STUDIA MATHEMATICA 121 (3) (1996)

Stable inverse-limit sequences, with application to Fréchet algebras

by

GRAHAM R. ALLAN (Cambridge)

Abstract. The notion of a *stable* inverse-limit sequence is introduced. It provides a sufficient (and, for sequences of *abelian* groups, necessary) condition for the preservation of exactness by the inverse-limit functor. Examples of stable sequences are provided through the abstract Mittag-Leffler theorem; the results are applied in the theory of Fréchet algebras.

1. Introduction. In a recent paper [2], a lemma was proved (Lemma 7) which may be interpreted as asserting the preservation of exactness in a certain inverse-limit process (which there arose in the study of Fréchet algebras). There is, in fact, a simple algebraic principle involved, which we isolate in the present paper by defining *stability*, for an inverse-limit sequence of groups; this property is discussed in §2. In §3, we give a number of applications, mostly in the theory of Fréchet algebras.

For inverse-limit sequences of *abelian* groups, stability of a sequence $\mathcal{G}$ is equivalent to the vanishing of a group $H^1(\mathcal{G})$ that appears in a simple cohomology theory of the inverse-limit functor. In terms of the "first derived functor of the inverse-limit functor", this idea has been around since about 1960 (see references in [16]); there is a convenient account in an appendix to [14]. (There is also a modern account of the derived-functor approach in §3.2 of [9], with reference back to [18].) However, since the present paper is primarily addressed to analysts, we have, in §4, given a brief account of this cohomology theory in the language of sheaves. The main features of the present paper are: (i) the use of the abstract Mittag-Leffler theorem as the primary source of examples of stable inverse-limit sequences, (ii) the fact that the results on stability do not require the groups to be abelian, and (iii) the applications in §3, which form the real motivation for the theory of §2. (We should remark that the account in §3.2 of [9] also makes use of

1991 *Mathematics Subject Classification*: Primary 46J05, 46M15, 46M40; Secondary 46M20.

the abstract Mittag-Leffler theorem; but the property that we have called *stability* is not made explicit and the types of application are quite different.) The cohomology theory of §4 is not used in the main discussion of the examples—though it occasionally yields a slightly more detailed comment.

We briefly recall basic ideas about inverse limits. Let $(X_n)_{n\geq 1}$ be a sequence of sets and, for each $n \geq 1$, let $f_n : X_{n+1} \to X_n$ be a mapping; we call $\mathcal{X} = (X_n; f_n)_{n\geq 1}$ an *inverse-limit sequence* (abbreviated to *IL-sequence*) of sets and mappings. The *inverse limit* of the sequence, $\varprojlim(X_n; f_n)$ (or, more briefly, $L(\mathcal{X})$), is the subset of $\prod_{n\geq 1} X_n$ consisting of all elements $x = (x_n)_{n\geq 1}$ such that $x_n = f_n(x_{n+1})$ for all $n \geq 1$. We write $\pi_m : \prod X_n \to X_m$ for the mth coordinate projection (and shall use the same symbol for the restriction of this mapping to $L(\mathcal{X})$). We may sometimes write $\mathcal{X}$ in a more extended form:

$$\mathcal{X} : X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \ldots \longleftarrow X_n \xleftarrow{f_n} X_{n+1} \longleftarrow \ldots$$

In most of the examples of IL-sequences to be considered, the sets (X_n) have some additional topological or algebraic structure (or both). If each X_n is a Hausdorff topological space and if each mapping f_n is continuous, then $L(\mathcal{X})$ is a closed subspace of $\prod X_n$ (in its product topology); $L(\mathcal{X})$ is then given the subspace topology. In this connection it is useful to observe that the collection of sets $\{\pi_n^{-1}(U_n) \cap L(\mathcal{X}) : n \geq 1,\ U_n$ an open subset of $X_n\}$ forms a *base* (i.e. not merely a sub-base) for the topology of $L(\mathcal{X})$. From this remark there is the following simple lemma, much used in the sequel.

LEMMA 1. *Let $\mathcal{X} = (X_n; f_n)$ be an IL-sequence of Hausdorff topological spaces and continuous maps. For any subset E of $L(\mathcal{X})$, let $E_n = \overline{\pi_n(E)}$ (closure in X_n). Then $\overline{E} = L(\mathcal{X}) \cap \bigcap_{n\geq 1} \pi_n^{-1}(E_n)$, so that $\overline{E} = \varprojlim(E_n; f_n \,|\, E_{n+1})$. Moreover, $\overline{f_n(E_{n+1})} = E_n$ for each n.*

Proof. Evidently, $\overline{E} \subseteq L(\mathcal{X}) \cap \bigcap_{n\geq 1} \pi_n^{-1}(E_n)$. Let $x \in L(\mathcal{X}) \setminus \overline{E}$; by the remark in the last paragraph, there is some $n \geq 1$ and an open neighbourhood U_n of $\pi_n(x)$ in X_n such that $\pi_n^{-1}(U_n) \cap E = \emptyset$, so also $U_n \cap \pi_n(E) = \emptyset$. Thus $\pi_n(x) \notin \overline{\pi_n(E)} = E_n$. Hence $x \notin \bigcap_{n\geq 1} \pi_n^{-1}(E_n)$. It follows that $\varprojlim(E_n; f_n \,|\, E_{n+1}) = L(\mathcal{X}) \cap \bigcap_{n\geq 1} \pi_n^{-1}(E_n) = \overline{E}$.

