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On the uniform ergodic theorem in Banach spaces that do not contain duals

by

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Abstract. Let T be a power-bounded linear operator in a real Banach space X. We study the equality

$$(*)\qquad (I-T)X = \Big\{z \in X : \sup_n \Big\| \sum_{k=0}^{n} T^k z \Big\| < \infty \Big\}.$$

For X separable, we show that if T satisfies $(*)$ and is not uniformly ergodic, then $\overline{(I-T)X}$ contains an isomorphic copy of an infinite-dimensional dual Banach space. Consequently, if X is separable and does not contain isomorphic copies of infinite-dimensional dual Banach spaces, then $(*)$ is equivalent to uniform ergodicity. As an application, sufficient conditions for uniform ergodicity of irreducible Markov chains on the (positive) integers are obtained.

1. Introduction. Von Neumann's mean ergodic theorem (e.g., [K, p. 4]) led to the study of operator ergodic theory: For $T : X \to X$ a linear operator in a Banach space X, study the convergence of the averages $n^{-1}\sum_{k=1}^{n} T^k$ (in the strong operator topology). The following results are well known [K, p. 73] for T power-bounded (i.e., $\sup_{n>0}\|T^n\| < \infty$—an assumption made throughout this note):

$$(1.1)\qquad \Big\{z \in X : n^{-1}\sum_{k=1}^{n} T^k z \to 0\Big\} = \overline{(I-T)X},$$

$$(1.2)\qquad \Big\{x \in X : \lim n^{-1}\sum_{k=1}^{n} T^k x \text{ exists}\Big\} = \{y \in X : Ty = y\} \oplus \overline{(I-T)X}.$$

We shall denote $\{y \in X : Ty = y\}$ by F. It follows from (1.2) that $F \oplus \overline{(I-T)X}$ is closed. Using the power-boundedness for the first inclu-

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sion below and (1.1) for the second, we obtain

$$(1.3)\qquad (I-T)X \subset \Big\{z\in X : \sup_n \Big\|\sum_{k=0}^{n} T^k z\Big\| < \infty\Big\} \subset \overline{(I-T)X}.$$

If the averages $n^{-1}\sum_{k=1}^{n} T^k$ converge in the strong operator topology (i.e., the subspace in (1.2) is all of X), we call T *mean ergodic* (ME). T is called *uniformly ergodic* (UE) if the averages converge in the uniform operator topology (i.e., in operator norm).

THEOREM 1.1. *Let T be power-bounded in X. Then the following are equivalent*:

(i) *T is uniformly ergodic.*
(ii) *$(I-T)X$ is closed.*
(iii) *$I-T$ is an isomorphism of $\overline{(I-T)X}$.*
(iv) *$\{z\in X : \sup_n \|\sum_{k=0}^{n} T^k z\| < \infty\}$ is closed.*

The equivalence of the first 3 conditions is in [L$_2$], and (ii) implies (iv) by (1.3). (iv) implies equality in the second inclusion of (1.3), so we can apply Corollary 1 of [L$_2$] to obtain (i).

Define $X_0 = \{z\in X : \sup_n \|\sum_{k=0}^{n} T^k z\| < \infty\}$. Thus, a power-bounded T is uniformly ergodic if and only if $X_0 = \overline{(I-T)X}$, and then also $X_0 = (I-T)X$ (so we have equalities in both inclusions of (1.3)). An example of T mean ergodic for which both inclusions of (1.3) are strict is given in [LS].

Butzer and Westphal [BuW] proved that for X reflexive and T power-bounded,

$$(*)\qquad (I-T)X = \Big\{z\in X : \sup_n \Big\|\sum_{k=0}^{n} T^k z\Big\| < \infty\Big\}.$$

This was extended in [L$_1$]: If X is a dual Banach space and T is a dual operator, then $(*)$ holds (independently of whether T is UE or not). Equality $(*)$ also holds for any contraction in $L_1(\mu)$ of a probability space [LS], for irreducible Markov operators in $C(K)$ [KoL], and for Markov transition operators on the space of bounded measurable functions [KoL]. Thus, in general, $(*)$ does not imply mean ergodicity, so $(*)$ and mean ergodicity are not comparable, nor do both of them together imply uniform ergodicity.

In this paper we prove that equality $(*)$ is equivalent to $(I-T)X$ being an F_σ set, and also to $\overline{(I-T)U} \subset (I-T)X$, where U is the unit ball of X. This result is used to show that for X separable, if T satisfies $(*)$ and is not uniformly ergodic, then $\overline{(I-T)X}$ contains an isomorphic copy of an infinite-dimensional dual Banach space. Consequently, if X is separable and does not contain an isomorphic copy of an infinite-dimensional dual Banach space, then $(*)$ is equivalent to uniform ergodicity of T (in contrast to the above mentioned result of [L$_1$]). As an application, we prove that an irreducible Markov matrix which preserves c is uniformly ergodic if and only if it does not preserve c_0.

2. Condition $(*)$ and the image of the unit ball under $I-T$. Throughout this section, T is a power-bounded linear operator in the real Banach space X, with $\sup_{n\ge 0}\|T^n\| = C$. The closed unit ball of X will be denoted by U, and we define $V_1 = \{z\in X : \sup_n \|\sum_{k=0}^{n} T^k z\| \le 1\}$.

PROPOSITION 2.1. $V_1 \subset \overline{(I-T)U} \subset (C+1)V_1$.

Proof. (i) Let $z\in\overline{(I-T)U}$. Then there is a sequence x_j with $\|x_j\|\le 1$ such that $z=\lim_j (I-T)x_j$. For $x^*\in X^*$ we have

$$|\langle x^*, z\rangle| = \lim_j |\langle x^*, (I-T)x_j\rangle| = \lim_j |\langle (I-T^*)x^*, x_j\rangle| \le \|(I-T^*)x^*\|.$$

Hence for every n we have

$$\Big|\Big\langle x^*, \sum_{k=0}^{n} T^k z\Big\rangle\Big| = \Big|\Big\langle \sum_{k=0}^{n} T^{*k}x^*, z\Big\rangle\Big| \le \Big\|(I-T^*)\sum_{k=0}^{n} T^{*k}x^*\Big\|$$
$$= \|(I-T^{*(n+1)})x^*\| \le (C+1)\|x^*\|.$$

Since this holds for any $x^*\in X^*$, we conclude that $\|\sum_{k=0}^{n} T^k z\| \le C+1$. Hence $z\in (C+1)V_1$.

