher-dimensional analogue is stated in the comments that follow Corollary 5.5 (it involves more complicated formulas and requires some extra assumptions on ψ).

Let m be a smooth bump function defined on $(0,\infty)$ which is concentrated around 1 and has suitably large support. Let us extend it to the whole real line by putting 0 on the negative axis. We define

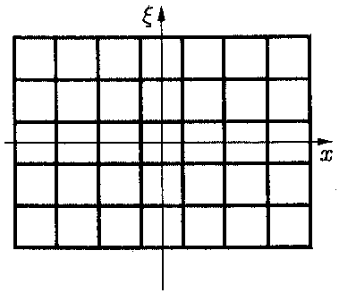
$$\begin{aligned}\psi_{k,l}^+&=(\zeta^{-1/2}\mathcal{F}^{-1}(\mathrm{m}_l\mathcal{F}(\xi^{1/2}\mathrm{m}_k(\xi)\hat\psi(\xi)))(\zeta))^\vee,\\ \psi_{k,l}^-&=(\zeta^{-1/2}\mathcal{F}^{-1}(\mathrm{m}_l\mathcal{F}(\xi^{1/2}\mathrm{m}_k(\xi)\hat\psi(-\xi)))(-\zeta))^\vee,\end{aligned} \tag{0.6}$$

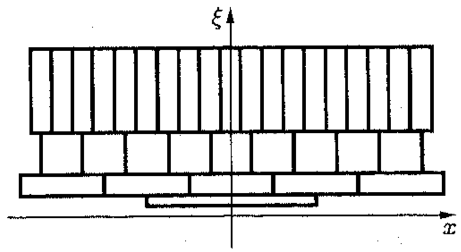
where $\mathrm{m}_k(\xi)=\mathrm{m}(\xi/e^k)$, k,l are integers, and $\mathcal{F}$ denotes the Mellin transform of the multiplicative half-line, i.e. the Fourier transform of the real line transferred to the positive half-line by the exponential map. Our main result about $\mathcal{T}_\psi$ asserts the following equivalence of norms, valid for all $0<p\le\infty$:

$$\begin{aligned}\|\mathcal{T}_\psi\|_{S^p}^p\cong{}&\sum_l\Big(\sum_{k\le0}\|\psi_{k,l}^+\|_{L^2}^2\Big)^p+\sum_l\Big(\sum_{k\le0}\|\psi_{k,l}^-\|_{L^2}^2\Big)^p\\ &+\sum_{k>0,\,l}\Big(\frac{1}{[e^k]}\sum_{r=0}^{[e^k]}\|\psi_{k,l+r}^+\|_{L^2}^2\Big)^p+\sum_{k>0,\,l}\Big(\frac{1}{[e^k]}\sum_{r=0}^{[e^k]}\|\psi_{k,l+r}^-\|_{L^2}^2\Big)^p.\end{aligned} \tag{0.7}$$

The norm on the right hand side of (0.7) seems to be new. Some examples are provided in Section 5.

Our results are illustrated by the phase space pictures presented below. The first one exhibits the building blocks $\phi^{k,n}$ corresponding to T_ϕ. The second one exhibits the blocks $\psi_{k,l}^+$, $\psi_{k,l}^-$ corresponding to $\mathcal{T}_\psi$. In the first case each block $\phi^{k,n}$ contributes to the l^p norm individually, in the second case $\psi_{k,l}^+$, $\psi_{k,l}^-$ have to be grouped properly before taking the l^p norm (as the formula (0.7) shows).





Instead of T_ϕ, $\mathcal{T}_\psi$ we consider the operators

$$R_{f,\phi} : L^2(\mathbb{R}^d, dy) \to L^2(\mathbb{R}^{2d}, dp dq),$$
$$\mathcal{R}_{\mathrm{f},\psi} : L^2(\mathbb{R}^d, dy) \to L^2\left(\mathbb{R}^d \times (0,\infty), du\frac{ds}{s^{d+1}}\right),$$

defined by the kernels

$$f(p,q)\overline{\varrho(p,q,0)\phi(y)}, \tag{0.8}$$
$$\mathrm{f}(u,s)\overline{\tau(u,s)\psi(y)}. \tag{0.9}$$

The operators $R_{f,\phi}$, $\mathcal{R}_{\mathrm{f},\psi}$ are the square roots of T_ϕ, $\mathcal{T}_\psi$ (provided $b = |f|^2$, $\mathrm{b} = |\mathrm{f}|^2$, where b is the symbol of T_ϕ, and b the symbol of $\mathcal{T}_\psi$). For this reason statements about singular values of $R_{f,\phi}$, $\mathcal{R}_{\mathrm{f},\psi}$ immediately translate to statements about eigenvalues of T_ϕ, $\mathcal{T}_\psi$ and vice versa.

Descriptions of the Schatten ideal norms of $R_{f,\phi}$ are contained in Theorem 4.3 and Corollary 4.4. The one-dimensional case of $\mathcal{R}_{\mathrm{f},\psi}$ is covered in Theorem 5.4 and Corollary 5.5, while the higher-dimensional case is discussed in the comments following Corollary 5.5.

To avoid technical problems in dealing with integral kernels we assume that the functions ϕ, ψ are of Schwartz class and we prove norm (quasi-norm) equivalence (denoted by $\cong$). The constant p that appears in our results varies in the range $(0,\infty)$ and we do not mention it separately in the text. As is traditional, the letter c stands for different constants in different places.

1. Historical background. We are going to describe three standard approaches to time-frequency localization. The first one is due to Landau, Slepian and Pollak. It consists in analyzing the operator

$$P_\Omega Q_T P_\Omega, \tag{1.1}$$

where Q_T and P_Ω are the orthogonal projections on $L^2(\mathbb{R})$ defined by the formulas

$$(Q_T f)(x) = \begin{cases} f(x) & \text{for } |x| < T, \\ 0 & \text{for } |x| > T, \end{cases}$$
$$(P_\Omega f)^\wedge(\xi) = \begin{cases} \hat{f}(\xi) & \text{for } |\xi| < \Omega, \\ 0 & \text{for } |\xi| > \Omega. \end{cases}$$

The eigenfunctions $h_0, h_1, \ldots$ and eigenvalues $\lambda_0, \lambda_1, \ldots$ of (1.1) have the following interpretation. The first eigenfunction h_0 is the most concentrated in the interval $[-T, T]$ among the bandlimited functions with the spectrum contained in $[-\Omega, \Omega]$. In general, h_n is the most concentrated bandlimited function that is orthogonal to $h_0, \ldots, h_{n-1}$. One of the important tasks was to approximate the dimension of the range of $P_\Omega Q_T P_\Omega$. It was accomplished by Landau, Slepian and Pollak and it was later applied by Landau to solve the problems of sampling and interpolation for bandlimited functions. We refer to the survey paper by Slepian [Sl] for more information.

The next two approaches are related to coherent states (both are described in [D]). The first one is based on the Schrödinger representation of the Heisenberg group (0.2) and the study of the operator

$$f \mapsto \int_W \langle f, \phi_{p,q}\rangle \phi_{p,q}\, dp\, dq, \tag{1.2}$$

where $\phi_{p,q}(x) = e^{2\pi i p x}\phi(x-q)$, and W is a compact subset of $\mathbb{R}^{2d}$. The other is related to the natural action of the "$ax+b$" group (0.3) on $L^2(\mathbb{R}^d)$ and the operator

$$f \mapsto \int_{\mathrm{W}} \langle f, \psi_{u,s}\rangle \psi_{u,s}\, du\, \frac{ds}{s^{d+1}}, \tag{1.3}$$

where $\psi_{u,s}(x) = s^{-d/2}\psi((x-u)/s)$, and W is a compact subset of $\mathbb{R}^n \times (0,\infty)$. Choosing $\phi(x) = 2^{1/4}e^{-\pi x^2}$ and ψ with the Fourier transform $\hat{\psi}(\xi) = \chi_{(0,\infty)}(\xi)\xi^{1/2}e^{-2\pi\xi}$ and the sets W, W to be balls in the Euclidean and hyperbolic metrics, Daubechies, Paul and Seip obtained formulas for the eigenfunctions and the eigenvalues of (1.2) and (1.3). Their study followed the direction of Landau–Slepian–Pollak and is surveyed in [D]. The result of Rochberg in [R3] showed that the behavior of the eigenvalues of (1.3) is different if the Bergman wavelet is replaced by the Haar function and that the rate of decay of the eigenvalues essentially depends on smoothness properties of ψ.

Operators (1.2) and (1.3) are particular cases of Toeplitz operators based on the representations ϱ and τ. Toeplitz operators based on τ are called *Calderón–Toeplitz operators*, their study was started by Rochberg in [R1].

The problem of finding Schatten ideal membership criteria for operators depending on function parameters has a long history. One of the early results in that direction is a well-known theorem by Peller and Semmes about Hankel operators acting on the Hardy space H^2. This theorem asserts that the Schatten ideal norm of the Hankel operator is equivalent to the Besov space norm of its symbol. The higher-dimensional analogue of this result was obtained by Janson and Wolf. Later on, Arazy, Axler, Berger, Coburn, Janson, Peetre, Rochberg, Zhu and many others investigated Toeplitz and Hankel operators on Fock and Bergman spaces. Among the results obtained were the Schatten ideal membership criteria. Criteria of that sort also occurred in the works of Janson, Peetre and Peng on paracommutators and in several papers by Birman and Solomyak on integral operators. Clearly, this is not the full list of papers about the Schatten ideal membership criteria.

Our paper may be thought of as another one in this series. On the other hand, however, it does not follow standard directions and it addresses the issues that were not investigated before.

We divide questions about Toeplitz operators based on ϱ and τ into two groups:

(i) global behavior, dependence on symbol functions b, b for wavelets ϕ, ψ fixed,

(ii) local behavior, dependence on wavelets ϕ, ψ for fixed compactly supported, continuous, nonnegative symbol.

The results in the direction (i) were obtained in [R2] and [N1].

Questions of the type (ii), specialized to the context of Schatten ideals, are studied in this paper. They are close in spirit to the work of Peng, Rochberg and Wu in [PRW] on the cut-off phenomenon. In their case, however, symbols are analytic and the cut-off has different character.