Also, for each n, $\overline{f_n(E_{n+1})} = E_n$, since $\pi_n(x) = f_n \pi_{n+1}(x)$ for all $x \in L(\mathcal{X})$.

We recall the Mittag-Leffler theorem on inverse limits.

THEOREM 1 (Mittag-Leffler theorem). *Let $\mathcal{X} = (X_n; f_n)$ be an IL-sequence of complete metrizable topological spaces and continuous mappings. Suppose that, for each n, $f_n(X_{n+1})$ is dense in X_n. Then, for each m, $\pi_m(L(\mathcal{X}))$ is dense in X_m. In particular, if each $X_n \neq \emptyset$, then also $L(\mathcal{X}) \neq \emptyset$.*

Proof. See e.g. [4], Theorem 2.4, or [10], Theorem 2.14.

Remark. This theorem includes (by giving each X_n the discrete topology) the essentially trivial set-theoretic condition for the non-emptiness of $L(\mathcal{X})$, i.e. that $\mathcal{X}$ is an IL-sequence of non-empty sets and *surjective* mappings. Even this case fails to generalize to IL-systems on an index set without a countable cofinal subset (e.g. [5], Chap. III, §7, Ex. 4).

One other standard condition for the non-emptiness of an inverse limit may be mentioned (unlike Theorem 1, this result *does* extend to more general index sets).

THEOREM 2 (Compactness criterion). *Let $\mathcal{X} = (X_n; f_n)$ be an IL-sequence of non-empty Hausdorff topological spaces and continuous mappings and suppose that, for each n, $f_n(X_{n+1})$ is a compact subset of X_n. Then $L(\mathcal{X})$ is a non-empty compact Hausdorff space.*

Remark. This follows easily from, e.g., [8], (3.2.13). Theorem 2 also includes a trivial set-theoretic result: the case in which each f_n has *finite* range—in particular, when each $f_n(X_{n+1})$ is a singleton; this case will occasionally be used.

There are a few remarks about subsequences. Let $\mathcal{X} = (X_n; f_n)$ be an IL-sequence (of sets and mappings) and let $n(1) < n(2) < \ldots$ be a strictly increasing sequence of positive integers; for each $k \geq 1$, define $f'_k : X_{n(k+1)} \to X_{n(k)}$ by $f'_k = f_{n(k)} f_{n(k)+1} \cdots f_{n(k+1)-1}$. Then $\mathcal{X}' = (X_{n(k)}; f'_k)$ will be called a *subsequence* of $\mathcal{X}$. It is trivial that any element $y = (y_k) \in L(\mathcal{X}')$ may be uniquely extended to an element of $\mathcal{X}$, i.e. to an element x such that $x_{n(k)} = y_k$ $(k \geq 1)$. This process clearly effects a natural bijection between $L(\mathcal{X})$ and $L(\mathcal{X}')$. It will be useful to record a simple consequence as a lemma:

LEMMA 2. *If $\mathcal{X}$ is an IL-sequence, then the following are equivalent*:

(i) $L(\mathcal{X}) \neq \emptyset$;
(ii) $L(\mathcal{X}') \neq \emptyset$ *for every subsequence* $\mathcal{X}'$ *of* $\mathcal{X}$;
(iii) $L(\mathcal{X}') \neq \emptyset$ *for some subsequence* $\mathcal{X}'$ *of* $\mathcal{X}$.

In case the sets (X_n) of an IL-sequence $\mathcal{X}$ are, for example, groups (or complex algebras) and the mappings (f_n) are homomorphisms, then it is simple to see that $L(\mathcal{X})$ inherits the same type of algebraic structure, being evidently a subgroup (respectively, subalgebra) of the direct product $\prod X_n$. In fact, taking, for example, *groups*, there is a category, which we shall write as **ILG**, of IL-sequences of groups and homomorphisms. If $\mathcal{G}$, $\mathcal{H}$ belong to **ILG**, then a *morphism* $\alpha : \mathcal{G} \to \mathcal{H}$ is a sequence $(\alpha_n)_{n\geq 1}$, where each $\alpha_n : G_n \to H_n$ is a group homomorphism, such that the following diagram is commutative:

$$\begin{tikzcd}
G_1 \arrow[d, "\alpha_1"'] & G_2 \arrow[l, "g_1"'] \arrow[d, "\alpha_2"'] & \cdots \arrow[l, "g_2"'] & G_n \arrow[l] \arrow[d, "\alpha_n"'] & G_{n+1} \arrow[l, "g_n"'] \arrow[d, "\alpha_{n+1}"'] & \cdots \arrow[l] \\
H_1 & H_2 \arrow[l, "h_1"'] & \cdots \arrow[l, "h_2"'] & H_n \arrow[l] & H_{n+1} \arrow[l, "h_n"'] & \cdots \arrow[l]
\end{tikzcd}$$