(ii) Let $z\in V_1$. Assume $z\notin\overline{(I-T)U}$. By the separation theorem, there exist a functional $x^*\in X^*$ and a number α such that $\langle x^*, z\rangle > \alpha$ and $\sup_{x\in U}\langle x^*, (I-T)x\rangle \le \alpha$. Hence $\|(I-T^*)x^*\| = \sup_{x\in U}\langle x^*, (I-T)x\rangle \le \alpha$. Clearly $\alpha\ge 0$. If $\alpha = 0$, then $T^*x^* = x^*$, so $\langle x^*, z\rangle = 0$ (since $\|n^{-1}\sum_{k=0}^{n} T^k z\|\to 0$)—a contradiction. Hence $\alpha>0$. Since $\sum_{k=0}^{n} T^k z\in U$, we have

$$\alpha \ge \Big\langle x^*, (I-T)\sum_{k=0}^{n} T^k z\Big\rangle = \langle x^*, (I-T^{n+1})z\rangle$$

for any $n\ge 0$. Averaging yields $\alpha \ge \langle x^*, z - N^{-1}\sum_{k=1}^{N} T^k z\rangle \to \langle x^*, z\rangle$, a contradiction. Hence $z\in\overline{(I-T)U}$.

COROLLARY 2.2. $\{z\in X : \sup_n \|\sum_{k=0}^{n} T^k z\| < \infty\} = \bigcup_{n>0} n\overline{(I-T)U}$.

THEOREM 2.3. *The following are equivalent for a power-bounded operator T in a Banach space X with unit ball U*:

(i) $\{z\in X : \sup_n \|\sum_{k=0}^{n} T^k z\| < \infty\} = (I-T)X$.
(ii) *$(I-T)X$ is an F_σ set.*
(iii) $\overline{(I-T)U} \subset (I-T)X$.

Proof. (i) implies (ii) by the previous corollary.

(ii)⇒(i). Let F_n be closed sets with $(I-T)X = \bigcup_n F_n$. Define $A = I-T$. Then $X = \bigcup_n A^{-1}(F_n)$, and by the Baire category theorem there exist $x_0 \in X$ and $r > 0$ such that $x_0 + rU \subset \overline{A^{-1}(F_m)}$ for some m. Hence $A(x_0+rU) \subset F_m$, and since F_m is closed, $\overline{A(U)} \subset r^{-1}(F_m - Ax_0)$. By Proposition 2.1, $V_1 \subset \overline{A(U)} \subset r^{-1}(F_m - Ax_0) \subset A(X)$. Since X_0 is the linear span of V_1, we have $X_0 \subset (\underline{I-T})X$. Together with (1.3) we obtain (i).

(i)⇒(iii). By Proposition 2.1, $\overline{A(U)} \subset X_0$, and $X_0 = (I-T)X$ is assumed in (i).

(iii)⇒(i). $V_1 \subset \overline{A(U)} \subset A(X)$ by Proposition 2.1 and (iii). Hence (i) holds.

Remarks. 1. If X is a dual space (with its dual norm), and T is a dual operator, then $(I-T)U$ is closed, by weak* compactness of U. In particular, if X is reflexive, then $(I-T)U'$ is closed for the unit ball U' of any equivalent norm.

2. We show below that uniform ergodicity does not imply that $(I-T)U$ is closed. Hence $(*)$ does not imply it.

3. The result is also true in complex Banach spaces. The modifications of the proof of Proposition 2.1 are obvious.

THEOREM 2.4. *Let $Y \neq \{0\}$ and Z be closed subspaces of a Banach space X such that $X = Y \oplus Z$, and let P be the corresponding projection onto Y. The image under P of the unit ball of any equivalent norm on X is closed if and only if Z is reflexive.*

Proof. (i) We assume that Z is reflexive, and fix a norm in X. Let $\|x_n\| \le 1$ with $Px_n \to y_0$ (clearly $y_0 \in Y$). Set $y_n = Px_n$ and $z_n = x_n - y_n$. Since $\{z_n\}$ is a bounded subset of the reflexive space Z, by the Eberlein–Shmulian theorem it has a weakly convergent subsequence, and by passing to the corresponding indices we may assume z_n weakly convergent, say to $z_0 \in Z$. Thus, $x_n \to y_0+z_0$ weakly. Clearly $\|y_0+z_0\| \le 1$, and $P(y_0+z_0) = y_0$.

(ii) Assume now Z is not reflexive. We have to find an equivalent norm with unit ball V such that $P(V)$ is not closed. For the norms $\|\cdot\|_Y$, $\|\cdot\|_Z$ on Y and Z respectively, we define the norm $\|x\| = \max\{\|Px\|_Y, \|(I-P)x\|_Z\}$ in X, and denote its unit ball by U. In this equivalent norm, $\|P\| = 1$.

Since Z is not reflexive, it contains a sequence of norm 1 vectors which is not weakly conditionally compact. Applying [Si, p. 53], we obtain in Z a basic sequence $\{z_n\}$ with $\|z_n\| = 1$ and a constant $a > 0$ such that for any finitely many $\alpha_1, \ldots, \alpha_m \ge 0$ we have $\|\sum_{j=1}^m \alpha_j z_j\| \ge a\sum_{j=1}^m \alpha_j$. By assumption there is $y \in Y$ with $\|y\| = 2$; define $y_n = \frac{n}{n+1}y$. We now define

$$V = \text{closed co}(\{\pm(y_n+z_n)\}_{n\ge1} \cup U).$$

Clearly V is a bounded closed convex symmetric body, so by a standard result (e.g., [BoR, p. 157]) there exists an equivalent norm on X with V as unit ball. We now show that $P(V)$ is not closed. Assume that it is. Then by definition, $y = \lim_n y_n = \lim_n P(y_n+z_n) \in P(V)$, so there exists $x \in V$ with $Px = y$. By the definition of V, there are

$$w_m \in U, \quad v_m \in \text{co}\{\pm(y_n+z_n)\}, \quad \text{and} \quad 0 \le t_m \le 1$$

such that the sequence $u_m := t_m v_m + (1-t_m)w_m$ converges to x. Then

$$y = Px = \lim_m Pu_m = \lim_m (t_m Pv_m + (1-t_m)Pw_m).$$

From $\|P\| = 1$, $\|y_n+z_n\| = \max\{\|y_n\|, \|z_n\|\} \le 2$ and $\|w_m\| \le 1$, we obtain $\|Pv_m\| \le 2$ and $\|Pw_m\| \le 1$. Since $\|y\| = 2$, we must have $t_m \to 1$, and thus $v_m \to x$.