2. Some heuristic arguments. The arguments that we present below stress the basic features of $R_{f,\phi}$ and $\mathcal{R}_{f,\psi}$ in the case when ϕ and ψ are rough functions which are not well localized in the phase space.

Let h be a function of Schwartz class and let $h_{m,n} = e^{2\pi i\omega_0 mx}h(x-t_0 n)$, with ω_0, t_0 fixed positive numbers, $m,n \in \mathbb{Z}$. Although the system $\{h_{m,n}\}$ cannot be an orthonormal basis of $L^2(\mathbb{R})$, for many choices of h it is a good substitute (if $\omega_0 t_0 < 1$, then one may choose h in such a way that $\{h_{m,n}\}$ constitute a tight frame; see [D], p. 84).

We express the kernel of $R_{f,\phi}R^*_{f,\phi}$ as follows:

$$f(p_1,q_1)\overline{f(p_2,q_2)}\sum_{m,n}\langle \phi_{p_2,q_2}, h_{m,n}\rangle\langle h_{m,n}, \phi_{p_1,q_1}\rangle. \tag{2.1}$$

If p, q are close to zero, then $e^{2\pi ipq}$ is close to 1, $h(x-q)$ is close to $h(x)$ and $\hat{h}(\xi-p)$ is close to $\hat{h}(\xi)$ (this is because h and $\hat{h}$ are both smooth). Making use of these observations we derive the following

$$\begin{aligned}
(2.2)\quad \langle \phi_{p,q}, h_{m,n}\rangle
&= \int e^{2\pi ipx}\phi(x-q)e^{-2\pi i\omega_0 mx}\overline{h(x-t_0n)}\,dx\\
&\approx e^{-2\pi i\omega_0 mq}\int e^{2\pi ipx}\phi(x)e^{-2\pi i\omega_0 mx}\overline{h(x-t_0n)}\,dx\\
&= e^{-2\pi i\omega_0 mq}\int \hat{\phi}(\xi-p)e^{2\pi it_0n\xi}e^{-2\pi i\omega_0 mt_0n}\overline{\hat{h}(\xi-\omega_0 m)}\,d\xi\\
&\approx e^{-2\pi i\omega_0 mq}e^{2\pi it_0np}\int \hat{\phi}(\xi)e^{2\pi it_0n\xi}e^{-2\pi\omega_0 mt_0n}\overline{\hat{h}(\xi-\omega_0 m)}\,d\xi\\
&= e^{-2\pi i\omega_0 mq}e^{2\pi it_0np}\langle\phi, h_{m,n}\rangle.
\end{aligned}$$

The above argument makes us think that for f close to the characteristic function of the unit square $[-1/(2\omega_0), 1/(2\omega_0)] \times [-1/(2t_0), 1/(2t_0)]$ the expression (2.1) is close to

$$\sum_{m,n}|\langle\phi, h_{m,n}\rangle|^2 H_{m,n}(p_1,q_1)\overline{H_{m,n}(p_2,q_2)}, \tag{2.3}$$

where $H_{m,n}(p,q) = f(p,q)e^{2\pi i\omega_0 mq}e^{-2\pi it_0np}$.

We conclude that for rough ϕ the numbers $|\langle\phi, h_{m,n}\rangle|^2$ should be closely related to the eigenvalues of $T_\phi = R^*_{f,\phi}R_{f,\phi}$ and that the basis diagonalizing $R_{f,\phi}R^*_{f,\phi}$ should share some common features with the two-dimensional trigonometric system.

The function $\phi^{k,n}$ (defined in (0.4)) is localized around the phase space point (n,k). Moreover, $\phi \approx \sum_{k,n}\phi^{k,n}$. We assume that $f(p,q) = f_1(p)f_2(q)$, where f_1, f_2 are nonnegative Schwartz class functions. Let $b_1(p) = (f_1(p))^2$, $b_2(q) = (f_2(q))^2$. We have

$$\begin{aligned}
(2.4)\quad R^*_{f,\phi}R_{f,\phi} &= \iint b_1(p)b_2(q)e^{2\pi ipx}\phi(x-q)e^{-2\pi ipy}\overline{\phi(y-q)}\,dp\,dq\\
&= \hat{b}_1(y-x)\int b_2(q)\phi(x-q)\overline{\phi(y-q)}\,dq\\
&\approx \hat{b}_1(y-x)\sum_{n_1,n_2,k_1,k_2}\int b_2(q)\phi^{k_1,n_1}(x-q)\overline{\phi^{k_2,n_2}(y-q)}\,dq\\
&\approx \sum_{n,k_1,k_2}\int b_2(q)\phi^{k_1,n}(x-q)\overline{\phi^{k_2,n}(y-q)}\,dq.
\end{aligned}$$

In the last step we made use of the fact that $\hat{b}_1(y-x)$ is localized around the diagonal $y = x$. Let F denote the Fourier transform. For fixed n we have

$$\begin{aligned}
(2.5)\quad &\sum_{k_1,k_2}\int b_2(q)\phi^{k_1,n}(x-q)\overline{\phi^{k_2,n}(y-q)}\,dq\\
&= \sum_{k_1,k_2}F^*\int b_2(q)e^{-2\pi iq\xi}(\phi^{k_1,n})^\wedge(\xi)e^{2\pi iq\eta}\overline{(\phi^{k_2,n})^\wedge(\eta)}\,dq\,F\\
&= F^*\sum_{k_1,k_2}\hat{b}_2(\xi-\eta)(\phi^{k_1,n})^\wedge(\xi)\overline{(\phi^{k_2,n})^\wedge(\eta)}\,F\\
&\approx F^*\sum_k(\phi^{k,n})^\wedge(\xi)\overline{(\phi^{k,n})^\wedge(\eta)}\,F = \sum_k\phi^{k,n}\otimes\phi^{k,n}.
\end{aligned}$$

We use the fact that $\hat{b}_2(\xi-\eta)$ is localized around the diagonal $\xi = \eta$. Combining (2.4) and (2.5) we obtain

$$R^*_{f,\phi}R_{f,\phi} \approx \sum_k \phi^{k,n}\otimes\phi^{k,n}.$$

The above arguments suggest the equivalence of norms stated in (0.5).

The pattern for $\mathcal{R}_{\mathrm{f},\psi}$ is more complicated. The blocks in the phase space that correspond to different eigenvalues may overlap. Moreover, one needs to use mixed norms, not only l^p norms, to describe the S^p criteria.

We only discuss a special case, namely when $\hat{\psi}$ is compactly supported, and we interpret the terms in (0.7) that correspond to $k \leq 0$. Let $\mathrm{f}(u,s) = \mathrm{f}_1(u)\mathrm{f}_2(s)$, where f_1 is a Schwartz class function and f_2 is a smooth, nonnegative, compactly supported function defined on $(0,\infty)$. We compose $\mathcal{R}_{\mathrm{f},\psi}$ with F^{-1} on the right and with $F \otimes \mathrm{id}$ on the left. This yields the integral kernel

$$\hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)s^{1/2}\overline{\hat{\psi}(s\eta)}. \tag{2.6}$$

Next we take a composition of (2.6) with its adjoint. We obtain

$$\hat{\mathrm{f}}_1^* * \hat{\mathrm{f}}_1(\xi-\eta)\int_0^\infty (f_2(s))^2\hat{\psi}(s\xi)\overline{\hat{\psi}(s\eta)}\,\frac{ds}{s}. \tag{2.7}$$

If $|\hat{\mathrm{f}}_1^* * \hat{\mathrm{f}}_1(\xi-\eta)| \geq \varepsilon > 0$ on the support of

$$\int_0^\infty (\mathrm{f}_2(s))^2\hat{\psi}(s\xi)\overline{\hat{\psi}(s\eta)}\,\frac{ds}{s}, \tag{2.8}$$

then the S^p norms of (2.7) and (2.8) are comparable. On the other hand,

$$\mathrm{f}_2(s)\overline{\psi(s\eta)} \tag{2.9}$$

is the kernel of a square root of (2.8). But (2.9) is just a convolution-product operator on the multiplicative line.

We conclude that in the case of compactly supported $\hat{\psi}$ we may forget about averaging over translations. Only dilations influence the picture, and $\mathcal{R}_{\mathrm{f},\psi}$ reduces to a convolution-product operator on the multiplicative half-line. The terms in (0.7) with $k \leq 0$ come from such a convolution-product operator.

3. Definitions and preliminary results. In this section we present basic definitions and several facts concerning the operators $R_{f,\psi}$, $\mathcal{R}_{\mathrm{f},\psi}$ (defined in (0.8), (0.9)), Schur multipliers and eigenvalue estimates for certain types of matrices.

We start by defining some partitions of unity and Hilbert spaces bases related to them.

Let $R_0 = [-1/2, 1/2]$, $R_1 = [0,1]$. For $n \in \mathbb{Z}^d$ and $\tau = (i_1,\ldots,i_d)$, $i_j = 0,1$, we define cubes

$$Q_n^\tau = n + R_{i_1} \times \ldots \times R_{i_d}.$$

We take two nonnegative C^∞ functions h_0, h_1 satisfying

$$\operatorname{supp} h_0 \subset (-1/2,1/2), \quad \operatorname{supp} h_1 \subset (0,1), \quad \sum_{n\in\mathbb{Z}}(h_0(x-n)+h_1(x-n)) \equiv 1.$$

We define

$$m_n^\tau(x) = \prod_{j=1}^d h_{i_j}(x_j - n_j) \quad \text{and} \quad m^\tau(x) = \sum_{n\in\mathbb{Z}^d} m_n^\tau(x).$$

It is clear that

$$\operatorname{supp} m_n^\tau \subset Q_n^\tau, \quad \operatorname{dist}(\operatorname{supp} m_n^\tau, \partial Q_n^\tau) \geq \delta > 0, \quad \sum_\tau m^\tau \equiv 1.$$

Each space $L^2(Q_n^\tau, dx)$ is equipped with the orthonormal trigonometric basis

$$e_l(x) = e^{2\pi i l x}, \quad l \in \mathbb{Z}^d.$$

We use the standard symbols $^\wedge$, $^\vee$, * to denote the Fourier transform, the inverse Fourier transform and the involution in $L^1(\mathbb{R}^d)$, i.e.

