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Stochastic continuity and approximation

by

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Abstract. This work is concerned with the study of stochastic processes which are continuous in probability, over various parameter spaces, from the point of view of approximation and extension. A stochastic version of the classical theorem of Mergelyan on polynomial approximation is shown to be valid for subsets of the plane whose boundaries are sets of rational approximation.

In a similar vein, one can obtain a version in the context of continuity in probability of the theorem of Arakelyan on the uniform approximation of continuous functions on a closed set by entire functions.

Locally bounded processes continuous in probability are characterized via operators from L^1-spaces to spaces of continuous functions. This characterization is utilized in a discussion of the problem of extension of the parameter space.

Introduction. The notion of a stochastic process which is continuous in probability (stochastically continuous in [16]) arises in numerous contexts in probability theory (see [4], [7], [8], [16], [28]). Indeed, the Poisson process is continuous in probability, and this notion plays a role in the study of generalizations of this process and, from a broader point of view, in the theory of processes with independent increments [16]. For instance, R. K. Getoor [15] showed that the Brownian escape process, in dimension at least three, is continuous in probability and has independent increments. The recent work of X. Fernique [13] on random right-continuous functions with left-hand limits (so-called cadlag functions) involves continuity in probability in an essential way.

The study of processes continuous in probability as a generalization of the notion of a continuous function began with the approximation theorems of K. Fan [11], [8, Thms. VI.III.III, VI.III.IV] and D. Dugué [8, Thm. VI.III.V] on the unit interval. These results were generalized to convex domains in higher dimensions in [18], where the problem was raised of describing all compact sets in the complex plane on which every random function continuous in probability can be uniformly approximated in probability by random

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polynomials. This problem, as well as the corresponding question for rational approximation, were taken up in [1]. Along with some stimulating examples, the authors of [1] prove that random polynomial approximation holds over Jordan curves and the closures of Jordan domains (when the probability space is nonatomic). In the latter case they assume that the function to be approximated is random holomorphic on the domain itself.

Motivated by [1], we shall show that random polynomial approximation obtains for a very large class of compact sets. Sets without interior in this class include those on which every continuous function can be uniformly approximated by rational functions (with poles off the set). For compact sets with interior, we show that random polynomials approximate uniformly in probability if this is the case over the boundary of the set. In particular, if K is a compact set with the property that every continuous function on ∂K can be uniformly approximated by rational functions, then every function continuous in probability on K (with respect to a nonatomic measure) and random holomorphic on the interior of K can be uniformly approximated in probability by random polynomials.

The paper concludes with a discussion of the extension of a process continuous in probability to one with the same property over a larger parameter space. These results hinge on the relationship between processes continuous in probability and operators mapping L^1-spaces to spaces of continuous functions.

1. Preliminaries. Consider a fixed probability space $(\Omega, \mathcal{A}, P)$, a measurable space $(X, \mathcal{E})$, and an index set S, which we take to be any topological space for the moment. We wish to study a stochastic process $\varphi = \varphi(s) = \varphi(s,\omega)$ taking values in X. Denote by $C(S)$ the space of all continuous, complex-valued functions on S, equipped as usual with the topology of uniform convergence on compacta. In concert with the point of view of the current work, we may refer to φ as a random function on S or a function on $S \times \Omega$. We shall identify functions φ and ψ if for every $s \in S$, $\varphi(s) = \psi(s)$ a.s.

DEFINITION 1.1. Let $\mathcal{F}$ be a class of functions $f : S \to X$. Assume that for all $\omega \in \Omega$, $\varphi(\cdot,\omega) \in \mathcal{F}$, and for all $s \in S$, $\varphi(s,\cdot)$ is $\mathcal{A}$-measurable. Then φ is called a *random element* of $\mathcal{F}$. Thus one speaks of random continuous functions or random polynomials. Unless otherwise specified, in the sequel we shall take X to be the complex numbers, and $\mathcal{E}$ to be the Borel sets in the plane.

DEFINITION 1.2. Let φ be a stochastic process taking values in the locally convex topological vector space X. We say that φ is *continuous in probability* at $s \in S$ if for every seminorm p on X and $\varepsilon > 0$ there is a neighborhood V of s in S such that $P[p(\varphi(t) - \varphi(s)) \geq \varepsilon] < \varepsilon$ for all $t \in V$. Recall, however, that if X is not separable, then the function $\varphi(t) - \varphi(s)$ need not be measurable with respect to the Borel sets in X. To avoid this difficulty, one assumes that for every $s \in S$, $\varphi(s)$ is measurable in the sense of Bochner, i.e., $\varphi(s)$ is the limit a.s. of a sequence of simple, X-valued, $\mathcal{A}$-measurable functions. We shall say that φ is *locally bounded* if for each $s \in S$ there is a neighborhood V of s such that $\varphi(V \times \Omega)$ is bounded in X. If φ is continuous in probability at every $s \in S$, then φ is called *continuous in probability* on S (or on $S \times \Omega$). If S is a metric space, then the function φ is called *uniformly continuous in probability* on S if it is continuous in probability on S, and for a given seminorm p and $\varepsilon > 0$, the neighborhoods V above can all be taken to be balls $B(s,\delta)$ for some $\delta > 0$. It is easy to see that if S is compact and metric, then any function continuous in probability on S is uniformly continuous in probability.

DEFINITION 1.3. Let φ_n, $n = 1, 2, \ldots$, and φ be random functions on S with values in X. We say φ_n *converges uniformly in probability* to φ if given p and ε as above, there exists $N > 0$ such that $P[p(\varphi_n(s) - \varphi(s)) \geq \varepsilon] < \varepsilon$ for all $s \in S$ and $n \geq N$. If $\mathcal{F}$ is a family of functions from S to X and φ is a process on S taking values in X, we shall say that φ *can be approximated by random elements of* $\mathcal{F}$ if there is a sequence of random elements of $\mathcal{F}$ converging uniformly in probability to φ.

PROPOSITION 1.4. *Let φ be continuous in probability on the compact space S, and let $\varphi_n = \min(|\varphi|, n)\operatorname{sgn}(\varphi)$. Then φ_n converges uniformly in probability to φ. In particular, every function continuous in probability on S is the uniform limit in probability of bounded functions continuous in probability on S.*

Proof. Given $\varepsilon > 0$, for each $s \in S$ choose a neighborhood $V(s)$ of s such that $P[|\varphi(t) - \varphi(s)| \geq 1] < \varepsilon$ for all $t \in V(s)$. By the continuity property of measures, there is a positive integer n_s such that $P[|\varphi(s)| \geq n_s] < \varepsilon$, and hence $P[|\varphi(t)| \geq n_s + 1] < 2\varepsilon$ for all $t \in V(s)$. The proposition now follows by taking the maximum N of the numbers $n_s + 1$ corresponding to a finite open cover of S by neighborhoods $V(s)$ and observing that we then have

$$P[|\varphi(s) - \varphi_n(s)| \geq \varepsilon] \leq P[|\varphi(s)| \geq N] < 2\varepsilon$$

for all $s \in S$ and $n \geq N$.