For each v_m we have a representation

$$v_m = \sum_{n=1}^{q_m} \alpha_n^{(m)}(y_n+z_n) \quad \text{with} \quad \sum_{n=1}^{q_m} |\alpha_n^{(m)}| = 1.$$

Since $Pv_m \to Px = y$, we obtain $\sum_{n=1}^{q_m} \alpha_n^{(m)} y_n \to y$ as $m \to \infty$. Since $y_n = \frac{n}{n+1}y$, we find that $\lim_{m\to\infty} \sum_{n=1}^{q_m} \alpha_n^{(m)} n/(n+1) = 1$. Since $\sum_{n=1}^{q_m} |\alpha_n^{(m)}| = 1$, we have $\lim_{m\to\infty} \sum_{n=1}^{q_m} |\alpha_n^{(m)}| n/(n+1) = 1$. Hence $\sum_{n=1}^{q_m} |\alpha_n^{(m)}|/(n+1) \to 0$ as $m \to \infty$, so for fixed k (when $k > q_m$ we have $\alpha_k^{(m)} = 0$)

$$|\alpha_k^{(m)}|/(k+1) \le \sum_{n=1}^{q_m} |\alpha_n^{(m)}|/(n+1) \xrightarrow[m\to\infty]{} 0.$$

Thus $\lim_{m\to\infty} \alpha_n^{(m)} = 0$ for every n. We also have $\lim_m \sum_{n=1}^{q_m} \alpha_n^{(m)} = 1$, since

$$\Big|\sum_{n=1}^{q_m} \alpha_n^{(m)} - \sum_{n=1}^{q_m} \alpha_n^{(m)} n/(n+1)\Big| \le \sum_{n=1}^{q_m} |\alpha_n^{(m)}|/(n+1) \xrightarrow[m\to\infty]{} 0.$$

The above implies that $q_m \to \infty$. Let $v'_m = \sum_{n=1}^{q_m} |\alpha_n^{(m)}|(y_n+z_n)$. Since $\|y_n+z_n\| \le 2$, we have

$$\|v'_m - x\| \le \|v_m - x\| + \Big\|\sum_{n=1}^{q_m} [|\alpha_n^{(m)}| - \alpha_n^{(m)}](y_n+z_n)\Big\|$$
$$\le \|v_m - x\| + 2\sum_{n=1}^{q_m} [|\alpha_n^{(m)}| - \alpha_n^{(m)}] \to 0.$$

Since $t_m \to 1$, also $t_m v'_m + (1-t_m)w_m \to x$, so if we replace $\alpha_n^{(m)}$ by $|\alpha_n^{(m)}|$, we may assume that $\alpha_n^{(m)} \ge 0$ for every n, m. From the above, $\lim_m \alpha_n^{(m)} = 0$ for every n.

We now construct a subsequence as follows. Let $m_1 = 1$. Given $m_k = j$, let

$$\widetilde{v}_m = \sum_{n=q_j+1}^{q_m} \frac{\alpha_n^{(m)}}{\sum_{i=q_j+1}^{q_m} \alpha_i^{(m)}} (y_n + z_n).$$

Since $\lim_m \alpha_n^{(m)} = 0$ for $1 \le n \le q_j$, we easily obtain $\lim_m \|v_m - \widetilde{v}_m\| = 0$. We pick m_{k+1} as a value of m with $\|v_m - \widetilde{v}_m\| < 2^{-k}$ (and $q_m > q_j$). As before, we replace v_{m_k} by $\widetilde{v}_{m_k}$, and keeping only the subsequence, we may now assume also that for each m we have $\alpha_n^{(m+1)} = 0$ for $1 \le n \le q_m$.

We also have

$$\sum_{n=q_{m-1}+1}^{q_m} \alpha_n^{(m)} z_n = \sum_{n=1}^{q_m} \alpha_n^{(m)} z_n = (I-P)v_m \to (I-P)x = x - y.$$

Since $\{z_n\}$ is a basic sequence, it follows (from [D, p. 88]) that there is a constant C such that

$$\Big\| \sum_{n=q_{m-1}+1}^{q_m} \alpha_n^{(m)} z_n \Big\| \le C \Big\| \sum_{n=q_{m-1}+1}^{q_m} \alpha_n^{(m)} z_n - \sum_{n=q_m+1}^{q_{m+1}} \alpha_n^{(m+1)} z_n \Big\|.$$

Since we have non-negative weights, the special property of the sequence $\{z_n\}$ yields that for every m,

$$\frac{a}{C} = \frac{a}{C} \sum_{n=q_{m-1}+1}^{q_m} \alpha_n^{(m)} \le \frac{1}{C} \Big\| \sum_{n=q_{m-1}+1}^{q_m} \alpha_n^{(m)} z_n \Big\|$$
$$\le \Big\| \sum_{n=q_{m-1}+1}^{q_m} \alpha_n^{(m)} z_n - \sum_{n=q_m+1}^{q_{m+1}} \alpha_n^{(m+1)} z_n \Big\|.$$

This contradicts the convergence of $\{\sum_{n=q_{m-1}+1}^{q_m} \alpha_n^{(m)} z_n\}$. Hence $P(V)$ is not closed.

THEOREM 2.5. *Let $T \ne I$ be a power-bounded uniformly ergodic operator in a Banach space X. The image under $I-T$ of the unit ball of any equivalent norm is closed if and only if the fixed-point space F is reflexive.*

Proof. Let $Y = (I-T)X$. By Theorem 1.1, Y is closed, and $X = Y \oplus F$. Let P be the corresponding projection onto Y. Fix a norm on X, and let U be its unit ball. For $x \in U$ we have $x = y + z$ with $y = Px \in P(U)$ and $z \in F$. Hence $(I-T)x = (I-T)y \in (I-T)P(U)$. Thus $(I-T)U = (I-T)P(U)$. Since $(I-T)_{|Y}$ is an isomorphism, $(I-T)P(U)$ is closed if and only if $P(U)$ is. The theorem now follows from the previous one, by taking $Z = F$.

EXAMPLE. Let Z be a non-reflexive Banach space, Y any non-zero Banach space, and let $X = Y \oplus Z$. Let T be the corresponding projection onto Z. Then T is obviously UE (so satisfies $(*)$), with fixed-point space Z. By the previous theorem, there exists an equivalent norm on X with unit ball V such that $(I-T)V$ is not closed.

THEOREM 2.6. *Let T be a power-bounded operator in a Banach space X satisfying $(*)$. Then there exists an equivalent norm in X, with unit ball U', such that $(I-T)U'$ is closed.*

Proof. By Theorem 2.3, $(I-T)X$ is an F_σ set. The result now follows from the following proposition (which extends a result of Saint-Raymond in [BoR, p. 156], where the operator is assumed one-to-one).