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\xi}\,dx, \quad \check{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{2\pi i x\xi}\,dx, \quad f^*(x) = \overline{f(-x)}.$$

We also need multiplicative, one-dimensional versions of Q_n^τ, m_n^τ, m^τ. We assume that $d=1$ and for $s>0$ we define

$$\mathrm{m}_k^\tau(s) = m_k^\tau(\log s), \quad \mathrm{m}^\tau(s) = m^\tau(\log s)$$

and

$$s \in \mathrm{Q}_k^\tau \quad \text{if and only if} \quad \log s \in Q_k^\tau.$$

In each space $L^2(\mathrm{Q}_k^\tau, ds)$ we take the orthonormal basis

$$\mathrm{e}_l(s) = s^{-1/2}s^{2\pi i l}, \quad l \in \mathbb{Z}.$$

By $\mathcal{F}f$ we denote the multiplicative Fourier transform of f, i.e.

$$\mathcal{F}f(s) = (f\circ\exp)^\wedge(\log s),$$

and

$$\widetilde{f}(l) = \mathcal{F}f(e^l) = \int_0^\infty f(t)t^{-2\pi i l}\,\frac{dt}{t}.$$

We recall that the *Schatten ideal* S^p consists of compact operators with pth power summable singular values. By S^∞ we denote the ideal of compact operators.

Functions f, f play the roles of the symbols of $R_{f,\phi}$, $\mathcal{R}_{\mathrm{f},\psi}$. Our next two propositions show that it is possible to switch them conveniently, namely we may replace $f(p,q)$ by $f_1(p)f_2(q)$ and $\mathrm{f}(u,s)$ by $\mathrm{f}_1(u)\mathrm{f}_2(s)$.

PROPOSITION 3.1. *Let f be a continuous, compactly supported function defined on $\mathbb{R}^d$ and let f_1, f_2 be Schwartz class functions defined on $\mathbb{R}^d$. Then*

$$\|R_{f,\phi}\|_{S^p} \cong \|R_{f_1\otimes f_2,\phi}\|_{S^p}.$$

The constants of equivalence do not depend on ϕ.

Proof. Let D be a ball in $\mathbb{R}^{2d}$ such that

$$|f(p,q)|^2 \ge \varepsilon > 0 \quad \text{for all } (p,q) \in D. \tag{3.1}$$

There is a constant $M > 0$ and a sequence $\{\alpha_{k,l}\}_{k,l\in M\mathbb{Z}^d}$, qth power summable for every $q > 0$, such that

$$|f_1(p)|^2|f_2(q)|^2 \le \sum_{k,l\in M\mathbb{Z}^d} \alpha_{k,l}\chi_{(k,l)+D}. \tag{3.2}$$

We observe that

$$\begin{aligned} &\langle R^*_{f_1\otimes f_2,\phi}R_{f_1\otimes f_2,\phi}h, h\rangle \\ &= \iint |f_1(p)|^2|f_2(q)|^2|\langle h, \phi_{p,q}\rangle|^2\,dp\,dq \\ &\le \sum_{k,l}\alpha_{k,l}\iint \chi_{(k,l)+D}(p,q)|\langle h, \phi_{p,q}\rangle|^2\,dp\,dq \\ &\le c\sum_{k,l}\alpha_{k,l}\iint |f(p-k,q-l)|^2|\langle h, \phi_{p,q}\rangle|^2\,dp\,dq \\ &\le c\sum_{k,l}\alpha_{k,l}\iint |f(p,q)|^2|\langle h_{-k,-l}, \phi_{p,q}\rangle|^2\,dp\,dq \\ &= c\sum_{k,l}\alpha_{k,l}\langle R^*_{f,\phi}R_{f,\phi}h_{-k,-l}, h_{-k,-l}\rangle, \end{aligned} \tag{3.3}$$

where $h \in L^2(\mathbb{R}^d)$. The relation (3.3) means

$$R^*_{f_1\otimes f_2,\phi}R_{f_1\otimes f_2,\phi} \le c\sum_{k,l}\alpha_{k,l}\varrho(k,l)R^*_{f,\phi}R_{f,\phi}\varrho(k,l)^*, \tag{3.4}$$

but clearly (3.4) implies

$$\|R_{f_1\otimes f_2,\phi}\|_{S^p} \le c\|R_{f,\phi}\|_{S^p}.$$

The reverse inequality follows in a similar manner.

PROPOSITION 3.2. *Let f be a continuous, compactly supported function defined on $\mathbb{R}^d \times (0,\infty)$, let f_1 be a Schwartz class function defined on $\mathbb{R}^d$ and let f_2 be a compactly supported, smooth function defined on $(0,\infty)$. Then*

$$\|\mathcal{R}_{\mathrm{f},\psi}\|_{S^p} \cong \|\mathcal{R}_{\mathrm{f}_1\otimes \mathrm{f}_2,\psi}\|_{S^p}.$$

Proof. The proof follows the same pattern as in Proposition 3.1 and we omit it.

Schur multipliers, i.e. pointwise multipliers of operators given by integral kernels, occur to be a useful technical tool in the context of questions we look at. In the following part of our preliminaries we collect several propositions about them.

PROPOSITION 3.3. *Let $f(x)$ be a Schwartz class function that satisfies*

$$|f(x)| \ge \delta > 0 \quad \textit{for } x \in Q - Q,$$

where Q is some cube in $\mathbb{R}^d$. Let $K(x,y)$ be a kernel of an S^p class operator acting on $L^2(Q)$. Then

$$\|f(x-y)K(x,y)\|_{S^p} \cong \|K(x,y)\|_{S^p}.$$

Proof. We extend the functions $f(x)$ and $1/f(x)$ outside $Q-Q$ to some C_c^∞ functions and expand them in a Fourier series on a cube containing the supports.

To get the estimate from above for the kernel $K(x,y)f(x-y)$ we expand $f(x-y)$ as indicated and we sum up the terms making use of the decay of the Fourier coefficients.

The estimate from below follows similarly. First we write

$$K(x,y) = f(x-y)K(x,y)\frac{1}{f(x-y)}$$

and then we expand $1/f(x-y)$ in a Fourier series.

PROPOSITION 3.4. *Let $K(p,q,y)$ be a kernel of an S^p class operator acting from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$. Then*

$$\|m^\tau(y-q)K(p,q,y)\|_{S^p} \le c\|K(p,q,y)\|_{S^p}.$$

Proof. It is enough to expand the multiplier m^τ in a Fourier series on Q_0^τ. This expansion is valid over the whole $\mathbb{R}^d$.

PROPOSITION 3.5. *Let $K(\xi,s,\eta)$ be a kernel of an S^p class operator acting from $L^2((0,\infty),d\eta)$ into $L^2(\mathbb{R}\times(0,\infty),d\xi ds)$. Then*

$$\|\mathrm{m}^\tau(s\eta)K(\xi,s,\eta)\|_{S^p} \le c\|K(\xi,s,\eta)\|_{S^p}.$$

Proof. It is enough to expand $\mathrm{m}^\tau(s)$ in a series $\sum_n a_n^\tau s^{2\pi in}$, which is obtained from the Fourier series expansion of m^τ by evaluating at $\log s$.

PROPOSITION 3.6. *There is a constant $\sigma > 1$ such that for all $0 < \varepsilon \le 1$ and all nonnegative integers k,*

$$\begin{aligned} \{(\xi,\eta) \in \mathrm{Q}_k^\tau \times \mathrm{Q}_k^\tau : |\xi-\eta| \le \sigma^{-1}\varepsilon\} &\subset \{(\xi,\eta) \in \mathrm{Q}_k^\tau \times \mathrm{Q}_k^\tau : |e^k\log\xi/\eta| \le \varepsilon\} \\ &\subset \{(\xi,\eta) \in \mathrm{Q}_k^\tau \times \mathrm{Q}_k^\tau : |\xi-\eta| \le \sigma\varepsilon\}. \end{aligned}$$

Proof. We assume that $\mathrm{Q}_k^\tau = [e^k, e^{k+1}]$. The argument for $\mathrm{Q}_k^\tau = [e^{k-1/2}, e^{k+1/2}]$ is the same.

Let $\xi,\eta \in [e^k, e^{k+1}]$. It is an easy geometric observation that

(i) if $|e^k \log \xi/\eta| \le \varepsilon$, then $|\xi - \eta| \le D_0/2$,

where D_0 is the distance between the points

$$(e^{-\varepsilon e^{-k}} e^{k+1}, e^{k+1}), \quad (e^{k+1}, e^{-\varepsilon e^{-k}} e^{k+1}),$$

(ii) if $|\xi - \eta| \le D_1/2$, then $|e^k \log \xi/\eta| \le \varepsilon$,

where D_1 is the distance between the points

$$(e^k, e^{\varepsilon e^{-k}} e^k), \quad (e^{\varepsilon e^{-k}} e^k, e^k).$$

Clearly

$$D_0 = \sqrt{2}\, e^{k+1}(1 - e^{-\varepsilon e^{-k}}) \le \sqrt{2}\, e\varepsilon, \quad D_1 = \sqrt{2}\, e^k (e^{\varepsilon e^{-k}} - 1) \ge \sqrt{2}\, \varepsilon,$$

and one can easily choose a suitable σ.

PROPOSITION 3.7. *Let σ be any constant satisfying the statement in Proposition* 3.6.

(a) *Let H, f be C^∞ functions satisfying the following conditions*:

(i) $\operatorname{supp} H \subset [-\varepsilon, \varepsilon]$, $\operatorname{supp} f \subset [-\delta, \delta]$, $\delta \le \sigma^{-1}$,

(ii) $|f(\xi)| \ge \kappa > 0$ *if* $|\xi| \le \sigma\varepsilon$.

If $K(\xi, \eta)$ is a kernel of an operator acting on $L^2(\mathrm{Q}_k^\tau, d\eta)$, $k \ge 0$, then

$$\|H(e^k \log \xi/\eta) K(\xi, \eta)\|_{S^p} \le c \|f(\xi - \eta) K(\xi, \eta)\|_{S^p}.$$

The constant c depends neither on k nor on $K(\xi, \eta)$.