The morphism α induces a group homomorphism $L(\alpha) : L(\mathcal{G}) \to L(\mathcal{H})$, by setting $L(\alpha)((x_n)_{n\ge1}) = (\alpha_n(x_n))_{n\ge1}$. The correspondence $\mathcal{G} \mapsto L(\mathcal{G})$, $\alpha \mapsto L(\alpha)$ is a covariant functor from **ILG** to the category of groups and homomorphisms. Evidently, the collection of IL-sequences of *abelian* groups forms a full subcategory (which we denote by **ILGA**) of **ILG**. The restriction of the functor L to **ILGA** is a covariant functor to the category of abelian groups. There is an extensive discussion of these categories (for systems over more general index sets; also of direct-limit systems) in Chapter 8 of [7]. In all the applications that we consider, the objects in the IL-sequences will be at least groups, usually with some extra algebraic or topological structure. The mappings in the sequences will usually then be homomorphisms also with respect to the additional structure; but a notion such as, for example, an exact sequence will be the one appropriate to the underlying groups. For that reason we shall discuss the general theory purely in the language of sequences of groups. Remark that if $\mathcal{X}$ is in **ILG** (or **ILGA**), then so is every subsequence $\mathcal{X}'$ of $\mathcal{X}$; the bijection $L(\mathcal{X}) \cong L(\mathcal{X}')$ (just before Lemma 2) is then a group isomorphism.

Let $0 \to \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{K} \to 0$ be a short exact sequence in **ILG**. That is to say (and this may be taken as a definition), we have the following commutative diagram of groups and homomorphisms:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
G_1 \arrow[d, "\alpha_1"'] & G_2 \arrow[l, "g_1"'] \arrow[d, "\alpha_2"'] & \cdots \arrow[l, "g_2"'] & G_n \arrow[l] \arrow[d, "\alpha_n"'] & G_{n+1} \arrow[l, "g_n"'] \arrow[d, "\alpha_{n+1}"'] & \cdots \arrow[l] \\
H_1 \arrow[d, "\beta_1"'] & H_2 \arrow[l, "h_1"'] \arrow[d, "\beta_2"'] & \cdots \arrow[l, "h_2"'] & H_n \arrow[l] \arrow[d, "\beta_n"'] & H_{n+1} \arrow[l, "h_n"'] \arrow[d, "\beta_{n+1}"'] & \cdots \arrow[l] \\
K_1 \arrow[d] & K_2 \arrow[l, "k_1"'] \arrow[d] & \cdots \arrow[l, "k_2"'] & K_n \arrow[l] \arrow[d] & K_{n+1} \arrow[l, "k_n"'] \arrow[d] & \cdots \arrow[l] \\
0 & 0 & & 0 & 0 &
\end{tikzcd}$$

Each row is an IL-sequence of groups, and each column is a short exact sequence. If we apply the inverse-limit functor L, then it is easy to see that the sequence $0 \to L(\mathcal{G}) \xrightarrow{L(\alpha)} L(\mathcal{H}) \xrightarrow{L(\beta)} L(\mathcal{K})$ is exact at $L(\mathcal{G})$ and $L(\mathcal{H})$, i.e. L is a *left-exact* functor. However, it is not always true that $\operatorname{im} L(\beta) = L(\mathcal{K})$ (e.g. [7], Chapter 8, Example 5.5). Following Theorem 3, we give an example in the context of Fréchet algebras.

The basic theory of Fréchet algebras (and of more general locally m-convex algebras) was introduced in [3] and [15]. A *Fréchet algebra A* is a complete metrizable topological algebra whose topology may be defined by a sequence $(p_n)_{n\ge1}$ of sub-multiplicative seminorms. Without loss of generality we take the sequence (p_n) to be increasing. The principal tool in the study of Fréchet algebras is the representation of such an algebra as the inverse limit of a sequence of Banach algebras. We briefly describe this, in order to establish notation that will be standard throughout the paper.

Thus, let A be a Fréchet algebra, with its topology defined by the increasing sequence $(p_n)_{n\ge1}$ of sub-multiplicative seminorms. For each n let $\pi_n : A \to A/\ker p_n$ be the quotient map; then $A/\ker p_n$ is naturally a normed algebra, normed by setting $\|\pi_n(x)\|_n = p_n(x)$ $(x \in A)$. We let $(A_n; \|\cdot\|_n)$ be its completion, so that A_n is a Banach algebra; henceforth we consider π_n as a mapping from A into A_n. (It is important to note that $\pi_n(A)$ is a dense subalgebra of A_n but that, in general, $\pi_n(A) \ne A_n$.) Since $p_n \le p_{n+1}$, there is a, naturally induced, norm-decreasing homomorphism $d_n : A_{n+1} \to A_n$ such that $d_n \circ \pi_{n+1} = \pi_n$, for all n. Since $\operatorname{im} d_n \supseteq \operatorname{im} \pi_n$, it follows that $d_n(A_{n+1})$ is dense in A_n for each n. For an element $x \in A$, we may write $x_n = \pi_n(x)$; it is then evident that, for each $x \in A$, the sequence $(x_n)_{n\ge1}$ is an element of $\varprojlim(A_n; d_n)$.