PROPOSITION 2.7. *Let S be a bounded linear operator from a Banach space X into a Banach space Z such that $S(X)$ is an F_σ set in Z. Then there exists an equivalent norm on X, with unit ball V, such that $S(V)$ is closed.*

Proof. Let $N = \mathrm{Ker}(S)$, define $\widetilde{X} = X/N$, and let $q : X \to \widetilde{X}$ be the quotient map. Since $x_1 \equiv x_2$ implies $Sx_1 = Sx_2$, we can define an operator $\widetilde{S} : \widetilde{X} \to Z$ such that $\widetilde{S}q = S$. Clearly $\widetilde{S}$ is one-to-one, and by the definition $\widetilde{S}(\widetilde{X}) = S(X)$, which is an F_σ set by assumption. We can apply to $\widetilde{S}$ the result of Saint-Raymond [BoR, p. 156], to obtain an equivalent norm on $\widetilde{X}$ with unit ball W such that $\widetilde{S}(W)$ is closed. Let $\widetilde{U}$ be the unit ball of $\widetilde{X}$ in the quotient norm induced by the norm of X. Without loss of generality, we may assume that $W \subset \widetilde{U}$.

Let U be the unit ball of X, and define $V = q^{-1}(W) \cap 2U$. Clearly V is a bounded closed convex symmetric body in X, so there is an equivalent norm on X which has V as unit ball. To complete the proof, we show that $S(V) = \widetilde{S}(W)$, which is closed. Since $\widetilde{S}q = S$, it is enough to show $q(V) = W$. By definition, $q(V) \subset W$. The definition of the quotient norm yields $\mathrm{int}\, \widetilde{U} \subset q(U)$, so

$$W \subset \widetilde{U} \subset \mathrm{int}\, 2\widetilde{U} \subset q(2U).$$

Thus, for $w \in W$ there is $x \in 2U$ with $q(x) = w$, so $x \in V$. Hence $q(V) = W$.

PROPOSITION 2.8. *Let T be a contraction of $L_1(\mu)$ of a σ-finite measure space. Then $(I-T)U$ is closed.*

Proof. We identify $L_1(\mu)$, via the Radon–Nikodym theorem, with the space $M(\mu)$ of countably additive finite signed measures $\ll \mu$. Let $\eta_n \in U$ with $(I-T)\eta_n \to \nu$. Since $L_\infty(\mu)^*$ is the space of finitely additive signed measures $\ll \mu$, we can find ϱ finitely additive, $\|\varrho\| \le 1$, such that $(I-T^{**})\varrho = \nu$, by taking any weak* limit point of $\{\eta_n\}$ as ϱ.

We now adapt some arguments of [LS]. Let $\varrho = \eta + \varrho_0$ be the Hewitt–Yosida [YH] decomposition of ϱ, with η countably additive and ϱ_0 a pure charge (i.e., $|\varrho_0|$ does not bound a countably additive measure). Then

$$\nu = (I - T^{**})\varrho = \eta - T^{**}\eta + \varrho_0 - T^{**}\varrho_0.$$

Since $T^{**}\eta = T\eta \in M(\mu)$, we conclude that $\nu' = \varrho_0 - T^{**}\varrho_0$ is countably additive. Since ν' (countably additive) and ϱ_0 (a pure charge) are mutually singular [YH], and $|T| \le 1$, we have

$$\|\varrho_0\| \ge \|T^{**}\varrho_0\| = \|\varrho_0 - \nu'\| = \|\varrho_0\| + \|\nu'\|.$$

Hence $\nu' = 0$, so $\nu = (I - T^{**})\eta = (I - T)\eta \in (I - T)U$, since $\|\eta\| \le \|\varrho\| \le 1$.

Let K be a compact Hausdorff space. A *Markov operator* on $C(K)$ is a positive contraction T with $T\mathbf{1} = \mathbf{1}$. The transition probability $P(x, A) = T^*\delta_x(A)$ yields an extension of T (still denoted by T) to all bounded Baire-measurable functions [K, p. 177]. A non-empty closed subset A is called *absorbing* if T^*m is supported in A whenever $m \in C(K)^*$ is supported in A (equivalently, if $T1_A \ge 1_A$). T is called *irreducible* if the only absorbing closed set is K.

PROPOSITION 2.9. *Let T be an irreducible Markov operator on $C(K)$ of a compact Hausdorff space. Then $(I - T)U$ is closed.*

Proof. Let $f = \lim(I - T)g_n$ with $g_n \in U$. By the main theorem of [KoL], condition (i) of Theorem 2.3 is satisfied. Hence $f \in (I - T)C(K)$, so there is $g \in C(K)$ with $(I - T)g = f$.

Now let μ be an extreme point of the set of all probabilities ν with $T^*\nu = \nu$ (which is non-empty; see, e.g., [K, p. 178]). Since $T^*\mu = \mu$, it follows that $M(\mu)$ is a T^*-invariant Banach space, and $Th = 0$ a.e. if $h = 0$ a.e. Let S be the restriction of T^* to $M(\mu)$. Then S^* on $L_\infty(\mu)$ is given by (the extended) T. It is well known that μ is an *ergodic* T-invariant probability (i.e., $Th = h$ a.e. for $h \in L_\infty$ implies that h is a.e. constant).

By weak* compactness of the unit ball of L_∞, a weak* limit point (in L_∞) of $\{g_n\}$ yields h bounded measurable with $\|h\|_{L_\infty} \le 1$ such that $(I - T)h = f$ a.e. Hence $T(g - h) = g - h$ a.e. By ergodicity of μ, there is a constant c with $g - h = c$ a.e. Hence $|g - c| \le 1$ a.e. Since T is irreducible, the support of μ is K, so continuity of $g - c$ implies that $g - c \in U$, and thus $f = (I - T)(g - c) \in (I - T)U$.

Wittmann has proved (see [KoL]) that $(*)$ is satisfied by the Markov operator T induced on the space of bounded measurable functions by a transition probability. We do not know if $(I - T)U$ is closed also in this case.

3. Uniform ergodicity and dual subspaces. In this section we study the relationship between uniform ergodicity and equality $(*)$ in real Banach spaces. We will need the following notion [LPP]:

DEFINITION. Let Z and X be Banach spaces. A *semi-embedding* (of Z in X) is a one-to-one bounded linear operator $S : Z \to X$ such that the image $S(U_Z)$ of the unit ball of Z is closed in X.

Theorem 2.6 and (1.2) yield that if T is ME and satisfies $(*)$, then for an equivalent norm $I - T$ becomes a semi-embedding of $\overline{(I - T)X}$ into itself. For uniform ergodicity it should be an isomorphism, by Theorem 1.1.