(b) *If H, f, K satisfy the same conditions as in* (a), *with* (i), (ii) *replaced by*

(i′) $\operatorname{supp} H \subset [-\varepsilon, \varepsilon]$, $\operatorname{supp} f \subset [-\delta, \delta]$, $\varepsilon \le 1$,

(ii′) $|H(x)| \ge \kappa > 0$ *if* $|x| \le \sigma\delta$,

then

$$\|f(\xi - \eta) K(\xi, \eta)\|_{S^p} \le c \|H(e^k \log \xi/\eta) K(\xi, \eta)\|_{S^p}.$$

The same comment about c as in (a) *applies.*

Proof. We only prove (a). The proof of (b) is similar.

We pick a C^∞ function $\widetilde{f}$ which satisfies

$$\widetilde{f}(\xi) = \begin{cases} 1/f(\xi) & \text{if } |\xi| \le \sigma\varepsilon, \\ 0 & \text{if } |\xi| \ge \delta. \end{cases}$$

By Proposition 3.6,

$$\{(\xi, \eta) \in \mathrm{Q}_k^\tau \times \mathrm{Q}_k^\tau : H(e^k \log \xi/\eta) \ne 0\} \subset \{(\xi, \eta) \in \mathrm{Q}_k^\tau \times \mathrm{Q}_k^\tau : |\xi - \eta| \le \sigma\varepsilon\},$$

thus

$$H(e^k \log \xi/\eta) K(\xi, \eta) = H(e^k \log \xi/\eta) \widetilde{f}(\xi - \eta) f(\xi - \eta) K(\xi, \eta). \tag{3.5}$$

We expand H in a Fourier series on the interval $[-\sigma\delta, \sigma\delta]$. This gives

$$H(e^k \log \xi/\eta) = \sum_m a_m e^{(\pi i e^k/(\sigma\delta)) m \log \xi/\eta}. \tag{3.6}$$

Since $\varepsilon < \sigma\delta$ the sequence a_m has fast decay.

We also expand $\widetilde{f}$ in a Fourier series on the interval $[-\delta, \delta]$. We obtain

$$\widetilde{f}(\xi - \eta) = \sum_n b_n e^{(\pi i/\delta) n (\xi - \eta)}. \tag{3.7}$$

Again the sequence b_n has fast decay. By Proposition 3.6,

$$\{(\xi, \eta) \in \mathrm{Q}_k^\tau \times \mathrm{Q}_k^\tau : |\xi - \eta| \le \delta\} \subset \{(\xi, \eta) \in \mathrm{Q}_k^\tau \times \mathrm{Q}_k^\tau : |e^k \log \xi/\eta| \le \sigma\delta\},$$

thus the Fourier series expansions are valid on a set where $f(\xi - \eta) \ne 0$. We combine (3.5)–(3.7) to get

$$\begin{aligned} &H(e^k \log \xi/\eta) K(\xi, \eta) \\ &\quad = \sum_m a_m e^{(\pi i e^k/(\sigma\delta)) m \log \xi/\eta} \sum_n b_n e^{(\pi i/\delta) n (\xi - \eta)} f(\xi - \eta) K(\xi, \eta). \end{aligned} \tag{3.8}$$

The sequences a_m and b_n are qth power summable with any $q > 0$ and our proposition follows easily from (3.8).

The next part of our preliminaries exhibits two classes of matrices for which one may explicitly characterize S^p membership conditions. By s_N we denote the Nth singular value of a given compact operator.

Let T be an operator acting on $l^2(\mathbb{Z}^2)$ defined by the matrix

$$a(l_2 - l_1) b(k_1, l_1) \overline{b(k_2, l_2)},$$

where (k_1, l_1) are column indices and (k_2, l_2) are those for rows. Let

$$c(l) = \sum_k |b(k, l)|^2,$$

and for $M, r \in \mathbb{Z}$,

$$c^{M,r}(l) = \sum_k |b(k, Ml + r)|^2.$$

The symbols c^*, $(c^{M,r})^*$ denote the nonincreasing rearrangements of the sequences c, $c^{M,l}$ respectively.

PROPOSITION 3.8. *If $a(l)$ is an absolutely summable, positive definite sequence and $a(0) > 0$, then*

(i′) *for some constant $C > 0$,*

$$s_N(T) \le C c^*(N),$$

(i″) *for some positive integer M and some constant $c > 0$,*

$$s_N(T) \ge c (c^{M,r})^*(N) \quad \textit{for all } r \in \mathbb{Z},$$

(ii)

$$\|T\|_{S^p} \cong \Big(\sum_l \Big(\sum_k |b(k,l)|^2\Big)^p\Big)^{1/p}.$$

Proof. (i′) Let m_i be the sequence corresponding to the nonincreasing rearrangement of $c(l)$, i.e.

$$c(m_i) = c^*(i).$$

Let

$$w_i(k,l) = \begin{cases} b(k,l) & \text{for } l = m_i, \\ 0 & \text{for } l \neq m_i, \end{cases}$$

and let $W_n = \operatorname{span}\{w_0, \ldots, w_{N-1}\}$.

We will show that for $w \in W_N^{\perp}$, $\|w\| = 1$,

$$\langle Tw, w\rangle \le Cc^*(N),$$

and this is enough to prove (i′). We have

$$\begin{aligned}\langle Tw, w\rangle &= \sum_{l_1,l_2} a(l_2 - l_1) \sum_{k_1} b(k_1,l_1)\overline{w(k_1,l_1)} \sum_{k_2} \overline{b(k_2,l_2)} w(k_2,l_2) \\ &\le C \sum_{l \notin V_N} \Big|\sum_k b(k,l)\overline{w(k,l)}\Big|^2 \\ &\le C \sum_{l \notin V_N} \sum_k |b(k,l)|^2 \sum_k |w(k,l)|^2 \le Cc^*(N),\end{aligned}$$

where $V_N = \{m_0, \ldots, m_{N-1}\}$.

(i″) We pick a positive integer M such that

$$a(0) > \sum_{l \neq 0} |a(Ml)|.$$

Let $r \in \mathbb{Z}$ be fixed and let m_i be such that

$$c^{M,r}(m_i) = (c^{M,r})^*(i).$$

Let

$$w_i^{M,r}(k, Ml + r) = \begin{cases} b(k, Ml + r) & \text{for } l = m_i, \\ 0 & \text{for } l \neq m_i, \end{cases}$$

$$W_{N+1}^{M,r} = \operatorname{span}\{w_0, \ldots, w_N\}, \quad w \in W_{N+1}^{M,r}, \quad \|w\| = 1,$$

$$w = \sum_i \lambda_i \frac{w_i}{\|w_i\|} \quad \text{and} \quad V_{N+1}^{M,r} = \{m_0, \ldots, m_N\}.$$

We have

$$\begin{aligned}\langle Tw, w\rangle &= \sum_{l_1,l_2} a(M(l_2 - l_1)) \sum_{k_1} b(k_1, Ml_1 + r)\overline{w(k, Ml_1 + r)} \\ &\quad \times \sum_{k_2} \overline{b(k_2, Ml_2 + r)} w(k_2, Ml_2 + r) \\ &\ge c \sum_{l \in V_{N+1}^{M,r}} \Big|\sum_k b(k, Ml + r)\overline{w(k, Ml + r)}\Big|^2 \\ &= c \sum_{l \in V_{N+1}^{M,r}} |\lambda_l|^2 \sum_k |b(k, Ml + r)|^2 \ge c(c^{M,r})^*(N).\end{aligned}$$

Clearly, the above computation proves (i″).

(ii) follows easily from (i′) and (i″).

Now let S be an operator acting on $l^2(\mathbb{Z})$ defined by the matrix

$$a(l_2 - l_1) \sum_k d(k) b(k, l_1)\overline{b(k, l_2)},$$

where $d(k) \ge 0$. Let

$$c(l) = \sum_k d(k)|b(k,l)|^2$$

and for $M, r \in \mathbb{Z}$,

$$c^{M,r}(l) = \sum_k d(k)|b(k, Ml + r)|^2.$$

PROPOSITION 3.9. *If $a(l)$ is an absolutely summable, positive definite sequence, $a(0) > 0$ and $d(k)$ is nonnegative, then*

(i′) *for some constant $C > 0$,*

$$s_N(S) \le Cc^*(N),$$

(i″) *for some positive integer M and some constant $c > 0$,*

$$s_N(S) \ge c(c^{M,r})^*(N) \quad \text{for all } r \in \mathbb{Z},$$

(ii) $$\|S\|_{S^p} \cong \Big(\sum_l \Big(\sum_k d(k)|b(k,l)|^2\Big)^p\Big)^{1/p}.$$

Proof. The proof is similar to that of Proposition 3.8 and we omit it.

4. Schatten ideal norm of $R_{f,\phi}$. This section contains descriptions of the expressions characterizing the S^p norms of the operators $R_{f,\phi}$. We study the dependence of $R_{f,\phi}$ on ϕ, provided f is a fixed continuous, compactly supported function. The main result is contained in Theorem 4.3. It is a direct consequence of Lemmas 4.1 and 4.2.

The following lemma exhibits the role of modulations in introducing separations in the time variable.

LEMMA 4.1. *Let f be a continuous, compactly supported function defined on $\mathbb{R}^{2d}$ and let f_1, f_2 be any two Schwartz class functions defined on $\mathbb{R}^d$. Then*

$$\|R_{f,\phi}\|_{S^p}^p \cong \sum_{\tau,n} \|R_{f_1\otimes f_2, m_n^\tau \phi}\|_{S^p}^p.$$

The constants of equivalence do not depend on ϕ.

Proof. In view of Proposition 3.4 and the fact that $\sum_\tau m^\tau \equiv 1$ we see that

$$\|R_{f,\phi}\|_{S^p}^p \cong \sum_{\tau} \|R_{f,m^\tau\phi}\|_{S^p}^p \tag{4.1}$$

We choose a nonzero Schwartz class function f_1 such that

$$\check{f}_1^* * \check{f}_1(x-y) = 0 \quad \text{if } |x-y| \geq \delta, \tag{4.2}$$

where $\operatorname{dist}(\operatorname{supp} m_n^\tau, \partial Q_n^\tau) \geq \delta$.