Remarks 1.5. (i) It is easy to see, using the sequential definition of continuity, that every random continuous function on a metric space is continuous in probability. Easy examples show that this is not true if the space is not metric. The converse is also false [1].

(ii) The usual notion of equivalence of stochastic processes involves the equality of the finite-dimensional distributions. It is clear that with respect to this notion of equivalence, all of the properties described in Definitions 1.2 and 1.3 are properties of equivalence classes. The reason we did not define equivalence in this way is that equivalent functions (processes) would not induce the same operators T_φ, as defined in the next section.

(iii) For any S, the space of all (equivalence classes of) functions on S continuous in probability with respect to P is an algebra over the space of (P-equivalence classes of) $\mathcal{A}$-measurable functions.

(iv) Continuity in measure could, of course, be introduced over any measure space. One should note, however, that no real increase in generality ensues from moving to that setting, at least if one assumes that the given measure μ is σ-finite. For in that case, there is a probability measure P such that μ and P are mutually absolutely continuous. The notions of continuity with respect to μ and P will then coincide.

Recall that a topological space S is said to be of *dimension* n if $n+1$ is the least integer m such that every open cover $\mathcal{U}$ of S has a refinement $\mathcal{V}$ with the property that the number of elements of $\mathcal{V}$ containing any point of S is at most m. In particular, $\mathbb{R}^n$ has dimension n, and any closed subset of $\mathbb{R}^n$ with void interior has dimension at most $n-1$. For a recent survey of dimension theory, see [12].

THEOREM 1.6. *Let S be a normal topological space of finite dimension and X be a locally convex topological vector space. If $\varphi : S \times \Omega \to X$ is continuous in probability, then φ can be approximated by random continuous functions.*

Proof. Let the dimension of S be denoted by N, and let $\varepsilon > 0$ and p be a continuous seminorm on X. For each $s \in S$, choose a neighborhood $V(s)$ of s such that $P[p(\varphi(t) - \varphi(s)) \geq \varepsilon] < \varepsilon$, $t \in V(s)$. By hypothesis we may choose a refinement $\{U_\alpha\}_{\alpha\in I}$ of $\{V(s) : s \in S\}$ such that for every $s \in S$, $\operatorname{card}\{\alpha : s \in U_\alpha\} \leq N+1$. Let $\{g_\alpha\}_{\alpha\in I}$ be a partition of unity subordinate to $\{U_\alpha\}_{\alpha\in I}$, and for each α choose s_α such that $U_\alpha \subset V(s_\alpha)$. Let

$$\psi(s,\omega) = \sum_{\alpha\in I} \varphi(s_\alpha,\omega) g_\alpha(s).$$

This is a sum of at most $N+1$ terms on a neighborhood of each $s \in S$. Thus ψ is a random continuous function on S, and

$$\begin{aligned} P[p(\varphi(s) - \psi(s)) \geq \varepsilon] &= P\Big[p\Big(\sum_{\alpha\in I} g_\alpha(s)(\varphi(s) - \varphi(s_\alpha))\Big) \geq \varepsilon\Big] \\ &\leq P\Big[\sum_{\alpha\in I} g_\alpha(s) p(\varphi(s,\omega) - \varphi(s_\alpha,\omega)) \geq \varepsilon\Big] \\ &< (N+1)\varepsilon. \end{aligned}$$

PROPOSITION 1.7. *Let S_α, $\alpha \in I$, be compact topological spaces, and set $S = \prod_\alpha S_\alpha$. Let X be a locally convex topological vector space and $\varphi : S \times \Omega \to X$ be continuous in probability. Then for every continuous seminorm p on X and $\varepsilon > 0$ there exists $\psi : S \times \Omega \to X$ such that:*

(i) *ψ is continuous in probability on S.*

(ii) *There exists a finite subset F of I such that $\psi(s) = \psi(t)$ whenever $s_\alpha = t_\alpha$ for all $\alpha \in F$.*

(iii) *$P[p(\psi(s) - \varphi(s)) \geq \varepsilon] < \varepsilon$, $s \in S$.*

Proof. Given $\varepsilon > 0$ and a continuous seminorm p on X, let $V_1, \ldots, V_n$ be an open cover of S by basic open sets such that for $j = 1, \ldots, n$ and $s, t \in V_j$, we have $P[p(\varphi(s) - \varphi(t)) \geq \varepsilon] < \varepsilon$. Let F be the union of the n finite subsets of I included in the description of $V_1, \ldots, V_n$; so that for $s, t \in S$, we have $t \in V_j$ if and only if $s \in V_j$, whenever $t_\alpha = s_\alpha$, $\alpha \in F$. Fix $s^0 \in S$, and for $s \in S$, let $s'_\alpha = s_\alpha$ for $\alpha \in F$ and $s'_\alpha = s^0_\alpha$ for $\alpha \notin F$. Set $\psi(s,\omega) = \varphi(s',\omega)$. Since $s \mapsto s'$ is continuous, it is easy to see that (i)–(iii) are satisfied.

COROLLARY 1.8. *Let S_α, S, and X be as in Proposition 1.7, and suppose that each S_α has finite dimension. Then every function continuous in probability on S with values in X can be uniformly approximated in probability by a random continuous function on S with values in X.*

Proof. Since a finite cartesian product of compact spaces of finite dimension has finite dimension, the corollary follows from Theorem 1.6 and Proposition 1.7.

For a generalization of this corollary to spaces S_α of possibly infinite dimension, see Section 4, in particular Theorem 4.5.

2. The operator T_φ.

Recall that a Hausdorff topological space S is called a *k-space* if every set in S which intersects every compact set of S in a closed set is itself closed. The class of k-spaces includes all locally compact spaces and all spaces that satisfy the first countability axiom, hence all metric spaces [20, Chap. 7].