THEOREM 3.1. *Let Z and X be Banach spaces, and $S : Z \to X$ a semi-embedding.*

(i) ([F₂], [F₃]) *If $S(Z)$ is not closed, then Z contains an infinite-dimensional closed subspace isometrically isomorphic to a dual Banach space.*

(ii) [F₄] *If Z is separable, then every closed subspace Y of Z such that $S(Y)$ is not closed contains an infinite-dimensional closed subspace isometrically isomorphic to a dual Banach space.*

When T is not assumed ME, (1.2) does not cover the whole space, so instead of the decomposition (1.2) we use quotient spaces. This requires the following lemma (its standard proof is left to the reader).

LEMMA 3.2. *Let Z_1 and Z_2 be closed subspaces of a Banach space X such that the direct sum $Z_1 \oplus Z_2$ is closed, and let q be the quotient map of X onto X/Z_1. Then $q_{|Z_2}$ is an isomorphism.*

THEOREM 3.3. *Let T be a power-bounded linear operator in a separable Banach space X. If T satisfies $(*)$ and is not uniformly ergodic, then $\overline{(I - T)X}$ contains an infinite-dimensional closed subspace isomorphic to a dual Banach space.*

Proof. Let $F = \{y \in X : Ty = y\}$ and consider the quotient Banach space $\widetilde{X} = X/F$, with quotient map q. Since $qx_1 = qx_2$ implies $(I - T)x_1 = (I - T)x_2$, we can define an operator $S : \widetilde{X} \to X$ such that $Sq = I - T$. Clearly S is bounded and one-to-one, and $S(\widetilde{X}) = (I - T)X$.

By (1.2), $F \oplus \overline{(I - T)X}$ is closed, so by Lemma 3.2, q is an isomorphism of $\overline{(I - T)X}$ and $Y = q(\overline{(I - T)X})$. Now $S(Y) = (I - T)(\overline{(I - T)X})$, so T is UE (by Theorem 1.1) if and only if $S(Y)$ is closed.

Assume that T is not UE. Since T satisfies $(*)$, $(I - T)X$ is an F_σ set by Theorem 2.3, so $S(\widetilde{X})$ is an F_σ set. By Proposition 2.7, there is an equivalent norm on $\widetilde{X}$ in which S is a semi-embedding of $\widetilde{X}$. Since by assumption $S(Y)$ is not closed, Theorem 3.1(ii) yields an infinite-dimensional closed subspace of Y isomorphic to a dual Banach space. Since Y and $\overline{(I - T)X}$ are isomorphic (Lemma 3.2), the proof is complete.

Remark. For X not separable, the proof of Theorem 3.3 shows that if T satisfies $(*)$ and is not UE, then $\widetilde{X}$ contains a closed infinite-dimensional subspace isomorphic to a dual Banach space (we apply Theorem 3.1(i), since $S(\widetilde{X})$ is not closed if T is not uniformly ergodic).

Condition $(*)$ is always necessary for uniform ergodicity (see the discussion following Theorem 1.1) but is not always sufficient.

COROLLARY 3.4. *Let X be a Banach space which does not contain any infinite-dimensional closed subspace isomorphic to a dual Banach space, and let T be a power-bounded operator.*

(i) *If X is separable, then T is uniformly ergodic if and only if it satisfies $(*)$.*

(ii) *If T has no non-zero fixed points, then T is uniformly ergodic if and only if it satisfies $(*)$.*

Proof. (i) is an immediate corollary of Theorem 3.3.

(ii) When $F = \{0\}$ and T satisfies $(*)$, $S = I - T$ is a semi-embedding in an equivalent norm by Theorem 2.6. If T is not UE, then $I - T$ is not an isomorphism, so by Theorem 3.1(i), X contains a closed infinite-dimensional subspace isomorphic to a dual space, a contradiction.

For a set Γ, we denote by $\ell_1(\Gamma)$ the space of absolutely summable functions on Γ, so $\ell_1 = \ell_1(\mathbb{N})$.

THEOREM 3.5. *Let X be a Banach space with $X^* = \ell_1(\Gamma)$, and let T be power-bounded.*

(i) *If $X^* = \ell_1$, then T is uniformly ergodic if and only if it satisfies $(*)$.*

(ii) *If T has no non-zero fixed points, then T is uniformly ergodic if and only if it satisfies $(*)$.*

Proof. Since $X^* = \ell_1(\Gamma)$, every infinite-dimensional closed subspace of X contains a subspace isomorphic to c_0 (see [F_1]). If X_1 is an infinite-dimensional subspace of X isomorphic to a dual space, then by [BP] it contains a subspace isomorphic to ℓ_∞, since it contains c_0. But this implies that every infinite-dimensional subspace of ℓ_∞ contains c_0—a contradiction, since ℓ_∞ contains a Hilbert space. Hence X has no infinite-dimensional subspace isomorphic to a dual space, so X satisfies the hypothesis of the previous corollary.

THEOREM 3.6. *Let K be a compact metric space. Then K is countable if and only if every power-bounded operator on $C(K)$ satisfying $(*)$ is uniformly ergodic.*

Proof. (i) If K is countable, then $C(K)^* = \ell_1$, and Theorem 3.5 yields the result.

(ii) Assume that K is uncountable. By Milyutin's theorem ([M], [P]), $C(K)$ of K compact metric uncountable is isomorphic to $C(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. Hence it suffices to exhibit T power-bounded on $C(\mathbb{T})$ satisfying $(*)$ but not UE.

Let $\phi : \mathbb{T} \to \mathbb{T}$ be an irrational rotation, and define T on $C(\mathbb{T})$ by $Tf(t) = f(\phi t)$. Since ϕ is a minimal homeomorphism, $(*)$ is satisfied by [GHe, p. 135] (see also [KoL]). However, T is not uniformly ergodic (see, e.g., [L_2]).

Remark. In part (i) of the proof of Theorem 3.6, Theorem 11 of [LPP] may be used instead of Theorem 3.5.

DEFINITION. A linear operator T is called *quasi-compact* if there exist an integer n and a compact operator S such that $\|T^n - S\| < 1$. Yosida and Kakutani [YKa] proved that a quasi-compact contraction is UE. A UE contraction with finite-dimensional fixed-point space need not be quasi-compact [L_2]. However, a UE power-bounded *positive* operator on a Banach lattice with finite-dimensional fixed-point space is quasi-compact [L_3].

Wittmann [W] proved that an irreducible Markov operator T on $C(K)$ is uniformly ergodic if and only if T^* is mean ergodic. For related results, see also [Lo].

THEOREM 3.7. *Every irreducible Markov operator on $C(Q)$ of a countable compact metric space Q is quasi-compact (hence uniformly ergodic).*

Proof. Let T be an irreducible Markov operator on $C(Q)$. Since irreducible Markov operators satisfy $(*)$ by [KoL], T is uniformly ergodic by Theorem 3.6. The fixed-point space is one-dimensional (an irreducible mean ergodic Markov operator on $C(K)$ is uniquely ergodic; see, e.g., [K, p. 178]). Hence, by [L_3], T is quasi-compact.