We take a nonzero Schwartz class function f_2 such that

$$f_2(q) m_n^\tau (x-q) = 0 \quad \text{unless } x \in Q_n^\tau. \tag{4.3}$$

Proposition 3.1 guarantees that

$$\|R_{f,m^\tau\phi}\|_{S^p} \cong \|R_{f_1\otimes f_2, m^\tau\phi}\|_{S^p}. \tag{4.4}$$

We compose $R_{f_1\otimes f_2, m^\tau\phi}$ with the inverse Fourier transform tensored with the identity on the left. This leads to the kernel

$$\check{f}_1(\xi - y) f_2(q) \overline{m^\tau \phi(y-q)}. \tag{4.5}$$

An application of the property (4.2) shows that the kernel of the composition of (4.5) with its adjoint has the form

$$\sum_n \check{f}_1^* * \check{f}_1(x-y) \int |f_2(q)|^2 m_n^\tau \phi(x-q) \overline{m_n^\tau \phi(y-q)}\, dq. \tag{4.6}$$

Since (4.3) holds, the terms in (4.6) act on orthogonal subspaces $L^2(Q_n^\tau)$. We obtain

$$\begin{aligned} &\|R_{f_1\otimes f_2, m^\tau\phi}\|_{S^p}^p \\ &\cong \sum_n \Big\| \check{f}_1^* * \check{f}_1(x-y) \int |f_2(q)|^2 m_n^\tau \phi(x-q)\overline{m_n^\tau\phi(y-q)}\, dq \Big\|_{S^{p/2}}^{p/2} \\ &= \sum_n \|\check{f}_1(\xi - y) f_2(q) \overline{m_n^\tau \phi(y-q)}\|_{S^p}^p = \sum_n \|R_{f_1\otimes f_2, m_n^\tau\phi}\|_{S^p}^p. \end{aligned} \tag{4.7}$$

The S^p norms of $R_{f_1\otimes f_2, m_n^\tau\phi}$ are equivalent for different choices of Schwartz class functions f_1, f_2 (Proposition 3.1) and we may forget about the restrictions on f_1, f_2. A combination of (4.1), (4.4), (4.7) finishes the proof.

The next lemma makes use of the fact that translations introduce separation in the frequency variable.

LEMMA 4.2. *For any two Schwartz class functions f_1, f_2,*

$$\|R_{f_1\otimes f_2, m_n^\tau\phi}\|_{S^p}^p \cong \sum_{\varrho,k} \|m_k^\varrho (m_n^\tau \phi)^\wedge\|_{L^2}^p.$$

The constants of equivalence depend neither on ϕ nor on n.

Proof. We start by making a convenient choice of f_1, f_2. We take the same f_2 as in the proof of Lemma 4.1, but this time we require that f_1 satisfies

$$|\check{f}_1^* * \check{f}_1(x-y)| \geq \kappa > 0 \quad \text{for } x, y \in Q_n^\tau. \tag{4.8}$$

We compose the kernel of ${}^\vee \otimes \mathrm{id} R_{f_1\otimes f_2, m_n^\tau\phi}$ with its adjoint. This operation yields

$$\check{f}_1^* * \check{f}_1(x-y) \int |f_2(q)|^2 m_n^\tau \phi(x-q) \overline{m_n^\tau \phi(y-q)}\, dq. \tag{4.9}$$

In view of Proposition 3.3 the $S^{p/2}$ norm of (4.9) is equivalent to the $S^{p/2}$ norm of

$$\int |f_2(q)|^2 m_n^\tau \phi(x-q) \overline{m_n^\tau \phi(y-q)}\, dq. \tag{4.10}$$

The kernel (4.10) is a composition of the kernel

$$f_2(q) \overline{m_n^\tau \phi(y-q)}, \tag{4.11}$$

acting on $L^2(\mathbb{R}^d)$, with its adjoint.

The above observations lead to the following equivalence of norms:

$$\|R_{f_1\otimes f_2, m_n^\tau\phi}\|_{S^p} \cong \|f_2(q)\overline{m_n^\tau\phi(y-q)}\|_{S^p}. \tag{4.12}$$

We will now work with the kernel (4.11), which appears on the right hand side of (4.12). The previous choice of f_2 was suitable to get (4.12). Since the S^p norms of (4.11) are equivalent for any nonzero Schwartz class functions, we may make another choice of f_2.

We choose f_2 satisfying

$$\hat{f}_1^* * \hat{f}_2(\xi - \eta) = 0 \quad \text{if } |\xi - \eta| \geq \delta. \tag{4.13}$$

We compose the kernel (4.11) with the inverse Fourier transform on the right and with the Fourier transform on the left. This yields

$$\hat{f}_2(\xi - \eta) \overline{(m_n^\tau \phi)^\wedge(\eta)}. \tag{4.14}$$

We observe that

$$\|\hat{f}_2(\xi-\eta)\overline{(m_n^\tau\phi)^\wedge(\eta)}\|_{S^p}^p \cong \sum_\varrho \|\hat{f}_2(\xi-\eta)\overline{m^\varrho (m_n^\tau\phi)^\wedge(\eta)}\|_{S^p}^p. \tag{4.15}$$

We compose the kernel

$$\hat{f}_2(\xi - \eta) \overline{m^\varrho (m_n^\tau \phi)^\wedge(\eta)} \tag{4.16}$$

with its adjoint. Since f_2 satisfies (4.13), we obtain

$$\sum_k \hat{f}_2^* * \hat{f}_2(\xi-\eta) m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}(\xi)\overline{m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}(\eta)}. \tag{4.17}$$

The terms in (4.17) act on orthogonal subspaces $L^2(Q_k^{\varrho})$. Combining this observation with (4.15) we get

$$\begin{aligned}\|f_2(q)\overline{m_n^{\tau}\phi(y-q)}\|_{S^p}^p &\cong \sum_{\varrho,k} \|\hat{f}_2^* * \hat{f}_2(\xi-\eta) m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}(\xi)\overline{m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}(\eta)}\|_{S^{p/2}}^{p/2}\\ &= \sum_{\varrho,k} \|\hat{f}_2(\xi-\eta)\overline{m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}(\eta)}\|_{S^p}^p.\end{aligned} \tag{4.18}$$

The last choice of f_2 was convenient to get (4.18). We still need yet another choice of f_2. We may change f_2 again since the S^p norms of the right hand side of (4.18) are equivalent for different choices of f_2.

We now take f_2 satisfying

$$|\hat{f}_2^* * \hat{f}_2(\xi-\eta)| \ge \kappa > 0 \quad \text{for } \xi,\eta \in Q_k^{\tau}.$$

An application of Proposition 3.3 yields

$$\begin{aligned}\|\hat{f}_2^* * \hat{f}_2(\xi-\eta) m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}(\xi)\overline{m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}(\eta)}\|_{S^{p/2}}^{p/2}\\ \cong \|m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}(\xi)\overline{m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}(\eta)}\|_{S^{p/2}}^{p/2} \cong \|m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}\|_{L^2}^p.\end{aligned} \tag{4.19}$$

A combination of (4.12), (4.18), (4.19) completes the proof.

THEOREM 4.3. *Let f be a continuous, compactly supported function defined on $\mathbb{R}^{2d}$. Then*

$$\|R_{f,\phi}\|_{S^p}^p \cong \sum_{\tau,n,\varrho,k} \|m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge}\|_{L^2}^p. \tag{4.20}$$

The constants of equivalence do not depend on ϕ.

The formula (4.20) is related to the following decomposition of ϕ:

$$\phi = \sum_{\tau,n,\varrho,k} (m_k^{\varrho}(m_n^{\tau}\phi)^{\wedge})^{\vee}. \tag{4.21}$$

The terms in (4.20) are the L^2 norms of the blocks in the formula (4.21). It is also possible to express the norm in (4.20) by sampling at suitable points.

COROLLARY 4.4. *Under the same assumptions as in Theorem 4.3,*

$$\|R_{f,\phi}\|_{S^p}^p \cong \sum_{\tau,n,k} |(m_n^{\tau}\phi)^{\wedge}(k)|^p. \tag{4.22}$$

Proof. This corollary follows immediately from Theorem 4.3 and standard Plancherel–Pólya type estimates for functions of exponential type.

Several aspects of the function space described in (4.20), (4.22) were studied by Feichtinger and Gröchenig in [F2], [FG].

5. Schatten ideal norm of $\mathcal{R}_{\mathrm{f},\psi}$. In this section we present expressions providing two-sided control of the S^p norms of $\mathcal{R}_{\mathrm{f},\psi}$. As in the case of $R_{f,\phi}$, we are interested in the dependence of $\mathcal{R}_{\mathrm{f},\psi}$ on ψ, provided that f is a fixed continuous, compactly supported function.

The following part of this section, leading to Theorem 5.4, deals with the one-dimensional case. The main result is given in Theorem 5.4. It is a direct consequence of Lemmas 5.2 and 5.3.

We start by reducing the problem to a class of operators which are easier to deal with.

PROPOSITION 5.1. *Let f be a continuous, compactly supported function defined on $\mathbb{R}\times(0,\infty)$, let f_1 be a Schwartz class function defined on $\mathbb{R}$ and let f_2 be a compactly supported, smooth function defined on $(0,\infty)$. Then*

$$\|\mathcal{R}_{\mathrm{f},\psi}\|_{S^p}^p \cong \sum_{\tau} \|S_+^{\tau}(\xi,s,\eta)\|_{S^p}^p + \|S_-^{\tau}(\xi,s,\eta)\|_{S^p}^p,$$

where

$$S_+^{\tau}(\xi,s,\eta) = \hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\mathrm{m}^{\tau}(s\eta)\overline{\hat{\psi}(s\eta)}, \tag{5.1}$$

$$S_-^{\tau}(\xi,s,\eta) = \hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\mathrm{m}^{\tau}(-s\eta)\overline{\hat{\psi}(s\eta)}, \tag{5.2}$$

and

$$S_+^{\tau} : L^2((0,\infty),d\eta) \to L^2(\mathbb{R}\times(0,\infty),d\xi ds),$$
$$S_-^{\tau} : L^2((-\infty,0),d\eta) \to L^2(\mathbb{R}\times(0,\infty),d\xi ds).$$

The constants of equivalence do not depend on ψ.