DEFINITION 2.1. Let φ be a locally bounded random function on S. For $f \in L^1(\Omega, P)$ and $s \in S$, set

$$T_\varphi f(s) = \mathbb{E}[\varphi(s,\cdot)f] = \int_\Omega \varphi(s,\omega) f(\omega)\, dP(\omega).$$

THEOREM 2.2. *Let φ be a locally bounded random function on S.*

(i) *If φ is continuous in probability on S, then $T_\varphi f$ is continuous on S for every $f \in L^1(\Omega, P)$, and T_φ is a continuous operator from $L^1(\Omega, P)$ to $C(S)$.*

(ii) *If S is a k-space, then φ is continuous in probability on S if and only if T_φ is a continuous operator from $L^1(\Omega,P)$ to $C(S)$ which maps weakly compact sets in $L^1(\Omega,P)$ into compact sets in $C(S)$.*

Proof. (i) Let φ be continuous in probability, and choose $\varepsilon>0$ and $f\in L^1(\Omega,P)$. Then there exists $\delta>0$ such that $\int_E|f|\,dP<\varepsilon$ whenever $P(E)<\delta$. For each $s\in S$, there is a neighborhood $V(s)$ of s and $M>0$ such that for all $t\in V(s)$, we have $|\varphi(t,\omega)|\le M$ for all $\omega\in\Omega$, and $E_t=\{\omega:|\varphi(t,\omega)-\varphi(s,\omega)|\ge\varepsilon\}$ has measure at most δ. Thus

$$\begin{aligned}|T_\varphi f(t)-T_\varphi f(s)|&\le\int_\Omega|\varphi(t)-\varphi(s)|\cdot|f|\,dP\\&\le 2M\int_{E_t}|f|\,dP+\int_{E_t^c}|f|\cdot|\varphi(t)-\varphi(s)|\,dP\\&\le 2M\varepsilon+\|f\|_1\varepsilon=(2M+\|f\|_1)\varepsilon.\end{aligned}$$

So $T_\varphi f$ is continuous on S. Clearly, T_φ is linear and $f\mapsto(T_\varphi f)|_K$ is bounded on $L^1(\Omega,P)$ for all compact $K\subset S$.

(ii) Let S be a k-space, and recall that a well-known theorem of Dunford [6, Thm. 15, p. 76] asserts that a subset of $L^1(\Omega,P)$ is relatively weakly compact if and only if it is bounded and uniformly integrable. If φ is continuous in probability and $\mathcal{W}$ is a weakly compact subset of $L^1(\Omega,P)$, then the elements of $\mathcal{W}$ are uniformly absolutely continuous with respect to P. The estimate above then shows that the image of $\mathcal{W}$ under T_φ is bounded and equicontinuous at each point of S, hence relatively compact [20, p. 234, Thm. 18].

Conversely, suppose φ is not continuous in probability on S, but T_φ maps $L^1(\Omega,P)$ into $C(S)$. Then there exist $\varepsilon>0$ and $s\in S$ such that for every neighborhood V of s there is an element $s_V\in V$ for which $P(E_V)\ge\varepsilon$, where $E_V=\{\omega:|\varphi(s_V,\omega)-\varphi(s,\omega)|\ge\varepsilon\}$. Let $f_V(\omega)=\overline{\operatorname{sgn}}(\varphi(s_V,\omega)-\varphi(s,\omega))$. Then f_V is uniformly bounded, hence relatively weakly compact in $L^1(\Omega,P)$. But $T_\varphi f_V$ is not equicontinuous at s, since

$$\begin{aligned}|T_\varphi f_V(s_V)-T_\varphi f_V(s)|&=|\mathbb{E}[(\varphi(s_V)-\varphi(s))f_V]|=\mathbb{E}[|\varphi(s_V)-\varphi(s)|]\\&\ge\int_{E_V}|\varphi(s_V)-\varphi(s)|\,dP\ge\varepsilon P(E_V)\ge\varepsilon^2.\end{aligned}$$

COROLLARY 2.3. *For S compact, the map $\varphi\mapsto T_\varphi$ defines a one-to-one linear map from the space of bounded functions continuous in probability on $S\times\Omega$ onto the space of operators $T:L^1(\Omega,P)\to C(S)$ which map weakly compact sets in $L^1(\Omega,P)$ to norm-compact sets in $C(S)$.*

Proof. We need only show the map $\varphi\mapsto T_\varphi$ is onto. If $T:L^1(\Omega,P)\to C(S)$ is as in the corollary, then for each $s\in S$, there is a function $\varphi(s)\in L^\infty(\Omega,P)$ such that $(Tf)(s)=\mathbb{E}[\varphi(s)f]$, $f\in L^1(\Omega,P)$. By Theorem 2.2, φ is continuous in probability, and clearly $T_\varphi=T$.

THEOREM 2.4. *Let φ_n, $n\ge1$, and φ be continuous in probability on S and uniformly bounded. Then φ_n converges uniformly in probability to φ if and only if $\|T_{\varphi_n}f-T_\varphi f\|\to0$ uniformly for f in any weakly compact subset of $L^1(\Omega,P)$.*

Proof. Assume that the φ_n and φ are uniformly bounded by $M>0$ and that φ_n converges uniformly to φ in probability. Let $\mathcal{W}$ be a weakly compact set in $L^1(\Omega,P)$, and choose $N>0$ such that $\|f\|_1\le N$ for all $f\in\mathcal{W}$. Given $\varepsilon>0$, choose $\delta>0$ such that $\int_A f\,dP<\varepsilon$ for all $f\in\mathcal{W}$ if $P(A)<\delta$. Then for all $s\in S$ and $f\in\mathcal{W}$, if we set $A_n=\{\omega:|\varphi_n(s,\omega)-\varphi(s,\omega)|\ge\varepsilon\}$, we have $P(A_n)<\delta$ for all sufficiently large n. For such n,

$$\begin{aligned}|T_{\varphi_n}f(s)-T_\varphi f(s)|&\le\int_\Omega|\varphi_n(s)-\varphi(s)|\cdot|f|\,dP\le\|f\|_1\varepsilon+2M\int_{A_n}|f|\,dP\\&\le\|f\|_1\varepsilon+2M\varepsilon\le(N+2M)\varepsilon.\end{aligned}$$

The converse proceeds by analogy with the proof of the converse in Theorem 2.2(ii); we omit the details.

3. Approximation. We now turn to the approximation questions raised in the Introduction. We begin with the analogues of Mergelyan's Theorem [21] in the current context. Throughout this section, the symbols F and K will denote, respectively, closed and compact subsets of $\mathbb{C}$. The interior, boundary, and complement of F are denoted by F°, ∂F, and F^c, respectively. Let $V_0,V_1,\ldots$ denote the components of F^c; for K compact, V_0 will be the unbounded component of K^c.

DEFINITION 3.1. The set K is called a *stochastic Mergelyan set* if for every nonatomic probability space (Ω,P), every function continuous in probability on $K\times\Omega$ and random holomorphic on $K^\circ\times\Omega$ can be approximated by random polynomials.