The elementary, but fundamental, structure theorem for Fréchet algebras is:

THEOREM 3 (Arens–Michael isomorphism). *Let A be a Fréchet algebra with a defining sequence of seminorms (p_n). Then, with the above notation, the mapping $x \mapsto (x_n)_{n\ge1}$ is a topological-algebra isomorphism of A with $\varprojlim(A_n; d_n)$.*

Proof. See [15], Theorem 5.1 (proved for more general locally multiplicatively convex algebras).

The main point of Theorem 3 should be emphasized: given elements $x_n \in A_n$ such that $x_n = d_n(x_{n+1})$ for all $n \ge 1$, there is a unique $x \in A$ such that $\pi_n(x) = x_n$ for all n. (It should be noted that what we write as A_n appears as $\overline{A_n}$ in [15].) The inverse-limit representation of A given by Theorem 3 will be called an *Arens–Michael representation* of A.

For an initial range of examples of (non-Banach) Fréchet algebras, we refer to the discussion of the examples $\mathbb{C}[[X]]$, $\mathcal{O}(U)$, $C(U)$, $L^1_{\mathrm{loc}}(\mathbb{R}^+)$, $L^1(\mathbb{R}^+; \mathbf{w})$, $C^\infty(\mathbb{R}^+)$ in [2] (following Theorem 3). There are other examples in [3] and [15].

We shall conclude this section with a simple example in which exactness is not preserved by the inverse-limit functor.

EXAMPLE. Let A be the Fréchet algebra $\mathcal{O}(\mathbb{C})$ of all entire functions in one variable. The standard topology on A (that of local uniform convergence) may be described by the seminorms $(p_n)_{n\geq 1}$, where $p_n(f) = \sup\{|f(z)| : z \in \Delta_n\}$, with $\Delta_n = \{z \in \mathbb{C} : |z| \leq n\}$. Then we have the corresponding Arens–Michael representation $A \cong \varprojlim(A(\Delta_n); d_n)$, where $A(\Delta_n)$ is the "disc algebra on Δ_n" (i.e. $A(\Delta_n) = \{f \in C(\Delta_n) : f|\mathrm{int}\,\Delta_n$ is holomorphic$\}$) and $d_n : A(\Delta_{n+1}) \to A(\Delta_n)$ is restriction.

For $n \geq 1$, let $I_n = z^n A(\Delta_n) = \{f \in A(\Delta_n) : f^{(k)}(0) = 0\ (k = 0, \ldots, n-1)\}$, which is a closed ideal of the Banach algebra $A(\Delta_n)$. Evidently, $d_n(I_{n+1}) \subseteq I_n$. Write $\overline{d}_n = d_n|I_{n+1}$ and $\widetilde{d}_n : A(\Delta_{n+1})/I_{n+1} \to A(\Delta_n)/I_n$ for the induced homomorphism between the quotient algebras. Then there are short exact sequences of complex algebras $0 \to I_n \xrightarrow{j_n} A(\Delta_n) \xrightarrow{q_n} A(\Delta_n)/I_n \to 0$ $(n \geq 1)$, where j_n, q_n are respectively canonical inclusions and quotient maps. Then writing $\mathcal{I}$, $\mathcal{A}$, $\mathcal{A}/\mathcal{I}$ respectively for the IL-sequences $(I_n; \overline{d}_n)$, $(A(\Delta_n); d_n)$, $(A(\Delta_n)/I_n; \widetilde{d}_n)$, we evidently have the short exact sequence

$$0 \longrightarrow \mathcal{I} \xrightarrow{j} \mathcal{A} \xrightarrow{q} \mathcal{A}/\mathcal{I} \longrightarrow 0,$$

of IL-sequences of commutative Banach algebras and continuous homomorphisms.

But $L(\mathcal{I}) \cong \{f \in A : f^{(k)}(0) = 0\ (k \geq 0)\} = 0$ and $L(\mathcal{A}) \cong A = \mathcal{O}(\mathbb{C})$. So L gives the sequence $0 \to 0 \xrightarrow{L(j)} A \xrightarrow{L(q)} L(\mathcal{A}/\mathcal{I}) \to 0$. But $A(\Delta_n)/I_n \cong \mathbb{C}[z]/(z^n)$ and $L(\mathcal{A}/\mathcal{I}) \cong \mathbb{C}[[z]]$, the algebra of *all* formal power series in z. In terms of this isomorphism, the mapping $L(q) : \mathcal{O}(\mathbb{C}) \to \mathbb{C}[[z]]$ assigns to each entire function its Taylor series about the origin, so it is certainly not surjective. (See Example 1 following Theorem 16 in §4 for the determination of $H^1(\mathcal{I})$.)