Proof. It was observed in Proposition 3.2 that replacing $\mathrm{f}(u,s)$ in $\mathcal{R}_{\mathrm{f},\psi}$ by $\mathrm{f}_1(u)\mathrm{f}_2(s)s^{1/2}$ provides a kernel with equivalent S^p norm. Thus we may consider the kernel

$$\mathrm{f}_1(u)\mathrm{f}_2(s)s^{-1}\overline{\psi((y-u)/s)},$$

acting from $L^2(\mathbb{R},dy)$ into $L^2(\mathbb{R}\times(0,\infty),duds)$, instead of $\mathcal{R}_{\mathrm{f},\psi}$.

We compose the above kernel with the inverse Fourier transform on the right and with the Fourier transform tensored with identity on the left. After performing those operations we obtain

$$\hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\overline{\hat{\psi}(s\eta)}. \tag{5.3}$$

It is clear that the S^p norm of the kernel (5.3) is equivalent to $\|\mathcal{R}_{\mathrm{f},\psi}\|_{S^p}$. We observe that

$$\begin{aligned}\|\hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\overline{\hat{\psi}(s\eta)}\|_{S^p}^p \cong \|\hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\overline{\hat{\psi}(s\eta)}\chi_{(0,\infty)}(\eta)\|_{S^p}^p\\ + \|\hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\overline{\hat{\psi}(s\eta)}\chi_{(-\infty,0)}(\eta)\|_{S^p}^p.\end{aligned}$$

The proof is completed by an application of Proposition 3.5 and the fact that $\sum_\tau \mathrm{m}^\tau(\eta) \equiv 1$.

In the above proposition we have reduced the study of $\mathcal{R}_{\mathrm{f},\psi}$ to the kernels S_+^τ, S_-^τ. These two kernels are treated in the same manner, and we present our arguments only in the case of S_+^τ. Let

$$\mathcal{M}^\tau = \sum_{k\le 0} \chi_{\mathrm{Q}_k^\tau}.$$

We choose a smooth, nonnegative f_2 with small support so that for all k,

$$\mathrm{f}_2(s)\mathrm{m}_k^\tau(s\eta) = 0 \quad \text{unless } \eta \in \mathrm{Q}_k^\tau.$$

We write the kernel (5.1) as the sum $\mathcal{K}_0^\tau + \mathcal{K}_1^\tau$, where

$$\begin{aligned}(5.4)\qquad \mathcal{K}_0^\tau(\xi,s,\eta) &= \hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\mathrm{m}^\tau(s\eta)\overline{\hat{\psi}(s\eta)}\mathcal{M}^\tau(\eta)\\ &= \sum_{k\le 0}\hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\mathrm{m}_k^\tau(s\eta)\overline{\hat{\psi}(s\eta)},\end{aligned}$$

$$\begin{aligned}(5.5)\qquad \mathcal{K}_1^\tau(\xi,s,\eta) &= \hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\mathrm{m}^\tau(s\eta)\overline{\hat{\psi}(s\eta)}(1-\mathcal{M}^\tau(\eta))\\ &= \sum_{k> 0}\hat{\mathrm{f}}_1(\xi-\eta)\mathrm{f}_2(s)\mathrm{m}_k^\tau(s\eta)\overline{\hat{\psi}(s\eta)}.\end{aligned}$$

It is clear that this decomposition of the kernel (5.1) reduces the problem of characterizing the S^p norm of $\mathcal{R}_{\mathrm{f},\psi}$ to the same problem for $\mathcal{K}_0^\tau$, $\mathcal{K}_1^\tau$.

The next two lemmas provide descriptions of the norms $\|\mathcal{K}_0^\tau\|_{S^p}$, $\|\mathcal{K}_1^\tau\|_{S^p}$. Their combination directly leads to a description of the S^p norm of $\mathcal{R}_{\mathrm{f},\psi}$.

LEMMA 5.2. *The following equivalence of norms (quasi-norms) holds*:

$$\|\mathcal{K}_0^\tau\|_{S^p} \cong \Big(\sum_l\Big(\sum_{k\le 0}|(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(\xi))^\sim(l)|^2\Big)^{p/2}\Big)^{1/p}.$$

The constants of equivalence do not depend on ψ.

Proof. It is clear that changing f_1 in $\mathcal{K}_0^\tau$ to any other nonzero Schwartz class function leads to a kernel with equivalent S^p norm. We pick f_1, a Schwartz class function, in such a way that

$$(5.6)\qquad |\hat{\mathrm{f}}_1^* * \hat{\mathrm{f}}_1(\xi-\eta)| \ge \kappa > 0 \quad \text{for all } \xi,\eta\in\bigcup_{k\le 0}\mathrm{Q}_k^\tau.$$

The kernel of $\mathcal{K}_0^{\tau *}\mathcal{K}_0^\tau$ is given by the formula

$$(5.7)\qquad \hat{\mathrm{f}}_1^* * \hat{\mathrm{f}}_1(\xi-\eta)\int_0^\infty \mathrm{b}_2(s)\mathrm{m}^{\tau,0}(s\xi)\hat{\psi}(s\xi)\mathrm{m}^{\tau,0}(s\eta)\overline{\hat{\psi}(s\eta)}\,ds,$$

where $\mathrm{b}_2(s) = (\mathrm{f}_2(s))^2$, $\mathrm{m}^{\tau,0}(\xi) = \sum_{k\le 0}\mathrm{m}_k^\tau(\xi)$. A combination of (5.6), (5.7) and Proposition 3.3 shows that

$$(5.8)\qquad \|\mathcal{K}_0^\tau\|_{S^p}^p \cong \Big\|\int_0^\infty \mathrm{b}_2(s)\mathrm{m}^{\tau,0}(s\xi)\hat{\psi}(s\xi)\mathrm{m}^{\tau,0}(s\eta)\overline{\hat{\psi}(s\eta)}\,ds\Big\|_{S^{p/2}}^{p/2}.$$

We compute the matrix of the operator given by the kernel

$$\int_0^\infty \mathrm{b}_2(s)\mathrm{m}^{\tau,0}(s\xi)\hat{\psi}(s\xi)\mathrm{m}^{\tau,0}(s\eta)\overline{\hat{\psi}(s\eta)}\,ds$$

with respect to the orthonormal basis $\{\chi_{\mathrm{Q}_k^\tau}\mathrm{e}_l(s)\}_{k,l}$ of the space $\bigoplus_{k\le 0}L^2(\mathrm{Q}_k^\tau, ds)$ on which this operator acts. A direct calculation shows that this matrix has the form

$$(5.9)\qquad \widetilde{\mathrm{b}}_2(l_2-l_1)(\xi^{1/2}\mathrm{m}_{k_1}^\tau(\xi)\hat{\psi}(\xi))^\sim(l_1)\overline{(\eta^{1/2}\mathrm{m}_{k_2}^\tau(\eta)\hat{\psi}(\eta))^\sim(l_2)}.$$

We apply Proposition 3.8, dealing with S^p norms of matrices, to get two-sided estimates of the S^p norm of the matrix (5.9). A combination of (5.8), (5.9) and that proposition shows that

$$\|\mathcal{K}_0^\tau\|_{S^p}^p \cong \sum_l\Big(\sum_{k\le 0}|(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(\xi))^\sim(l)|^2\Big)^{p/2}.$$

LEMMA 5.3. *The following norms (quasi-norms) are equivalent*:

$$\|\mathcal{K}_1^\tau\|_{S^p} \cong \Big(\sum_{k>0,\,l}\Big(\frac{1}{2e^k}\sum_r \hat{H}\Big(\frac{r}{2e^k}\Big)\Big|(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(\xi))^\sim\Big(l-\frac{r}{2}\Big)\Big|^2\Big)^{p/2}\Big)^{1/p}.$$

Here H is any smooth, nonnegative, positive definite function with sufficiently small support (depending on the constant σ in Proposition 3.6). *The constants of equivalence do not depend on* ψ.

Proof. It is clear that we may change f_1, f_2 in $\mathcal{K}_1^\tau$ as long as f_1 is a Schwartz class function and f_2 any smooth function with compact support contained in $(0,\infty)$ (Proposition 3.2). We pick a nonnegative function f_2 with the above properties and intervals $P_k^\tau \subset \mathrm{Q}_k^\tau$, $k>0$, in such a manner that

$$(5.10)\qquad \mathrm{f}_2(s)\mathrm{m}_k^\tau(s\xi) = 0 \quad \text{unless } \xi\in P_k^\tau,$$

$$(5.11)\qquad \operatorname{dist}(P_k^\tau, \partial\mathrm{Q}_k^\tau) \ge \delta > 0 \quad \text{for } k>0.$$

Now we take a Schwartz class function f_1 satisfying

$$(5.12)\qquad \hat{\mathrm{f}}_1^* * \hat{\mathrm{f}}_1(\xi-\eta) = 0 \quad \text{if } |\xi-\eta|\ge\delta.$$

We put those functions into $\mathcal{K}_0^\tau$. Since f_1, f_2 satisfy (5.10)–(5.12) we obtain

$$(5.13)\qquad \mathcal{K}_1^{\tau *}\mathcal{K}_1^{\tau} = \hat{\mathrm{f}}_1^* * \hat{\mathrm{f}}_1(\xi-\eta)\sum_{k>0}\int_0^\infty \mathrm{b}_2(s)\mathrm{m}_k^\tau(s\xi)\hat{\psi}(s\xi)\mathrm{m}_k^\tau(s\eta)\overline{\hat{\psi}(s\eta)}\,ds.$$

The terms in the sum in (5.13) act on orthogonal subspaces $L^2(\mathrm{Q}_k^\tau, ds)$ and this implies

$$(5.14)\qquad \|\mathcal{K}_1^\tau\|_{S^p}^p \cong \sum_{k>0}\Big\|\hat{\mathrm{f}}_1^* * \hat{\mathrm{f}}_1(\xi-\eta)\int_0^\infty \mathrm{b}_2(s)\mathrm{m}_k^\tau(s\xi)\hat{\psi}(s\xi)\mathrm{m}_k^\tau(s\eta)\overline{\hat{\psi}(s\eta)}\,ds\Big\|_{S^{p/2}}^{p/2}.$$

The formula (5.14) reduces the problem for $\mathcal{K}_1^\tau$ to the two-sided S^p estimates for the kernels

$$(5.15)\qquad \hat{\mathrm{f}}_1^* * \hat{\mathrm{f}}_1(\xi-\eta)\int_0^\infty \mathrm{b}_2(s)\mathrm{m}_k^\tau(s\xi)\hat{\psi}(s\xi)\mathrm{m}_k^\tau(s\eta)\overline{\hat{\psi}(s\eta)}\,ds$$

acting on $L^2(\mathrm{Q}_k^\tau, ds)$.