LEMMA 3.2. *Let φ be continuous in probability on $K\times\Omega$ and let $\varepsilon>0$. Suppose that there exist compact sets $K_1,\ldots,K_n$, pairwise disjoint measurable subsets $\Omega_1,\ldots,\Omega_n$ of Ω, and random polynomials $p_1,\ldots,p_n$ on Ω such that $K=K_1\cup\ldots\cup K_n$, $\Omega=\Omega_1\cup\ldots\cup\Omega_n$, and for all $1\le j\le n$, we have $0<P(\Omega_j)<\varepsilon$ and $P[|\varphi(z)-p_j(z)|\ge\varepsilon\mid\Omega_j]<\varepsilon$ for $z\in\bigcup_{i\ne j}K_i$. Then there is a random polynomial p on $K\times\Omega$ such that $P[|\varphi(z)-p(z)|\ge\varepsilon]<2\varepsilon$, $z\in K$.*

Proof. Let p be the random polynomial given by $p(z,\omega)=p_j(z,\omega)$, $\omega\in\Omega_j$, $j=1,\ldots,n$. For any $z\in K$, choose $j=j(z)$ such that $z\in K_j$.

Then for all $z \in K$,

$$P[|\varphi(z) - p(z)| \geq \varepsilon] = \sum_{i=1}^{n} P[|\varphi(z) - p_i(z)| \geq \varepsilon \mid \Omega_i]P(\Omega_i)$$
$$\leq P(\Omega_j) + \sum_{i \neq j} P[|\varphi(z) - p_i(z)| \geq \varepsilon \mid \Omega_i]P(\Omega_i)$$
$$< \varepsilon + \sum_{i \neq j} \varepsilon P(\Omega_i) < 2\varepsilon.$$

COROLLARY 3.3. *Suppose that for each n there exist compact sets $K_1, \ldots, K_n$ such that $K = K_1 \cup \ldots \cup K_n$ and for all $1 \leq j \leq n$, $\bigcup_{i \neq j} K_i$ is a stochastic Mergelyan set. Then K is a stochastic Mergelyan set.*

Proof. For each $\varepsilon > 0$, choose a measurable partition $\Omega_1, \ldots, \Omega_n$ of Ω with $0 < P(\Omega_j) < \varepsilon$ for all j. The hypotheses of Lemma 3.2 are then satisfied.

DEFINITION 3.4. If U and V are disjoint open sets in a topological space S, then a *strip connecting U and V* is the image of a homeomorphism h of a closed disk D of $\mathbb{C}$ into S such that $h(\partial D) \cap U \neq \emptyset$ and $h(\partial D) \cap V \neq \emptyset$. For an open set U of S, let $\mathcal{C}(U)$ denote the set of connected components of U. Then U is *connectable* (*in* S) if either U is connected or to each $V \in \mathcal{C}(U)$ there corresponds a strip S_V in S connecting V to some other element of $\mathcal{C}(U)$ such that the S_V, $V \in \mathcal{C}(U)$, are pairwise disjoint and $\bigcup\{S_V : V \in \mathcal{C}(U)\} \cup U$ is connected. We shall say that a closed set F has *finite connectivity weight* if there is a sequence $F_0 \subset F_1 \subset \ldots \subset F_m = F$ of closed sets such that F_0^c is connected, and for all $j > 0$, $V \setminus F_j$ is connectable in V for every component V of F_{j-1}^c. The least m for which these conditions hold will be called the *connectivity weight* of F.

EXAMPLE 3.5. (1) If F^c has finitely many components, then F has a connectable complement and hence connectivity weight 0 or 1. The converse is false, since compact sets with connectable complements having infinitely many components are easily constructed. For instance, the following two examples of varying geometric nature are easily verifiable.

(2) The set consisting of 0 and all circles with center 0 and radius n^{-1}, $n = 1, 2, \ldots$, has a connectable complement.

(3) Let C denote the Cantor ternary set, let $I_1, I_2, \ldots$ be the open intervals which constitute the components of C^c in [0,1], and set $K = C \cup \bigcup_{n=1}^{\infty} C_n$, where C_n is the circle with diameter I_n. Then K has a connectable complement.

(4) It is not difficult to see by induction that if $F^\circ = \emptyset$ and F has finite connectivity weight, then so does any closed subset of F.

(5) Let K denote the union of all circles of radius $1 - n^{-1}$ with center at the origin, $n = 1, 2, \ldots$, along with the unit circle and all points $re^{i\theta}$ in the closed unit disk with

$$1 - \frac{1}{n} \leq r \leq 1, \quad \theta = \frac{2\pi i k}{n}, \quad k = 0, 1, \ldots, n-1;\ n = 2, 3, \ldots$$

Then K does not have a connectable complement, since any homeomorphic image D of the unit disk which meets V_0 and some V_j with $j \geq 1$ must contain V_i for some i. But it is easy to see that K has connectivity weight 2.

THEOREM 3.6. *Let K be a compact set with finite connectivity weight and void interior. Then K is a stochastic Mergelyan set.*

Proof. Theorem 1.6 implies that φ can be approximated uniformly in probability by random continuous functions. Suppose first that K^c is connected. Then each of these random continuous functions is the uniform limit over $K \times \Omega$ of random polynomials by [5, Thm. 3.3].

Suppose K has a connectable complement. It is clear that each of the strips in Definition 3.4 can be considered as the image of a homeomorphism

$$h_j : [0,1] \times [0,1] \to S_j \subset \mathbb{C}$$

such that $h_j([0,1] \times \{0\}) \subset V_j$ and $h_j([0,1] \times \{1\}) \subset V_{r(j)}$ for some $r(j) \geq 1$. Fix n, and for each $i = 1, \ldots, n$, let

$$L_i = K \cap \bigcup_j h_j\left(\left[\frac{i-1}{n}, \frac{i}{n}\right] \times [0,1]\right).$$

Set $K_0 = [K \setminus \bigcup_j S_j]^-$. Then each of the sets $K_i = K_0 \cup L_i$, $i = 1, \ldots, n$, is closed, and for $i = 1, \ldots, n$, $\bigcup_{k \neq i} K_k$ has a connected complement and is thus a stochastic Mergelyan set. Our theorem now follows in this case from Corollary 3.3.

Finally, suppose K has connectivity weight m. Assume that our theorem is known to be valid for all sets of connectivity weight less than m, and let $K_0, \ldots, K_m$ be as in Definition 3.4. Then proceeding as in the previous argument simultaneously on all of the components of K_{m-1}^c, we can again apply Corollary 3.3, this time to finite unions of sets of connectivity weight at most $m - 1$, to conclude that K is a stochastic Mergelyan set.

DEFINITION 3.7. By a *closed Jordan region* we shall mean a (finitely connected) compact subset K of $\mathbb{C}$ such that K° is connected, K is the closure of K°, and the boundary of K is the disjoint union of finitely many Jordan curves.