We now obtain another version of the previous results, which may be applicable to non-separable spaces. For separable spaces the previous results are stronger.

PROPOSITION 3.8. *Let T be a power-bounded mean ergodic operator on X satisfying $(*)$, and let Y be a closed T-invariant subspace. Then $T_{|Y}$ also satisfies $(*)$.*

Proof. Let $y \in Y$ satisfy $\sup_n \|\sum_{k=0}^n T^k y\| < \infty$. Then by $(*)$ for T in X, there is $x \in X$ with $(I - T)x = y$. Since T is mean ergodic, we can assume (using (1.2)) that $\lim_{n\to\infty} n^{-1}\sum_{k=1}^n T^k x = 0$. Hence

$$\frac{1}{N}\sum_{n=1}^{N}\sum_{k=0}^{n-1} T^k y = \frac{1}{N}\sum_{n=1}^{N}\sum_{k=0}^{n-1} T^k(I-T)x = \frac{1}{N}\sum_{n=1}^{N}(I-T^n)x$$
$$= x - \frac{1}{N}\sum_{n=1}^{N} T^n x \to x.$$

This shows that $x \in Y$, so $T_{|Y}$ satisfies $(*)$.

Remark. Without mean ergodicity, the previous result is false [LS].

THEOREM 3.9. *Let T be a power-bounded mean ergodic linear operator in a Banach space X. If T satisfies $(*)$ and is not uniformly ergodic, then $\overline{(I-T)X}$ contains a separable infinite-dimensional closed subspace isomorphic to a dual Banach space.*

Proof. Since T is not uniformly ergodic, $(I-T)X$ is not closed. Hence there exists a sequence $\{x_n\}$ with $(I-T)x_n \to z \notin (I-T)X$. Let Y be the closed linear manifold generated by $\{T^j x_n : j \geq 0,\ n > 0\}$. Then Y is a separable T-invariant subspace, and since $(I-T)Y$ is also not closed, $T_{|Y}$ is not uniformly ergodic. Since T is assumed mean ergodic and satisfies $(*)$, also $T_{|Y}$ satisfies $(*)$ by the previous proposition. Applying Theorem 3.3 to $T_{|Y}$ we conclude that $\overline{(I-T)Y}$ contains an infinite-dimensional closed subspace isomorphic to a dual space, which is separable since Y is.

COROLLARY 3.10. *Let X be a Banach space which does not contain any infinite-dimensional separable closed subspace isomorphic to a dual Banach space, and let T be a power-bounded operator. Then T is uniformly ergodic if and only if it is mean ergodic and satisfies $(*)$.*

Remark. An example to which Corollary 3.10 applies, but Corollary 3.4 does not, must be (or contain) a non-separable dual space which does not contain separable duals. The existence of such an example is an open problem.

4. Uniform ergodicity of Markov chains. Markov chains with $\mathbb{N}$ as state space are defined by a Markov transition matrix $P = (p_{ij})_{i,j>0}$. Such a matrix defines a contraction P on ℓ_∞, by multiplying column vectors (on the left) by P. In this section we study the uniform ergodicity and mean ergodicity of operators on ℓ_∞ defined by certain Markov transition matrices. We first need the following lemma.

LEMMA 4.1. *The operator defined by a Markov matrix $P = (p_{ij})_{i,j>0}$ preserves c (the space of convergent sequences) if and only if $\lim_{i\to\infty} p_{ij}$ exists for every j. It preserves c_0 if and only if $\lim_{i\to\infty} p_{ij} = 0$ for every j.*

Proof. Let e_j be the jth unit vector written as a column. Then Pe_j is the jth column of P. Since $P\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the vector with all components 1, c (c_0) is invariant if and only if $Pe_j \in c$ (c_0) for every j. The result is now immediate.

Remark. The corollary that c is P-invariant if c_0 is P-invariant follows directly from $P\mathbf{1} = \mathbf{1}$.

A Markov matrix $P = (p_{ij})_{i,j>0}$ is called *irreducible* if for every non-empty subset A there exists $i \in A$ with $\sum_{j\in A} p_{ij} < 1$.

THEOREM 4.2. *Let P be an irreducible Markov matrix such that $\phi_j := \lim_{i\to\infty} p_{ij}$ exists for every j. If $\sum_{j>0}\phi_j > 0$, then P has a (unique) invariant probability vector, and the operator induced by P on ℓ_∞ is quasi-compact and UE.*

Proof. By Lemma 4.1, c is an invariant subspace for multiplication by P, and the restriction to c is represented by a Markov operator on $C(Q)$, where $Q = \mathbb{N}\cup\{\infty\}$ is the one-point compactification. In this representation, ϕ_j is the probability of passing from ∞ to j, so the condition of the theorem means that ∞ is not an absorbing state, and, together with the irreducibility of the original matrix, yields the irreducibility of the Markov operator on $C(Q)$. Uniform ergodicity on c follows from Theorem 3.7. Hence the operator of multiplication of ℓ_1 rows by the matrix P is also UE. Since $P\mathbf{1} = \mathbf{1}$, the limit projection preserves the set of probability row vectors. Finally, the dual of the operator on ℓ_1 rows is P on ℓ_∞, which is now also UE. By [L3] (or even [L1]), P is also quasi-compact.

Remarks. 1. A result analogous to Theorem 4.2 holds for Markov chains on $\mathbb{Z}$, given by matrices $(p_{ij})_{i,j\in\mathbb{Z}}$. The proof uses the two-point compactification of $\mathbb{Z}$.

2. The assumptions of Theorem 4.2 do not imply uniformity (in j) of the convergence to ϕ_j, as seen in the example $p_{i1} = p_{i,i+1} = 1/2$.

3. Y. Derriennic has remarked that Doeblin's theory can be used to prove a stronger result:

Let $P = (p_{ij})_{i,j>0}$ be an irreducible Markov matrix. If there are $j > 0$ and $\delta > 0$ such that $\{i : p_{ij} < \delta\}$ is finite, then P on ℓ_∞ is quasi-compact and UE.

Proof. We note that P is UE if and only if $\frac{1}{2}(I+P)$ is UE (by Theorem 1.1(ii)), so we may assume $p_{jj} > 0$, and by changing δ, also $p_{jj} \geq \delta$.