Proposition 3.2 allows us to change f_1 in (5.15) to any nonzero Schwartz class function. This observation together with Proposition 3.7 shows that the $S^{p/2}$ norm of the kernel (5.15) is equivalent to the $S^{p/2}$ norm of the kernel

$$(5.16)\qquad H(e^k\log\xi/\eta)\int_0^\infty \mathrm{b}_2(s)\mathrm{m}_k^\tau(s\xi)\hat{\psi}(s\xi)\mathrm{m}_k^\tau(s\eta)\overline{\hat{\psi}(s\eta)}\,ds,$$

where H is any nonnegative, positive definite C^∞ function with sufficiently small support ($\operatorname{supp} H \subset (-\sigma^{-2},\sigma^2)$, where σ is any constant satisfying the condition in Proposition 3.6, is enough). The constants of equivalence depend neither on k nor on ψ.

We compute the matrix of (5.16) with respect to the basis $\{\chi_{\mathrm{Q}_k^\tau}\mathrm{e}_l\}_l$. We obtain

$$(5.17)\qquad \widetilde{\mathrm{b}}_2(l_2-l_1)\times\int_0^\infty\int_0^\infty H(e^k\log\xi/\eta)\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(\xi)\xi^{-2\pi i l_1}\eta^{1/2}\mathrm{m}_k^\tau(\eta)\overline{\hat{\psi}(\eta)}\eta^{2\pi i l_2}\,\frac{d\xi}{\xi}\,\frac{d\eta}{\eta}.$$

Now we expand the function H in (5.17) in a Fourier series on the interval $[-e^{-k}, e^k]$ and (5.17) becomes

$$(5.18)\qquad \frac{1}{2e^k}\widetilde{\mathrm{b}}_2(l_2-l_1)\times\sum_r \hat{H}\Big(\frac{r}{2e^k}\Big)(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(\xi))^\sim(l_1-r/2)\overline{(\eta^{1/2}\mathrm{m}_k^\tau(\eta)\hat{\psi}(\eta))^\sim(l_2-r/2)}.$$

Proposition 3.9 shows that the $S^{p/2}$ norm of the matrix (5.18) (and thus also of the kernel (5.15)) is equivalent to

$$(5.19)\qquad \Big\|\frac{1}{2e^k}\sum_r \hat{H}\Big(\frac{r}{2e^k}\Big)|(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(\xi))^\sim(l-r/2)|^2\Big\|_{l^{p/2}}.$$

Combining (5.14) and (5.19) we obtain the lemma.

We may combine the results we have obtained so far to obtain the final form of the expressions controlling the S^p norm of $\mathcal{R}_{\mathrm{f},\psi}$ in the one-dimensional case.

THEOREM 5.4. *Let* f *be a continuous, compactly supported function defined on* $\mathbb{R}\times(0,\infty)$. *Then*

$$(5.20)\qquad \|\mathcal{R}_{\mathrm{f},\psi}\|_{S^p}^p \cong \sum_{\tau,l}\Big(\sum_{k\le 0}|(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(\xi))^\sim(l)|^2\Big)^{p/2} + \sum_{\tau,l}\Big(\sum_{k\le 0}|(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(-\xi))^\sim(l)|^2\Big)^{p/2}$$
$$+\sum_{\tau,k>0,l}\Big(\frac{1}{2e^k}\sum_r \hat{H}\Big(\frac{r}{2e^k}\Big)|(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(\xi))^\sim(l-r/2)|^2\Big)^{p/2} + \sum_{\tau,k>0,l}\Big(\frac{1}{2e^k}\sum_r \hat{H}\Big(\frac{r}{2e^k}\Big)|(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(-\xi))^\sim(l-r/2)|^2\Big)^{p/2},$$

where H is a smooth, nonnegative, positive definite function with sufficiently small support. The constants of equivalence do not depend on ψ.

We illustrate the above theorem with two examples.

EXAMPLES. (i) If $\hat{\psi}\in C^\infty(\mathbb{R}\setminus\{0\})$ and $\hat{\psi}(\xi)=|\xi|^{-\alpha}$ for large $|\xi|$ and $\hat{\psi}(\xi)=|\xi|^\beta$ for $|\xi|$ close to 0, then the expression (5.20) is finite if and only if $\alpha>1/p$ and $\beta>-1/2$.

(ii) Let $\psi=\chi_{(0,1)}-\chi_{(-1,0)}$ be the Haar function. The expression (5.20) is finite if and only if $p>1$.

Both of these examples are not hard to check directly and we omit the computations justifying our statements. The second example, with the Haar function, was originally studied by Rochberg in [R3].

The expression (5.20) is an analogue of the expression (4.22) obtained for $R_{f,\phi}$. It is also possible to formulate the result in a form analogous to (4.20) and this is what we do next.

We take a compactly supported smooth function $\mathrm{m}^\dagger$ defined on $(0,\infty)$ such that

$$\sum_{k\in\mathbb{Z}} \mathrm{m}_k^\dagger(s) \equiv 1, \tag{5.21}$$

where $\mathrm{m}_k^\dagger(s) = \mathrm{m}^\dagger(s/e^k)$. We define

$$\psi^+ = (\hat{\psi}\chi_{(0,\infty)})^\vee, \qquad \psi^- = (\hat{\psi}\chi_{(-\infty,0)})^\vee \tag{5.22}$$

and

$$\psi_{k,l}^+ = (\zeta^{-1/2}\mathcal{F}^{-1}(\mathrm{m}_l^\dagger\mathcal{F}(\xi^{1/2}\mathrm{m}_k^\dagger(\xi)(\psi^+)^\wedge(\xi)))(\zeta))^\vee,$$
$$\psi_{k,l}^- = (\zeta^{-1/2}\mathcal{F}^{-1}(\mathrm{m}_l^\dagger\mathcal{F}(\xi^{1/2}\mathrm{m}_k^\dagger(\xi)(\psi^-)^\wedge(-\xi)))(-\zeta))^\vee.$$

Clearly,

$$\psi = \sum_{k,l}\psi_{k,l}^+ + \sum_{k,l}\psi_{k,l}^-. \tag{5.23}$$

The formula (5.23) is an analogue of (4.21).

COROLLARY 5.5. *Under the same assumptions as in Theorem* 5.4,

$$\begin{aligned}\|\mathcal{R}_{\mathrm{f},\psi}\|_{S^p}^p &\cong \sum_l\Big(\sum_{k\le 0}\|\psi_{k,l}^+\|_{L^2}^2\Big)^{p/2} + \sum_l\Big(\sum_{k\le 0}\|\psi_{k,l}^-\|_{L^2}^2\Big)^{p/2}\\ &\quad + \sum_{k>0,\,l}\Big(\frac{1}{[e^k]}\sum_{r=0}^{[e^k]}\|\psi_{k,l+r}^+\|_{L^2}^2\Big)^{p/2} + \sum_{k>0,\,l}\Big(\frac{1}{[e^k]}\sum_{r=0}^{[e^k]}\|\psi_{k,l+r}^-\|_{L^2}^2\Big)^{p/2}.\end{aligned} \tag{5.24}$$

Proof. We prove the estimates for ψ^+. Those for ψ^- follow in the same way.

First we prove the estimate from below. The proof is based on the following reproducing formula. We take a C_c^∞ function M^τ defined on $(0,\infty)$ such that $\mathrm{M}^\tau \equiv 1$ on Q_0^τ and $\operatorname{supp}\mathrm{M}^\tau \cap \mathrm{m}_k^\tau = \emptyset$ for $k \ne 0$. We obtain

$$(\psi^+)^\wedge(\xi) = \sum_{\tau,k,l}(\eta^{1/2}\mathrm{m}_k^\tau(\eta)\hat{\psi}(\eta))^\sim(l)\mathrm{M}_k^\tau(\xi)\xi^{2\pi i l}\xi^{-1/2}. \tag{5.25}$$

A simple computation shows that

$$\|\psi_{k,l}^+\|_{L^2}^2 = \int_0^\infty |\mathcal{F}(\xi^{1/2}\mathrm{m}_k^\dagger(\xi)\hat{\psi}(\xi))(\eta)\mathrm{m}_l^\dagger(\eta)|^2\,\frac{d\eta}{\eta}.$$

After substituting (5.25) in the above formula we obtain

$$\|\psi_{k,l}^+\|_{L^2} \le \sum_{\tau,k',l'} K(k-k', l-l', \tau)|a_{k',l'}^\tau|, \tag{5.26}$$

where

$$K(k-k', l-l', \tau) = \Big(\int_0^\infty |\mathcal{F}(\mathrm{m}_k^\dagger(\xi)\mathrm{M}_{k'}^\tau(\xi)\xi^{2\pi i l'})(\eta)\mathrm{m}_l^\dagger(\eta)|^2\,\frac{d\eta}{\eta}\Big)^{1/2}, \tag{5.27}$$

$$a_{k,l}^\tau = (\xi^{1/2}\mathrm{m}_k^\dagger(\xi)\hat{\psi}(\xi))^\sim(l). \tag{5.28}$$

The functions $\mathrm{m}_0^\dagger$, M^τ have compact supports, and this implies

$$K(k-k', l-l', \tau) = 0 \quad \text{if } |k-k'| > C, \tag{5.29}$$

where C is some positive constant. It is also easy to see that

$$K(k-k', l-l', \tau) \le O(|l-l'|^{-N}) \tag{5.30}$$

for all positive integers N. Estimates (5.29) and (5.30) show that the matrix K has good decay properties. It is a matter of a standard computation to use (5.26) together with (5.29) and (5.30) to see that

$$\begin{aligned}\sum_l\Big(\sum_{k\le 0}\|\psi_{k,l}^+\|_{L^2}^2\Big)^{p/2} &+ \sum_{k>0,\,l}\Big(\frac{1}{[e^k]}\sum_{r=0}^{[e^k]}\|\psi_{k,l+r}^+\|_{L^2}^2\Big)^{p/2}\\ &\le c\sum_{\tau,l}\Big(\sum_{k\le C}|a_{k,l}^\tau|^2\Big)^{p/2} + c\sum_{\tau,\,k>-C,\,l}\Big(\sum_r \hat{H}_k(r)|a_{k,l-r}^\tau|^2\Big)^{p/2},\end{aligned} \tag{5.31}$$

where $\hat{H}_k(r) = (1/(2e^k))\hat{H}(r/(2e^k))$ and H is a function as in Theorem 5.4. Theorem 5.4 and (5.31) complete the proof of the estimate from below.