2. Stable inverse-limit sequences. Let G, H be groups and $f : G \to H$ be a homomorphism. For any $\eta \in H$ then, in multiplicative notation, let $\eta.f : G \to H$ be the mapping defined by $(\eta.f)(\gamma) = \eta f(\gamma)$ for all $\gamma \in G$. If $\mathcal{G} = (G_n; g_n)_{n\geq 1}$ is a sequence in **ILG** then, for any $\gamma = (\gamma_n) \in \prod_{n\geq 1} G_n$, we define $\gamma.\mathcal{G}$ to be the IL-sequence $(G_n; \gamma_n.g_n)$; we call $\gamma.\mathcal{G}$ a *perturbed sequence* of $\mathcal{G}$. (Of course, the mappings in the perturbed sequence are not generally homomorphisms.)

We shall say that a sequence $\mathcal{G}$ in **ILG** is *stable* if and only if every perturbed sequence of $\mathcal{G}$ has a non-empty inverse limit. Thus, the sequence

$$G_1 \xleftarrow{g_1} G_2 \xleftarrow{g_2} \ldots \longleftarrow G_n \xleftarrow{g_n} G_{n+1} \longleftarrow \ldots$$

in **ILG** is stable if and only if, for every choice of $\gamma_n \in G_n$ $(n \geq 1)$, we may simultaneously solve the equations $x_n = \gamma_n g_n(x_{n+1})$ for $x_n \in G_n$ $(n \geq 1)$.

Remark. In the definition of stability there is an asymmetry, in that we have considered only multiplication on the left. Evidently, there is a corresponding "right stability" (considering "right perturbations" $g_n.\gamma_n$) and even two-sided stability (with "double perturbations" of the form $\gamma_n.g_n.\gamma_n'$). In fact, in all the classes of stable sequences that we consider, the stronger property of "double stability" holds. However, all the applications in the paper require only the weaker condition that we have called "stability". Of course, for sequences of abelian groups the notions are, anyway, the same.

Trivial examples of stable sequences are provided by two opposite extreme cases. Let $\mathcal{G} = (G_n; g_n)$ be a sequence in **ILG**. Then:

(i) If $g_n(G_{n+1}) = G_n$ for each n, then also $(\gamma_n.g_n)(G_{n+1}) = G_n$, for every choice of $\gamma_n \in G_n$. Hence (see the Remark following Theorem 1), $L(\gamma.\mathcal{G}) \neq \emptyset$ for every perturbed sequence, i.e. $\mathcal{G}$ is stable.

(ii) If each g_n is the trivial homomorphism $g_n(x) = 1_n$ $(x \in G_{n+1})$ then, for every choice of $\gamma_n \in G_n$, we solve $x_n = \gamma_n.g_n(x_{n+1})$ $(n \geq 1)$ by putting $x_n = \gamma_n$ for all n; so again $\mathcal{G}$ is stable.

In fact, these trivial examples are also included as special cases of the following result, which is the main source of examples of stable sequences.

THEOREM 4. *Let $\mathcal{G} = (G_n; g_n)$ be a sequence in* **ILG**. *Then $\mathcal{G}$ is stable provided it satisfies either of the following conditions*:

(i) *each G_n is a complete metrizable topological group and each homomorphism g_n is continuous with $g_n(G_{n+1})$ dense in G_n;*

(ii) *each G_n is a Hausdorff topological group and each homomorphism g_n is continuous with $g_n(G_{n+1})$ compact.*

Proof. (i) Evidently, in any perturbed sequence $\gamma.\mathcal{G}$ of $\mathcal{G}$, each mapping $\gamma_n.g_n$ is continuous with range a dense subset of G_n. It follows from the Mittag-Leffler theorem, Theorem 1, that $L(\gamma.\mathcal{G}) \neq \emptyset$. Thus $\mathcal{G}$ is stable.

(ii) The proof is similar to (i), but using Theorem 2 instead of the Mittag-Leffler theorem.

By a *Mittag-Leffler sequence* we shall mean an IL-sequence $(G_n; g_n)$ where each G_n is a complete metrizable topological group, and, for each $n \geq 1$, g_n is a continuous homomorphism with $\overline{g_n(G_{n+1})} = G_n$. Thus, part (i) of Theorem 4 states that every Mittag-Leffler sequence is stable. [Note that in [14], Definition A.16, the term "Mittag-Leffler condition" is used in a purely algebraic sense, which is basically an amalgam of our two trivial cases (preceding Theorem 4), combined in a way analogous to Corollary 1 below.]

It is, in fact, sufficient that the conditions for stability should be verified on some subsequence. We shall need the following lemma.