THEOREM 3.8. *Let K be a finite disjoint union of closed Jordan regions. Then K is a stochastic Mergelyan set.*

Proof. If K is simply connected, this is [1, Thm. 3.2].

Suppose K is the disjoint union of simply connected closed Jordan regions $K_1, \ldots, K_n$. Let φ be continuous in probability on K and random holomorphic on K°. Given $\varepsilon > 0$, for each $j = 1, \ldots, n$ there is a random polynomial p_j on $K_j \times \Omega$ such that $P[|\varphi(z) - p_j(z)| \geq \varepsilon] < \varepsilon$, $z \in K_j$. Let $\psi(z) = p_j(z)$, $z \in K_j$, $j = 1, \ldots, n$. Then ψ and K satisfy the hypotheses of [5, Thm. 3.3], so there is a random polynomial p on K such that $P[|\psi(z) - p(z)| \geq \varepsilon] < \varepsilon$, and hence $P[|\varphi(z) - p(z)| \geq 2\varepsilon] < 2\varepsilon$, $z \in K$.

Now let K be a closed Jordan region. Then K has connectivity weight either zero or one, since it is finitely connected. And the removal from K of the interiors of any finite collection of strips connecting the components of K^c leaves a finite union of simply connected closed Jordan regions, to which the argument in the proof of Theorem 3.6 applies.

Finally, if K is the disjoint union of finitely many closed Jordan regions $K_1, \ldots, K_n$, then there is a disjoint collection of open sets $U_1, \ldots, U_n$ such that $K_j \subset U_j$, $1 \leq j \leq n$. Hence K has connectivity weight at most one.

THEOREM 3.9. *Let φ be a function on $K \times \Omega$ that is continuous in probability on K and random holomorphic on K°. If the restriction of φ to the boundary of K can be approximated by random polynomials, then so can φ itself.*

Proof. By hypothesis, given $\varepsilon > 0$, there exits a random polynomial p such that $P[|\varphi(z) - p(z)| \geq \varepsilon/2] < \varepsilon/2$, $z \in \partial K$. Since K is compact, $\varphi - p$ is uniformly continuous in probability on K. Hence there is a neighborhood V of ∂K such that $P[|\varphi(z) - p(z)| \geq \varepsilon] < \varepsilon$, $z \in V \cap K$. Assuming that $K^\circ \neq \emptyset$, let $K' = K \backslash V$, and consider K' as a compact subset of K°. It is well known that $V \cap K^\circ$ contains a finite collection $\gamma_1, \ldots, \gamma_m$ of oriented Jordan curves, each consisting of a finite union of line segments parallel to the axes, which define a finitely connected region—a finite union of disjoint, closed Jordan regions—which contains K' (e.g., see [25, p. 269]). For $1 \leq j \leq m$, let γ_j' be a Jordan curve in $V \cap K^\circ$ consisting of segments parallel to those of γ_j, oriented in the same direction as γ_j, such that $\gamma_j \cap \gamma_j' = \emptyset$, $j = 1, \ldots, m$, and the closed regions defined by $\gamma_1', \ldots, \gamma_m'$ contain those defined by $\gamma_1, \ldots, \gamma_m$. Let K_j be the closed annular set bounded by γ_j' and $-\gamma_j$.

Let D be a closed disk whose interior contains K. The set

$$L = K \setminus (K_1^\circ \cup \ldots \cup K_m^\circ)$$

consists of the finite union L_1 of closed Jordan regions contained in K° and bounded by $\gamma_1, \ldots, \gamma_m$, along with the intersection with K of the finite union L_2 of closed Jordan regions defined by $\partial D, -\gamma_1', \ldots, -\gamma_m'$; in particular, $L_2 \cap K \subset V$. Thus $L^* = L_1 \cup L_2$ is a finite disjoint union of closed Jordan regions, hence a stochastic Mergelyan set by Theorem 3.8, and $L_1 \cap L_2 = \emptyset$.

Let

$$\varrho(z) = \begin{cases} \varphi(z), & z \in L_1, \\ p(z), & z \in L_2. \end{cases}$$

Then ϱ is continuous in probability on L^* and random holomorphic on $L^{*\circ}$, so there is a random polynomial q such that $P[|\varrho(z) - q(z)| \geq \varepsilon] < \varepsilon$, $z \in L^*$, and hence $P[|\varphi(z) - q(z)| \geq 2\varepsilon] < 2\varepsilon$, $z \in L$.

For $j = 1, \ldots, m$, and for each positive integer n, K_j can be written as the union of n closed annular regions K_{ji}, $1 \leq i \leq n$, each bounded by curves parallel to γ_j and γ_j'. By the argument above, for each $i = 1, \ldots, n$ there is a random polynomial q_i such that

$$P[|\varphi(z) - q_i(z)| \geq \varepsilon] < \varepsilon, \quad z \in K \setminus \bigcup_{j=1}^{m} K_{ji}^\circ = K \setminus \Big(\bigcup_{j=1}^{m} K_{ji}\Big)^\circ.$$

Our theorem now follows from Lemma 3.2.

Combining Theorems 3.6 and 3.9, we have the following.

THEOREM 3.10. *If the boundary of K has finite connectivity weight, then K is a stochastic Mergelyan set.*

For an infinite compact set K in $\mathbb{C}$, let $A(K)$ and $R(K)$ denote the spaces of all continuous functions on K which are holomorphic in K° (if it is nonvoid) and which are uniform limits over K of restrictions of rational functions with poles in K^c, respectively. Recall that the celebrated theorems of A. G. Vitushkin [29], [30] (cf. [14, Chap. 5], [31]) provide necessary and sufficient conditions for a compact set K in $\mathbb{C}$ to have the property that $R(K) = A(K)$. As we shall see, this property also has implications for the problem of random polynomial approximation.

The characterizations developed by Vitushkin involve notions of *capacity*, and we shall not reproduce them here. Let us cite, however, several consequences of and motivating results for Vitushkin's work:

(1) (Mergelyan) If every point of ∂K lies in the boundary of a component of K^c, then $R(K) = A(K)$. In particular, if K^c has finitely many components, then $R(K) = A(K)$.

(2) (Hartogs–Rosenthal) If K has planar measure zero, then $R(K) = C(K)$.

(3) (Garnett) Let F be the set of all points in K each neighborhood of which intersects infinitely many components of K^c. If F is countable, then $R(K) = A(K)$.

(4) (Mergelyan) If the diameters of the components of K^c are bounded away from zero, then $R(K) = A(K)$.

(5) If $R(K) = A(K)$, then $R(\partial K) = C(\partial K)$, but not conversely.