Now we prove the estimate from above. We need another reproducing formula. We take a function $\mathrm{M}^\dagger$ defined on $(0,\infty)$ such that $\mathcal{F}^{-1}(\mathrm{M}^\dagger) \in C_c^\infty(0,\infty)$ and $\sum_l \mathrm{M}_l^\dagger \equiv 1$. We have

$$(\psi^+)^\wedge(\xi) = \sum_{k,l}\mathcal{F}^{-1}(\mathcal{F}(\eta^{1/2}\mathrm{m}_k^\dagger(\eta)\hat{\psi}(\eta))(\zeta)\mathrm{M}_l^\dagger(\zeta))(\xi)\xi^{-1/2}, \tag{5.32}$$

and

$$|(\xi^{1/2}\mathrm{m}_k^\tau(\xi)\hat{\psi}(\xi))^\sim(l/2)| \le \sum_{k',l'} K(k,l,\tau,k',l'), \tag{5.33}$$

where this time

$$K(k,l,\tau,k',l') = |(\mathrm{m}_k^\tau(\xi)\mathcal{F}^{-1}(\mathcal{F}(\eta^{1/2}\mathrm{m}_{k'}^\dagger(\eta)\hat{\psi}(\eta))(\zeta)\mathrm{M}_{l'}^\dagger(\zeta))(\xi))^\sim(l/2)|.$$

We want to control the terms of the right hand side of (5.33) by an expression depending on $\|\psi_{k',l'}^+\|_{L^2}$. Since m_0^τ, $\mathrm{m}_0^\dagger$ and $\mathcal{F}^{-1}(\mathrm{M}^\dagger)$ have compact supports,

$$K(k,l,\tau,k',l') = 0 \quad \text{if } |k-k'| > C, \tag{5.34}$$

where C is some positive constant. We take a C_c^∞ function $\mathcal{M}$ defined on $(0,\infty)$ such that $\mathcal{M} \equiv 1$ on $\operatorname{supp}\mathrm{m}^\dagger$. In the following estimate we use the decomposition of $M_{l'}^{\dagger}$ into compactly supported pieces $\sum_n m_{n+l'}^{\dagger} M_{l'}^{\dagger}$. We obtain

$$(5.35)\quad K(k,l,\tau,k',l') \le \sum_n \|\mathcal{F}(m_k^\tau)(\xi^{-1})\mathcal{M}_{n+l'-l/2}(\xi)\|_{L^\infty} \|\mathcal{F}(\eta^{1/2} m_{k'}^{\dagger}(\eta)\hat{\psi}(\eta)) m_{n+l'}^{\dagger} M_{l'}^{\dagger}\|_{L^1} \le \sum_n h(n+l'-l/2)h(n)\|\psi_{k',n+l'}^{+}\|_{L^2},$$

where for every positive integer N,

$$(5.36)\quad h(n) = O(n^{-N}).$$

It is a matter of a standard argument to check that (5.33)–(5.36) imply

$$(5.37)\quad \sum_{\tau,l}\Big(\sum_{k\le 0} |(\xi^{1/2} m_k^\tau(\xi)\hat{\psi}(\xi))^{\sim}(l/2)|^2\Big)^{p/2} + \sum_{\tau,\,k>0,\,l}\Big(\sum_r \hat{H}_k(r)\Big|(\xi^{1/2} m_k^\tau(\xi)\hat{\psi}(\xi))^{\sim}\Big(\frac{l-r}{2}\Big)\Big|^2\Big)^{p/2} \le c\sum_l \Big(\sum_{k\le C}\|\psi_{k,l}^{+}\|_{L^2}^2\Big)^{p/2} + c\sum_{k>-C,\,l}\Big(\sum_r \hat{H}(r)\|\psi_{k,(l-r)/2}^{+}\|_{L^2}^2\Big)^{p/2},$$

where H is a function as in Theorem 5.4. Another standard argument allows us to rewrite the inner summation in the form given in (5.24). This observation together with Theorem 5.4 and (5.37) complete the proof of the estimate from above.

Comments. (i) The operator T_ϕ involves averages of translations and modulations of ϕ, while $\mathcal{T}_\psi$ is based on averages of translations and dilations of ψ. It is these averaging processes that produce the patterns described in our results.

It is natural to compare what happens if only translations are involved, i.e. we take T_ϕ with the symbol supported in the plane $p = 0$. An argument similar to those given for $R_{f,\phi}$ (see also [N3]) shows that the S^p norm in this case is controlled by the expression

$$(5.38)\quad \Big(\sum_{\tau,k}\Big(\sum_n |(m_n^\tau\phi)^\wedge(k)|^2\Big)^p\Big)^{1/p}.$$

If one takes a symbol supported in a plane which is a rotated plane $p = 0$, then there is a corresponding phase space rotation in the expression (5.38). In particular, taking the plane $q = 0$ corresponds to switching the roles of n and k.

(ii) It is possible to extend the method of Theorem 5.4 to higher dimensions provided that $\hat{\psi}$ factors out in polar coordinates. We formulate this extension below. The proof essentially follows the steps of the one-dimensional case and we do not include it.

Let $\Phi^j : U^j \to \mathbb{R}^{d-1}$, $j = 1,\ldots,N$, be a system of coordinate maps of the $(d-1)$-dimensional sphere S^{d-1} and let M^j, $j = 1,\ldots,N$, be a family of smooth, nonnegative functions on S^{d-1} satisfying

(1) $\operatorname{supp} M^j \subset U^j$,
(2) $\sum_j M^j \equiv 1$.

Let f be a continuous, compactly supported function defined on $\mathbb{R}^d \times (0,\infty)$. Let us assume that $\hat{\psi}$ factors out in polar coordinates, i.e.

$$\hat{\psi}(\xi) = R(r)\Omega(\omega),$$

where $\xi = r\omega$, $r > 0$, $\omega \in S^{d-1}$. Let $\Omega^j(\Phi^j(\omega)) = M^j(\omega)\Omega(\omega)$ and let $\Omega^j(x) = 0$ for $x \in \mathbb{R}^{d-1}\setminus\Phi^j(U^j)$. Then

$$\|\mathcal{R}_{f,\psi}\|_{S^p}^p \cong \|\Omega\|_{L^2(S^{d-1})}^p \sum_{\tau,l}\Big(\sum_{k\le 0}|(r^{d/2} m_k^\tau(r)R(r))^{\sim}(l)|^2\Big)^{p/2} + \sum_{\tau,\,k>0,\,l}\Big(\frac{1}{2e^k}\sum_n \hat{H}\Big(\frac{n}{2e^k}\Big)|(r^{d/2} m_k^\tau(r)R(r))^{\sim}(l-n/2)|^2\Big)^{p/2} \times \sum_{j,\varrho,m}\|m_m^\varrho \Omega_k^j\|_{L^2}^p,$$

where $\Omega_k^j(x) = e^{-(d-1)k/2}\Omega^j(x/e^k)$ is the L^2 normalized dilation of Ω^j by the exponential factor e^k and H is a smooth, nonnegative, positive definite function with sufficiently small support.

Acknowledgements. The main part of this work was done during the author's stay at the University of Vienna. The author would like to thank Professor Hans Feichtinger and the members of the Department of Mathematics of the University of Vienna for their hospitality. He also thanks Professor Andrzej Hulanicki and Professor Richard Rochberg for several comments on the early version of this paper.

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Received June 8, 1995
Revised version January 15, 1996
(3484)

STUDIA MATHEMATICA 119 (1) (1996)

Decomposable embeddings, complete trajectories, and invariant subspaces

by

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Abstract. We produce closed nontrivial invariant subspaces for closed (possibly unbounded) linear operators, A, on a Banach space, that may be embedded between decomposable operators on spaces with weaker and stronger topologies. We show that this can be done under many conditions on orbits, including when both A and A^* have nontrivial non-quasi-analytic complete trajectories, and when both A and A^* generate bounded semigroups that are not stable.

0. Introduction. We produce closed nontrivial invariant subspaces for a closed (possibly unbounded) linear operator A, on a Banach space X, by "sandwiching" it between two slightly better operators. Specifically, we embed A between a decomposable operator, acting on a smaller space continuously embedded in X, and an operator, acting on a larger space in which X is continuously embedded, whose local spectral subspaces are closed. In addition, we need either slightly better behavior of the restricted operator, including, but not limited to, generating a polynomially bounded group (see Proposition 2.2), or having an element where the local spectrum of A contains at least two points (Proposition 2.3).

We show that these conditions are satisfied when A^* has a nontrivial non-quasi-analytic complete trajectory and A has a complete nontrivial non-quasi-analytic trajectory that either grows more slowly (polynomial growth is sufficient) or has spectrum that contains at least two points (Theorem 2.4). By a *complete trajectory* we mean a mild solution of the reversible abstract Cauchy problem (see Definition 1.4). When A generates a strongly continuous bounded semigroup that is not stable, then it is sufficient for A to have a non-quasi-analytic complete trajectory (Corollary 2.8; weaker conditions on the semigroup are sufficient—see Theorem 2.6). It is also sufficient

1991 *Mathematics Subject Classification*: Primary 47A15; Secondary 47D03, 47A11, 47B40.

We would like to thank Shengwang Wang and Michael Neumann for introducing us to the subject of decomposable operators.