Define $A = \{i : p_{ij} < \delta\}$. For $i \notin A$, we have $p_{ij}^{(n)} \geq \delta^n$ for every n, since one of the paths is to go from i to j in the first step, and then stay $n-1$ steps at j. For $i \in A$ there is a k with $p_{ij}^{(k)} > 0$ by irreducibility. Hence $p_{ij}^{(n)} > 0$ for every $n \geq k$ (one path is to go from i to j in k steps, and then stay at j for the remaining steps). Since A is finite, there is a k with $p_{ij}^{(k)} > 0$ for every $i \in A$, so finiteness again yields that there is $\varepsilon > 0$ with $p_{ij}^{(k)} \geq \varepsilon$ for every i. Hence P satisfies Doeblin's condition [Loe, p. 451] (with $\mu = \delta_j$), so it is quasi-compact and thus UE.

PROPOSITION 4.3. *Let P be an irreducible Markov matrix. If the operator induced by P on ℓ_∞ is ME, then c_0 is not invariant under P.*

Proof. Assume that P preserves c_0. By mean ergodicity on ℓ_∞, P has an invariant probability distribution (for any initial distribution row vector μ, the averages $N^{-1}\sum_{k=1}^{N}\mu P^k$ form a weak Cauchy sequence). By irreducibility, there is a unique invariant probability distribution λ, and $\lambda_i > 0$ for every i. By the ergodic theorem, $N^{-1}\sum_{k=1}^{N} P^k f$ converges to the constant vector $\int f\,d\lambda$ for each $f \in \ell_\infty$. For $0 < f \in c_0$ the limit is not zero. But if P preserves c_0, the limit must be in c_0—and a constant, so the limit is 0, a contradiction.

COROLLARY 4.4. *Let P be an irreducible Markov matrix which preserves c. Then the following are equivalent*:

(i) *The operator induced on ℓ_∞ is quasi-compact and UE.*
(ii) *The operator induced on ℓ_∞ is ME.*
(iii) *c_0 is not P-invariant.*

Remarks. 1. When c_0 is invariant under the irreducible matrix P (so P is not ME on ℓ_∞), the Markov chain may be transient, null-recurrent or positive recurrent:

(i) Let $p_{11} = p_{12} = 1/2$, and $p_{i,i+1} = p_{i,i-1} = 1/2$ for $i > 1$, with the remaining entries zero. Then the only fixed row vector is constant, so P has no invariant probability. However, P is recurrent (null-recurrent).

(ii) Let $p_{1j} = 2^{-j}$ for $j > 0$, and $p_{i,i-1} = 1$ for $i > 1$, with remainnig entries zero. Then P is irreducible and preserves c_0, and has invariant probability distribution $(\lambda_i = 2^{-i})_{i>0}$.

2. If P is an irreducible Markov matrix preserving c_0, then T, the restriction to c of the operator induced by P, does not satisfy $(*)$.

3. The proof of Theorem 4.3 shows that if P is an irreducible Markov matrix preserving c_0 and the Markov chain is positive recurrent, then the restriction of P to c is not ME.

4. The equivalence of (i) and (ii) in Corollary 4.4 is true for any irreducible Markov matrix (even if c is not P-invariant): (ii) implies that the Markov chain is necessarily positive recurrent (see proof of 4.3), and [Ho] implies quasi-compactness and UE.

5. Irreducible random walks on $\mathbb{Z}$ can be taken as P preserving c_0, such that the restriction to c is ME, but not UE. For other examples, see Proposition 4.6 below.

The following example [R] shows that P may preserve c_0 and have countably many absorbing sets, and still the restriction to c will not be ME. Our proof is entirely different.

EXAMPLE. Let $P = (p_{ij})_{i,j>0}$ be a lower triangular Markov matrix (i.e., $p_{ij} = 0$ for $j > i$) which preserves c_0 and has $\{i : p_{ii} = 1\}$ finite. Then the restriction of P to c is not ME.

Proof. For every j, the set $\{1,\ldots,j\}$ is absorbing. The only recurrent states are the absorbing ones, since it is impossible to return to a state once it is left. Denote by C the set of absorbing states, which is finite by assumption. Then the sequence defined by $a_j = 1_C(j)$ is in c_0. But clearly $\lim_{n\to\infty} P^n 1_C(j) = 1$ for every j. Thus, $N^{-1}\sum_{k=1}^{N} P^k 1_C$ cannot converge uniformly to a c_0-sequence, so the restriction of P to c_0 is not ME.

A Markov transition matrix $P = (p_{ij})_{i,j>0}$ is *tridiagonal* if $p_{ij} = 0$ for $|i-j| > 1$. By Lemma 4.1, a tridiagonal matrix preserves c_0.

PROPOSITION 4.5. *Let P be a tridiagonal Markov matrix with $p_{i+1,i} \neq 0$ for every $i \in \mathbb{N}$, such that*

$$\limsup_{i\to\infty} p_{i,i+1}/p_{i+1,i} < 1. \tag{4.1}$$

Then the restriction of P to c is not ME.

Proof. Let T be the restriction of P to c_0. Then T^* operates on ℓ_1 by multiplying row vectors on the right by P.

CLAIM 1. *T has no non-zero fixed points.*

Let $x = (x_1, x_2, \ldots)$ be a bounded sequence satisfying $Px = x$ as a column vector. Then $p_{11}x_1 + p_{12}x_2 = x_1$ implies $p_{11}(x_1 - x_2) = 0$, and $p_{j,j-1}x_{j-1} + p_{jj}x_j + p_{j,j+1}x_{j+1} = x_j$ implies

$$p_{j,j-1}(x_j - x_{j-1}) = p_{j,j+1}(x_{j+1} - x_j) \quad \forall j > 1. \tag{4.2}$$

If $p_{j,j+1}(x_{j+1} - x_j) = 0$, then $x_j = x_{j-1}$ since $p_{j,j-1} \neq 0$. Hence $x_i = x_j$ for all $1 \le i \le j$. Thus, if $p_{j,j+1}(x_{j+1} - x_j) = 0$ for infinitely many j, then x is a constant vector. Since the only constant vector in c_0 is 0, the claim is proved in this case. We now deal with the case where $p_{j,j+1}(x_{j+1} - x_j) \neq 0$ for $j \ge N$. Multiplying the equations (4.2) for $N \le j \le m$ and cancelling the factors $x_{j+1} - x_j$ (which are non-zero by assumption), we obtain

$$\prod_{j=N}^{m} p_{j,j-1}(x_N - x_{N-1}) = \prod_{j=N}^{m} p_{j,j+1}(x_{m+1} - x_m).$$

Hence

$$x_{m+1} - x_m = \frac{p_{N,N-1}}{p_{m,m+1}}(x_N - x_{N-1}) \prod_{j=N}^{m-1} \frac{p_{j+1,j}}{p_{j,j+1}}.$$

We may take N large enough so that by (4.1), $p_{j,j+1}/p_{j+1,j} \le \alpha < 1$ for $j \ge N$. Hence

$$|x_{m+1} - x_m| \ge \frac{p_{N,N-1}}{p_{m,m+1}}|x_N - x_{N-1}|\alpha^{N-m} \to \infty \quad \text{as } m \to \infty,$$

contradicting the boundedness of (x_j). Hence we are in the previous case, and x is a constant sequence.