By a *random rational function* on $K \times \Omega$ we mean a random element, in the sense of Definition 1.1, of the class $\mathcal{F}$ of restrictions to K of rational functions. In this case we can take X to be the Riemann sphere $\mathbb{C}^*$ and $\mathcal{E}$ to be its Borel σ-field. For a countable subset Z of K^c, let $R_Z(K)$ denote the algebra of functions on K which are the restrictions of rational functions with poles in Z.

LEMMA 3.11. *Let $r = r(z,\omega)$ be a random element of $R_Z(K)$. For each ω, also denote by $r = r(\cdot,\omega)$ the unique extension to $\mathbb{C}$ of the given random function r on K. Then r is a random rational function on $\mathbb{C}$.*

Proof. Let z_0 be a cluster point of K. If f is a random function on K which is a.s. holomorphic at z_0, and we write $f(z) = \sum_{k=0}^{\infty} a_k(z-z_0)^k$, then each of the coefficients a_k is $\mathcal{A}$-measurable. Indeed, $a_0 = f(z_0)$ is measurable. If $z_m \in K$ such that $z_m \to z_0$, and if $a_0, \ldots, a_{n-1}$ are known to be measurable, set $g(z) = f(z) - \sum_{k=0}^{n-1} a_k(z-z_0)^k$. Then $g(z)$ is a measurable function for all $z \in K$, and

$$\lim_{n\to\infty} \frac{g(z_m) - g(z_0)}{(z_m - z_0)^n} = a_n,$$

so a_n is measurable.

For each monic polynomial q all of whose roots lie in Z (including $q \equiv 1$), it follows from the remark above that

(1) the set

$$E_q = \{\omega : q(\cdot)r(\cdot,\omega) \text{ is a polynomial}\}$$

is measurable, and

(2) the function $q(z)r(z,\cdot)$ is a (relatively) measurable function on E_q for all $z \in \mathbb{C}$.

The set of all such q is countable, so by choosing, for each $\omega \in \Omega$, the monic polynomial of lowest degree such that $q(\cdot)r(\cdot,\omega)$ is a polynomial, we can write Ω as a disjoint union, $\Omega = \bigcup_{j=1}^{\infty} E_j$, where $E_j \subset E_{q_j}$ for some monic polynomial q_j as above.

Let $z \in \mathbb{C}$. If $q_j(z) = 0$, then $r(z,\omega) \equiv \infty$, $\omega \in E_j$. If $q_j(z) \neq 0$, then $q_j(z)r(z,\omega)$ is measurable on E_j, so $r(z)$ is measurable on E_j. Hence $r(z)$ is measurable for all $z \in \mathbb{C}$.

LEMMA 3.12. *Let Z be a countable subset of K^c. Every random element of $R_Z(K)$ can be approximated by random polynomials.*

Proof. Identify a rational function with its restriction to K. Let $r = r(z,\omega)$ be a random element of $R_Z(K)$. For any subset W of Z, Lemma 3.11 implies that $\Delta_W = \{\omega : r(\cdot,\omega)^{-1}(\infty) \subset W\} \in \mathcal{A}$. Thus, given $\varepsilon > 0$, there is a finite subset F of Z such that $P(\Delta_F^c) < \varepsilon$. Let us assume that $F \neq \emptyset$ and write $F = \{z_1, \ldots, z_n\}$. Let $D_1, \ldots, D_n$ be pairwise disjoint, closed disks such that $z_i \in D_i$ and $D_i \cap K = \emptyset$, $i = 1, \ldots, n$. Let D be a sufficiently large, closed disk so that $K \subset D$ and $D_i \subset D^\circ$, $i = 1, \ldots, n$. Set $K' = D \setminus \bigcup_{i=1}^n D_i^\circ$. Then $K \subset K'$, K' is a closed Jordan region, and the restriction of r to $K' \times \Delta_F$ is a random continuous function which is random holomorphic on $(K')^\circ \times \Delta_F$. By Theorem 3.8, there is a random polynomial p' on $K' \times \Delta_F$ such that for all $z \in K'$, $P[|r(z) - p'(z)| \geq \varepsilon \mid \Delta_F] < \varepsilon$. Letting $p(z,\omega) = p'(z,\omega)$ for $\omega \in \Delta_F$ and $p(z,\omega) = 0$ for $\omega \in \Delta_F^c$, we have for all $z \in K'$, so *a fortiori* on K,

$$P[|r(z) - p(z)| \geq \varepsilon] \leq P[|r(z) - p'(z)| \geq \varepsilon \mid \Delta_F]P(\Delta_F) + P(\Delta_F^c) < 2\varepsilon.$$

The following lemma is the analogue in the current context of [5, Thm. 3.3].

LEMMA 3.13. *Let K be a compact set for which $R(K) = A(K)$, and let φ be a random element of $A(K)$. Let Z consist of one element chosen from each component of K^c, with ∞ chosen from V_0. For every $\varepsilon > 0$, there is a random element r of $R_Z(K)$ such that $\|\varphi - r\|_K < \varepsilon$ a.s.*

Proof. Recall that the hypothesis on K and Runge's Theorem ([26], [25, Thm. 13.6]) imply that every element of $A(K)$ is the uniform limit over K of elements of $R_Z(K)$. By [5, Prop. 1.1, Lemma 3.2], there is a subset Ω' of Ω such that $P(\Omega') = 1$ and

$$H = \{(r,\omega) : r \in R_Z(K),\ \omega \in \Omega',\ \|\varphi(\cdot,\omega) - r\|_K < \varepsilon\}$$

is product-measurable with respect to the restrictions of $\mathcal{A}$ to Ω' and the Borel σ-field of $C(K)$ to $R_Z(K)$. By hypothesis, $H(\omega) = \{r : (r,\omega) \in H\} \neq \emptyset$ for all ω. Our lemma now follows from the selection theorem [17, Thm. 5.7], once we demonstrate that $R_Z(K)$ is a Souslin space.

Now, the space $\mathcal{P}(K)$ of polynomial functions on K is a Souslin space as the countable union of finite-dimensional spaces. Similarly, $q^{-1}\mathcal{P}(K)$ is a Souslin space for each monic polynomial q all of whose roots lie in Z. The set Q of all such polynomials q is countable, and $R_Z(K) = \bigcup_{q\in Q} q^{-1}\mathcal{P}(K)$. Thus $R_Z(K)$ is Souslin, and the proof is complete.

THEOREM 3.14. *If $R(K) = C(K)$, then K is a stochastic Mergelyan set.*

Proof. Let φ be continuous in probability on $K \times \Omega$. By Theorem 1.6, we may assume φ is a random continuous function. Since our hypothesis implies that $K^\circ = \emptyset$, by Lemma 3.13 there is a countable set Z and a sequence of random elements of $R_Z(K)$ converging uniformly in probability to φ. Our theorem now follows from Lemma 3.12.