LEMMA 3. *Let $\mathcal{G}$ be a sequence in* **ILG**. *Then the following are equivalent*:

(i) *$\mathcal{G}$ is stable*;

(ii) *every subsequence of $\mathcal{G}$ is stable*;

(iii) *some subsequence of $\mathcal{G}$ is stable.*

Proof. (i)⇒(ii). Let $\mathcal{G}' = (G_{n(k)}; g'_k)$ be a subsequence of $\mathcal{G} = (G_n; g_n)$. Let $\gamma.\mathcal{G}' = (G_{n(k)}; \gamma_k.g'_k)$ be a perturbed sequence of $\mathcal{G}'$. We then define $\delta_{n(k)} = \gamma_k$ $(k \geq 1)$, $\delta_n = 1_n$ (the identity of G_n) for $n \notin \{n(k) : k \geq 1\}$. Then $\gamma.\mathcal{G}'$ is a subsequence of $\delta.\mathcal{G}$, so that $L(\gamma.\mathcal{G}') \neq \emptyset$, by Lemma 2 and the stability of $\mathcal{G}$. Hence $\mathcal{G}'$ is stable.

(ii)⇒(iii). Trivial.

(iii)⇒(i). Let $\mathcal{G}' = (G_{n(k)}; g'_k)$ be a subsequence of $\mathcal{G} = (G_n; g_n)$, where $\mathcal{G}'$ is stable; let $\gamma.\mathcal{G}$ be a perturbed sequence of $\mathcal{G}$, $\gamma = (\gamma_n)_{n\geq 1}$. For each $k \geq 1$, define

$$\delta_k = \gamma_{n(k)} g_{n(k)}(\gamma_{n(k)+1}) \ldots (g_{n(k)} \cdots g_{n(k+1)-2})(\gamma_{n(k+1)-1}).$$

Then $\delta.\mathcal{G}'$ is a subsequence of $\gamma.\mathcal{G}$, so that, by Lemma 2 and the stability of $\mathcal{G}'$, $L(\gamma.\mathcal{G}) \neq \emptyset$. Hence $\mathcal{G}$ is stable.

The point of the idea of stability lies in the following theorem.

THEOREM 5. *Let $0 \to \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{K} \to 0$ be a short exact sequence in* **ILG**. *If $\mathcal{G}$ is stable, then the sequence*

$$0 \longrightarrow L(\mathcal{G}) \xrightarrow{L(\alpha)} L(\mathcal{H}) \xrightarrow{L(\beta)} L(\mathcal{K}) \longrightarrow 0$$

is also exact.

Remark. Later (Theorem 16) it will be shown that for sequences of *abelian* groups, the stability of $\mathcal{G}$ is also a necessary condition for exactness always to be preserved.

Proof of Theorem 5. We write the given short exact sequence in extended form:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & \\
G_1 \arrow[d, "\alpha_1"'] & G_2 \arrow[l, "g_1"'] \arrow[d, "\alpha_2"'] & \ldots \arrow[l, "g_2"'] & G_n \arrow[l] \arrow[d, "\alpha_n"'] & G_{n+1} \arrow[l, "g_n"'] \arrow[d, "\alpha_{n+1}"'] & \ldots \arrow[l] \\
H_1 \arrow[d, "\beta_1"'] & H_2 \arrow[l, "h_1"'] \arrow[d, "\beta_2"'] & \ldots \arrow[l, "h_2"'] & H_n \arrow[l] \arrow[d, "\beta_n"'] & H_{n+1} \arrow[l, "h_n"'] \arrow[d, "\beta_{n+1}"'] & \ldots \arrow[l] \\
K_1 \arrow[d] & K_2 \arrow[l, "k_1"'] \arrow[d] & \ldots \arrow[l, "k_2"'] & K_n \arrow[l] \arrow[d] & K_{n+1} \arrow[l, "k_n"'] \arrow[d] & \ldots \arrow[l] \\
0 & 0 & & 0 & 0 &
\end{tikzcd}$$

We know that the sequence $0 \to L(\mathcal{G}) \xrightarrow{L(\alpha)} L(\mathcal{H}) \xrightarrow{L(\beta)} L(\mathcal{K}) \to 0$ is exact at $L(\mathcal{G})$ and $L(\mathcal{H})$, without any special conditions. We have to show that, with the assumption that $\mathcal{G}$ is stable, $\operatorname{im} L(\beta) = L(\mathcal{K})$.

Thus, let $\zeta = (z_n)_{n\geq 1} \in L(\mathcal{K})$; i.e. $z_n \in K_n$ and $z_n = k_n(z_{n+1})$ for all n. Each mapping β_n is surjective, so we may pick, say, $y_n \in H_n$ with $\beta_n(y_n) = z_n$, for all n. Then

$$\beta_n(y_n^{-1} h_n(y_{n+1})) = z_n^{-1}(k_n\beta_{n+1})(y_{n+1}) = z_n^{-1} k_n(z_{n+1}) = 1_n,$$

i.e. $y_n^{-1} h_n(y_{n+1}) \in \ker \beta_n = \operatorname{im} \alpha_n$, for all n. Thus, for each n, there is a unique $u_n \in G_n$ such that $\alpha_n(u_n) = y_n^{-1} h_n(y_{n+1})$.