CLAIM 2. *T^* has non-zero fixed points.*

The fixed points of T^* are the solutions in ℓ_1 of the equation $yP = y$ for row vectors $y = (y_1, y_2, \ldots)$. The first equation of the system, $y_1 = p_{11}y_1 + p_{21}y_2$, yields $p_{21}y_2 = p_{12}y_1$. By induction we obtain $p_{j+1,j}y_{j+1} = p_{j,j+1}y_j$ (using $y_j = p_{j-1,j}y_{j-1} + p_{jj}y_j + p_{j+1,j}y_{j+1}$ and the induction hypothesis). Hence (again by induction)

$$y_{j+1} = y_1 \prod_{i=1}^{j} \frac{p_{i,i+1}}{p_{i+1,i}}.$$

By the ratio test and (4.1), $\sum_j |y_j|$ converges, and for $y_1 = 1$ we have a non-zero solution.

End of proof of Proposition 4.5. The two claims show that the fixed points of T do not separate the fixed points of T^*. By Sine's criterion [K, p. 74], T is not ME.

PROPOSITION 4.6. *Let P be a tridiagonal Markov matrix with $p_{i,i+1} \neq 0$ for every $i \in \mathbb{N}$ such that*

$$\limsup_{i\to\infty} p_{i,i-1}/p_{i,i+1} < 1. \tag{4.3}$$

Then the restriction of P to c is ME, and is not UE.

Proof. Let T be the restriction of P to c_0. For $z \in c_0$ the equation $(I-T)x = z$ (with column vectors) leads to the system of equations

$$p_{12}(x_1 - x_2) = z_1, \quad p_{i,i-1}(x_i - x_{i-1}) - p_{i,i+1}(x_{i+1} - x_i) = z_i, \quad i > 1.$$

Thus, any value of x_1 yields a solution sequence (x_j). Assume now that $z_i = 0$ for every $i > N$. Then for $i > N$ the equations become $x_{i+1} - x_i = (x_i - x_{i-1})p_{i,i-1}/p_{i,i+1}$. By (4.3), $\sum_i (x_{i+1} - x_i)$ is absolutely convergent, so the solution sequence is in c. By subtracting a constant vector we obtain a solution in c_0. Hence $(I-T)c_0$ is dense in c_0, so T is ME. Hence also the restriction of P to c is ME.

In order to show that $I - T$ is not invertible on c_0, we now show that the sequence $(1/j)_{j\geq 1}$ is not in its range (and in fact is not in $(I-P)\ell_\infty$). Set $b_i = 1/p_{i,i+1}$, and $t_i = p_{i+1,i}/p_{i,i+1}$. Clearly for any sequence z we can obtain a solution sequence x of $(I-T)x = z$ with $x_1 = 0$. The system of equations yields $x_2 = -b_1 z_1$, and $x_{i+1} = x_i + (x_i - x_{i-1})p_{i,i-1}/p_{i,i+1} - b_i z_i$. By induction we obtain

$$x_j = -\sum_{i=1}^{j-1} z_i(b_i + t_i b_{i+1} + \ldots + t_i t_{i+1} \ldots t_{j-2} b_{j-1}).$$

For $z_j = 1/j$, we obtain $-x_j \geq \sum_{i=1}^{j-1} b_i(1/i) \geq \sum_{i=1}^{j-1}(1/i)$, since $b_i \geq 1$. Hence the solution sequence (x_j) is not bounded. If we had a bounded solution x', by subtracting a constant vector we obtain a bounded solution x with $x_1 = 0$—a contradiction.

5. Problems. The results of Section 3 raise some questions. The first is about the converse of Corollary 3.4(i). Since the assumption on X there is inherited by all subspaces, we formulate the question as follows.

PROBLEM 1. *Let X be a separable Banach space such that every power-bounded operator on any closed subspace of X satisfying $(*)$ is uniformly ergodic. Is it true that X does not contain any infinite-dimensional closed subspace isomorphic to a dual Banach space?*

In an attempt to solve Problem 1, we were led to the following problem.

PROBLEM 2. *Does every infinite-dimensional Banach space have a power-bounded operator which is not uniformly ergodic?*

Remark. A positive solution to Problem 2 will show that every infinite-dimensional dual Banach space has a power-bounded operator satisfying $(*)$ which is not UE, yielding a positive solution to Problem 1.

THEOREM 5.1. *Let Y be a Banach space with an unconditional basis. Then there exists a power-bounded operator on Y which is not uniformly ergodic.*

Proof. Let $\{y_i\}$ be an unconditional basis in Y with $\|y_i\| = 1$. Define an operator T on Y by $T(\sum a_i y_i) = \sum(1 - i^{-1})a_i y_i$ (for $\sum a_i y_i \in Y$). By the well-known properties of unconditional bases (see [D], (4)b on p. 79 and Theorem 1 on p. 95), it follows that T maps Y into itself, and is power-bounded. By the uniqueness of the coefficients, T has no fixed points. T is mean ergodic since the averages converge strongly (to 0) for each y_i. Hence, by (1.2), $Y = \overline{(I-T)Y}$. However, $I-T$ is not invertible (1 is in the spectrum of T), so by Theorem 1.1, T is not UE.

Remark. If Y in Theorem 5.1 is not reflexive, then in $X = Y^*$ there is an equivalent norm with unit ball V such that for T constructed above, $(I - T^*)V$ is not closed (and T^* clearly satisfies $(*)$).

It is well known that every power-bounded operator in a reflexive space is mean ergodic. An old question is the following.

PROBLEM 3. *Let X be a Banach space such that every power-bounded operator is mean ergodic. Is X reflexive?*

The original (still unsolved) question [Su] assumed mean ergodicity only for contractions. The answer is negative if we assume only mean ergodicity of all isometries—Davis [Da] constructed an equivalent norm on the real ℓ_1 for which the only isometries are $\pm I$. Zaharopol [Z] has shown that if

X is a countably order complete Banach lattice such that every *positive* power-bounded operator is mean ergodic, then X is reflexive, and deduced a positive answer for dual Banach lattices. Brunel and Sucheston [$BrSu_1$], [$BrSu_2$] showed that super-ergodicity is equivalent to super-reflexivity (we refer the reader to these papers for the definitions. Super-reflexivity characterizes the existence of an equivalent norm which makes the space uniformly convex, and also characterizes the existence of an equivalent norm which makes the space uniformly smooth [D, p. 169]).

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