The notion of connectivity weight was used to prove Theorem 3.14. On the other hand, Theorem 3.14 implies Theorem 3.6, as shown by the following proposition. We are grateful to P. de Paepe for suggesting the utilization of the Bishop Splitting Lemma in the manner in which it appears below.

PROPOSITION 3.15. *Let K be a compact set with finite connectivity weight and empty interior. Then $R(K) = C(K)$.*

Proof. If the connectivity weight m of K is 0, then Mergelyan's Theorem says $R(K) = C(K)$.

Suppose $R(K') = C(K')$ for all K' of connectivity weight less than m. There is a sequence S_j of strips connecting components of K^c such that $K' = K\setminus\bigcup_j S_j^o$ has connectivity weight at most $m-1$.

Let μ be a measure on K which annihilates $R(K)$. Since the S_j are disjoint, no component of K^c can be contained in any S_j^o. So if $z \in S_j^o$, there is a neighborhood U of z such that the complement of $\overline{U} \cap K$ is connected. If V is any open set not containing z such that $K \subset U \cup V$, then the Bishop Splitting Lemma [14, Chap. II, Lemma 10.2] implies that we may write $\mu = \mu_1 + \mu_2$, where the closed supports of μ_1 and μ_2 are contained in U and V, respectively, μ_1 annihilates $R(\overline{U}\cap K)$, and μ_2 annihilates $R(\overline{V^c}\cap K)$. But $R(\overline{U} \cap K) = C(\overline{U} \cap K)$, so $\mu_1 = 0$; i.e., $\mu = \mu_2$. By the choice of z, we see that μ is supported on K' and annihilates $R(K')$, which by hypothesis implies that $\mu = 0$, and the proposition has been proven.

Applying Theorems 3.14 and 3.9, we obtain the following:

THEOREM 3.16. *Let K be a compact set such that $R(\partial K) = C(\partial K)$. Then K is a stochastic Mergelyan set. In particular, any set K for which $R(K) = A(K)$ is a stochastic Mergelyan set.*

Remarks 3.17. (i) For examples of sets K for which $R(K) \neq A(K)$ but $R(\partial K) = C(\partial K)$, see the three examples in [14, Chap. VIII, Sec. 9].

(ii) It remains open to determine whether there exist any compact sets in $\mathbb{C}$ that are not stochastic Mergelyan sets. In searching for such sets, in light of Theorem 3.16, it suffices to examine sets K with empty interior such that $R(K) \neq C(K)$. The natural first candidates are the appropriate "Swiss cheese" sets [14, pp. 25–26].

(iii) The assumption that φ be a random holomorphic function on K^o is not necessary for φ to be approximable on K by random polynomials. Indeed, there is an example in [1] of an approximable random function φ on the closed unit disk such that for each ω, $\varphi(\cdot,\omega)$ is only holomorphic off a line segment in the open disk. Lemma 3.2 could be invoked to allow us to assume in Theorems 3.9 and 3.16 that each $\varphi(\cdot,\omega)$ is holomorphic on an appropriate subset of K that depends on ω. In fact, the argument used to develop the example of Andrus and Nishiura is a prototype of the one used in proving Lemma 3.2. It would be interesting to determine the degree of analyticity

inherent in a function φ that is approximable by random polynomials on K, when K is the closure of its interior.

In a similar way, one can formulate analogues of the classical theorem of Arakelyan [2] in the current context. Let $\mathcal{H}$ denote the space of all entire functions. Recall that a subset A of $\mathbb{C}$ is *locally connected at infinity* if for every neighborhood U of ∞ in $\mathbb{C}$ there exists a neighborhood V of ∞ contained in U such that $A \cap V$ is contained in the connected component of ∞ in $A \cap U$. Corresponding to Theorem 3.6 we have the following:

THEOREM 3.18. *Let F be a closed set with finite connectivity weight and void interior. Then every function continuous in probability on F can be approximated by random elements of $\mathcal{H}$.*

Proof. Let f be continuous in probability on F. By Theorem 1.6, we may assume that f is a random continuous function on F.

Suppose first that F^c is connected. Let $\{W_n\}$ be any sequence of open annuli centered at the origin with disjoint closures and inner radii $\{r_n\}$ tending to infinity. Set $E = F \cap \bigcap_{n=1}^{\infty} W_n^c$. Then E^c is connected and locally connected at infinity. Indeed, F^c is dense in $\mathbb{C}$, so $W_n \cap F^c \neq \emptyset$ for all n, hence E^c is connected. And if U is any neighborhood of ∞ in $\mathbb{C}$, then $V = \{z : |z| > r_{n_0}\} \subset U$ for some n_0. Any connected component of $V \cap E^c$ which is disjoint from W_{n_0} would be an open, connected set whose closure is contained in V, hence a connected component of E^c. Thus every component of $V \cap E^c$ meets W_{n_0}. Hence $V \cap E^c$ is connected and unbounded, so it lies in the connected component of ∞ in $E^c \cap U$.

As pointed out on p. 121 of [5], one may argue as in [5, Sec. 3] to conclude that there is a sequence of random elements of $\mathcal{H}$ that converge to f a.s. If we vary the choice of the sequence $\{W_n\}$ and modify Lemma 3.2 and Corollary 3.3 (replacing compact sets by closed sets and polynomials by entire functions), we may proceed as in the proof of Theorem 3.6 to conclude that our theorem holds if F^c is connected: For each k and n, choose n disjoint, open, concentric annuli such that the union of their closures is $\overline{W}_k$. The remainder of the proof then proceeds like that of Theorem 3.6.

4. Extension.

Given a stochastic process $\varphi = \varphi(s,\omega)$ continuous in probability over a parameter space S, and a space $T \supset S$, when can φ be extended to a process $\widetilde{\varphi}$ continuous in probability on T? One can also add the requirement that the extended sample functions $\widetilde{\varphi}(\cdot,\omega)$ lie in some prescribed space of functions on T. The deterministic version of this question has a long history (see [3], [9], [19], [22], [23], [24], [27]). Our approach to this problem relies on the operators T_φ introduced in Section 2 and results of A. Pełczyński on linear operators of extension.

We begin with the following definitions from [24]. In the sequel, all maps will be assumed to be continuous. If $h : S \to T$, then $h^\circ : C(T) \to C(S)$ is the canonical induced homomorphism, i.e., $h^\circ(f) = f \circ h$.