Since $\mathcal{G}$ is assumed stable, the perturbed sequence $(G_n; u_n.g_n)$ has a non-empty inverse limit; i.e. there is some $v_n \in G_n$ $(n \geq 1)$ such that $v_n = u_n g_n(v_{n+1})$, for all n. But then

$$\begin{aligned} h_n(y_{n+1}\alpha_{n+1}(v_{n+1})) &= h_n(y_{n+1})(\alpha_n g_n)(v_{n+1}) \\ &= h_n(y_{n+1})\alpha_n(u_n^{-1} v_n) = y_n \alpha_n(v_n). \end{aligned}$$

Thus the sequence, say, $\eta \equiv (y_n\alpha_n(v_n)) \in L(\mathcal{H})$ and

$$L(\beta)(\eta) = (\beta_n(y_n)(\beta_n\alpha_n)(v_n)) = (\beta_n(y_n)) = (z_n) = \zeta.$$

Hence $L(\beta)$ maps $L(\mathcal{H})$ onto $L(\mathcal{K})$, and the proof is complete.

We have the following result on the preservation of stability. In the case of sequences of abelian groups, it could also be deduced as an immediate consequence of Theorem 15 in §4. However, since the result is also true in the non-abelian case, we give the simple direct proof.

PROPOSITION 1. *Let $0 \to \mathcal{G} \xrightarrow{\alpha} \mathcal{H} \xrightarrow{\beta} \mathcal{K} \to 0$ be a short exact sequence in* **ILG**. *Then*:

(i) *if $\mathcal{H}$ is stable, then $\mathcal{K}$ is stable*;

(ii) *if $\mathcal{G}$ and $\mathcal{K}$ are stable, then $\mathcal{H}$ is stable.*

Proof. (i) With an obvious notation, let $\gamma_n \in K_n$ $(n \geq 1)$. For each n choose $\gamma'_n \in H_n$ such that $\beta_n(\gamma'_n) = \gamma_n$. By the assumed stability of $\mathcal{H}$, there are elements $y_n \in H_n$ such that $y_n = \gamma'_n h_n(y_{n+1})$ $(n \geq 1)$. Then, setting $z_n = \beta_n(y_n)$ $(n \geq 1)$, we easily deduce that $z_n = \gamma_n k_n(z_{n+1})$ for all n. Thus $\mathcal{K}$ is stable.

(ii) We now suppose that $\mathcal{G}$ and $\mathcal{K}$ are stable. Let $\gamma_n \in H_n$ $(n \geq 1)$ and consider the sequence $(\beta_n(\gamma_n)) \in \prod_{n\geq 1} K_n$.

Since $\mathcal{K}$ is stable, there are $\delta_n \in K_n$ $(n \geq 1)$ with $\delta_n = \beta_n(\gamma_n)k_n(\delta_{n+1})$ for all n. For each n, let $y_n \in H_n$ be such that $\beta_n(y_n) = \delta_n$. Then

$$\beta_n(y_n) = \beta_n(\gamma_n)(k_n\beta_{n+1})(y_{n+1}) = \beta_n(\gamma_n h_n(y_{n+1})),$$

so that $y_n^{-1}\gamma_n h_n(y_{n+1}) \in \ker \beta_n = \operatorname{im} \alpha_n$. For each n, let $u_n \in G_n$ be such that $y_n^{-1}\gamma_n h_n(y_{n+1}) = \alpha_n(u_n)$, i.e. $\gamma_n h_n(y_{n+1}) = y_n \alpha_n(u_n)$.

But $\mathcal{G}$ is stable, so there are $x_n \in G_n$ with $x_n = u_n g_n(x_{n+1})$ $(n \geq 1)$. Hence $\alpha_n(x_n) = \alpha_n(u_n)(h_n\alpha_{n+1})(x_{n+1})$. Set $w_n = y_n\alpha_n(x_n)$ for all n. Then $w_n = y_n\alpha_n(u_n)(\alpha_n g_n)(x_{n+1}) = \gamma_n h_n(y_{n+1})(h_n\alpha_{n+1})(x_{n+1}) = \gamma_n h_n(w_{n+1})$. Thus $(w_n) \in L(\gamma.\mathcal{H})$, and so $\mathcal{H}$ is stable.

COROLLARY 1. *Let $\mathcal{G} = (G_n; g_n)$ be a sequence in* **ILG**. *Suppose that each G_n has a normal subgroup H_n such that*:

(i) $g_n(G_{n+1}) \subseteq H_n$ $(n \geq 1)$;
(ii) *the sequence $\mathcal{H} = (H_n; \overline{g}_n)$ is stable* $(\overline{g}_n = g_n|H_{n+1} : H_{n+1} \to H_n)$.

Then $\mathcal{G}$ is stable.

Proof. For each n, the induced mapping $\widetilde{g}_n : G_{n+1}/H_{n+1} \to G_n/H_n$ is the trivial homomorphism, so that $\mathcal{G}/\mathcal{H} \equiv (G_n/H_n; \widetilde{g}_n)$ is stable (see the second trivial example before Theorem 4). But $\mathcal{H}$ is given to be stable, so the stability of $\mathcal{G}$ follows from Proposition 1(ii) applied to the canonical short exact sequence $0 \to \mathcal{H} \to \mathcal{G} \to \mathcal{G}/\mathcal{H} \to 0$.