DEFINITION 4.1. Let S and T be compact spaces and $h : S \to T$ be injective. An *extension operator* for the triple (h, S, T) is a bounded operator $u : C(S) \to C(T)$ such that $h^\circ u(f) = f$ for $f \in C(S)$, i.e., $h^\circ u h^\circ = h^\circ$. If X is a closed subspace of $C(T)$ and u is an extension operator such that $u(f) \in X$ for all $f \in C(S)$, then we say u is an *extension operator taking values in* X.

DEFINITION 4.2. A compact space S is an *L-extensor* if for every compact space T and injection $h : S \to T$, the triple (h, S, T) admits an extension operator.

Remarks 4.3. (i) As shown in [24], a compact space S is an L-extensor if and only if there is an injection $h : S \to [0,1]^m$, for some cardinal m, such that the triple $(h, S, [0,1]^m)$ admits an extension operator.

(ii) Examples of L-extensors include compact metric spaces, compact absolute neighborhood retracts, and any cartesian product of L-extensors.

(iii) In [24], L-extensors are called "almost Dugundji spaces". The term "Dugundji space" in [24] is reserved for those L-extensors for which the extension operators u can be chosen so that $u(f) \geq 0$ whenever $f \geq 0$. The term "L-extensor" appears in [27]. Remark (ii) remains valid if "L-extensors" is replaced by "Dugundji spaces".

We shall need the following analogue of Tietze's Extension Theorem.

THEOREM 4.4. *Let S be an L-extensor and $h : S \to T$ be an injection of compact spaces. If φ is bounded and continuous in probability on $S \times \Omega$, then there exists a bounded function $\widetilde{\varphi}$ continuous in probability on $T \times \Omega$ such that $\widetilde{\varphi} \circ (h \times \mathrm{id}_\Omega) = \varphi$.*

Proof. Let u be an extension operator for the triple (h, S, T), and let T_φ be the operator defined in Definition 2.1. Set $Uf = u(T_\varphi f)$, $f \in L^1(\Omega, P)$. Then the operator $U : L^1(\Omega, P) \to C(T)$ maps weakly compact subsets of $L^1(\Omega, P)$ to strongly precompact sets in $C(T)$. For $t \in T$, the functional $f \mapsto Uf(t)$ is bounded, so there exists $\widetilde{\varphi}(t) \in L^\infty(\Omega, P)$ such that $\|\widetilde{\varphi}(t)\| \leq \|U\|$, $t \in T$, and $Uf(t) = \int_\Omega f \widetilde{\varphi}(t)\, dP$, $f \in L^1(\Omega, P)$.

Clearly, $T_{\widetilde{\varphi}} = U$, so $\widetilde{\varphi}$ is continuous in probability on $T \times \Omega$, by Theorem 2.2. If $s \in S$, then $Uf(h(x)) = u((T_\varphi f) \circ h(x)) = T_\varphi f(x)$. Hence $\widetilde{\varphi} \circ h(s) = \varphi(s)$ a.s., $s \in S$.

A corollary to Theorem 4.4 and Corollary 1.8 is the following result on approximability by random continuous functions (in the sense of Definition 1.3).

THEOREM 4.5. *If S is an L-extensor, then every function continuous in probability on $S \times \Omega$ can be approximated by random continuous functions.*

Proof. In accordance with Remark 4.3(i), let $h : S \to [0,1]^m$ be an injection, for some cardinal m, and let φ be continuous in probability on $S \times \Omega$. Proposition 1.4 says that we may assume that φ is bounded on $S \times \Omega$. Then by Theorem 4.4, there is a function $\widetilde{\varphi}$ continuous in probability on $[0,1]^m \times \Omega$ such that $\widetilde{\varphi} \circ (h \times \mathrm{id}_\Omega) = \varphi$. We may now choose, in accordance with Corollary 1.8, a sequence $\widetilde{\varphi}_n$ of random continuous functions on $[0,1]^m$ converging uniformly in probability to $\widetilde{\varphi}$. The random continuous functions $\varphi_n = \widetilde{\varphi}_n \circ (h \times \mathrm{id}_\Omega)$ then converge uniformly in probability to φ on S.

When X is a Dirichlet subalgebra of $C(T)$, Pełczyński obtained the following interesting generalization of the well-known Rudin–Carleson Theorem:

THEOREM 4.6 [23]. *Let $h : S \to T$ be an injection of compact spaces and X be a Dirichlet subalgebra of $C(T)$. Then an extension operator for the triple (h, S, T) with values in X and norm one exists if and only if $\{f \circ h : f \in X\} = C(S)$.*

COROLLARY 4.7 [23]. *Let S be a closed subset of Lebesgue measure zero of the unit circle Γ, and denote by A the disk algebra, considered as an algebra of functions on Γ. Then there exists an extension operator $u : C(S) \to A$ with $\|u\| = 1$.*

Arguing as in the proof of Theorem 4.4 and using Theorem 4.6, we obtain the following analogue of Theorem 4.6 for functions continuous in probability:

THEOREM 4.8. *Let $h : S \to T$ be an injection of compact spaces and X be a Dirichlet subalgebra of $C(T)$ such that $\{f \circ h : f \in X\} = C(S)$. If φ is bounded and continuous in probability on $S \times \Omega$, then there exists ψ bounded and continuous in probability on $T \times \Omega$ such that $\psi \circ (h \times \mathrm{id}_\Omega) = \varphi$ and $T_\psi f \in X$, $f \in L^1(\Omega, P)$.*

COROLLARY 4.9. *Let S, A, and Γ be as in Corollary 4.7, and let φ be bounded and continuous in probability on $S \times \Omega$. Then there is a bounded function ψ continuous in probability on $\Gamma \times \Omega$ such that:*

(i) $T_\psi f \in A$, $f \in L^1(\Omega, P)$,

(ii) $\psi(\cdot, \omega) \in H^\infty$ *a.s.*

Proof. Let ψ be the extension of φ whose existence is asserted by Theorem 4.8 with $X = A$, so that (i) holds. By [7, Chap. 2, Thm. 2.6] we may assume that ψ is measurable on $\Gamma \times \Omega$. Then for all $f \in L^1(\Omega, P)$ and $n = 0, 1, \ldots,$

$$0 = \int_{\Gamma} T_{\psi} f(z) z^n \, dz = \int_{\Gamma} \int_{\Omega} \psi(z,\omega) f(\omega) z^n \, dP(\omega) \, dz$$
$$= \int_{\Omega} \Big[\int_{\Gamma} \psi(z,\omega) z^n \, dz \Big] f(\omega) \, dP(\omega).$$

Hence for each n, $\int_{\Gamma} \psi(z,\omega) z^n \, dz = 0$ a.s. But then this holds a.s. simultaneously for all $n \geq 0$, so (ii) follows.

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