3. Examples and applications

3.1. *The Mittag-Leffler and Weierstrass theorems of classical complex analysis.*

3.1.1. *The Mittag-Leffler theorem.* The abstract Mittag-Leffler theorem, Theorem 1, takes its name from the classical Mittag-Leffler theorem on meromorphic functions. In fact, the viewpoint of the present paper may also be used to rearrange the proof in a way that seems illuminating.

Let U be an open subset of the complex plane and let Z be a discrete subset of U; thus Z is a countable subset of U that has no cluster point in U, say $Z = \{a_n : n = 1, 2, \ldots\}$, where, despite the notation, the finite case is not excluded. For each k, let g_k be a given element of $\mathcal{O}(U \setminus \{a_k\})$ (a "principal part at a_k"). Then the classical Mittag-Leffler theorem asserts the existence of some $f \in \mathcal{O}(U \setminus Z)$ such that, for each k, $f - g_k$ has a removable singularity at a_k (i.e. f has the prescribed principal part at each a_k). We will explain how to reformulate this statement so that it will appear as an immediate consequence of the ideas of the last section. (We would emphasize that the proof we give is simply a rearrangement of the classical proofs.)

With U and Z as above, the restriction mapping $\mathcal{O}(U) \to \mathcal{O}(U \setminus Z)$ is injective; it is convenient to regard $\mathcal{O}(U)$ as a subgroup (additive) of $\mathcal{O}(U \setminus Z)$. We then define $M(U; Z) = \mathcal{O}(U \setminus Z)/\mathcal{O}(U)$ (it might be called the *Mittag-Leffler group* of the pair (U, Z)). We regard an element of $M(U; Z)$ as a "principal part on Z", since the set of functions in $\mathcal{O}(U \setminus Z)$, having prescribed principal parts at the points of Z, is precisely a coset of $\mathcal{O}(U)$ in $\mathcal{O}(U \setminus Z)$.

Next, in the special case where Z is *finite*, say $Z = \{a_1, \ldots, a_N\}$, we may, given $g_k \in \mathcal{O}(U \setminus \{a_k\})$ $(k = 1, \ldots, N)$, define $f(z) = \sum_{k=1}^{N} g_k(z)$ $(z \in U \setminus Z)$. Then clearly $f - g_k$ has a removable singularity at a_k, for each $k = 1, \ldots, N$. Thus, in case Z is finite, to specify a principal part at each point of Z is equivalent to specifying an element of $M(U; Z)$.

Now let Z be countably infinite. Write $U = \bigcup_{n \geq 1} U_n$, where, for each n,

(i) U_n is open and has compact closure $\overline{U}_n \subset U_{n+1}$;
(ii) every bounded component of $\mathbb{C} \setminus U_n$ meets $\mathbb{C} \setminus U$.

(This may, for example, be achieved by letting—for all sufficiently large n—$U_n = \{z \in U : |z| < n,\ \mathrm{dist}(z, \mathbb{C} \setminus U_n) > 1/n\}$.)

Then by Runge's theorem (e.g. in the form of [17], Chapter 5, §3, Theorem 3), for each n, $\mathcal{O}(U)|U_n$ is dense in $\mathcal{O}(U_n)$ (for the usual Fréchet topologies). In particular, writing $d_n : \mathcal{O}(U_{n+1}) \to \mathcal{O}(U_n)$ for restriction, the IL-sequence $\mathcal{O} \equiv (\mathcal{O}(U_n); d_n)_{n \geq 1}$ is a Mittag-Leffler sequence (in the sense defined after Theorem 4) with inverse-limit group $L(\mathcal{O}) \cong \mathcal{O}(U)$.

For each n, $Z_n \equiv Z \cap U_n$ is finite, and $U \setminus Z = \bigcup_{n \geq 1}(U_n \setminus Z_n)$; so also if $\mathcal{O}_Z \equiv (\mathcal{O}(U_n \setminus Z_n); \overline{d}_n)_{n \geq 1}$ (where $\overline{d}_n$ is the obvious restriction map) then $L(\mathcal{O}_Z) \cong \mathcal{O}(U \setminus Z)$ as well. If, for each n, $\widetilde{d}_n : M(U_{n+1}; Z_{n+1}) \to M(U_n; Z_n)$ is the homomorphism naturally induced on the quotient groups and we write $\mathcal{M} \equiv (M(U_n; Z_n); \widetilde{d}_n)_{n \geq 1}$, then there is the short exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_Z \longrightarrow \mathcal{M} \longrightarrow 0$$

in **ILGA**, with the sequence $\mathcal{O}$ stable, by Theorem 4(i).

Since each Z_n is finite, it should be clear, from the discussion of the case "Z finite" above, that to give a prescription of principal parts at the points of Z is precisely equivalent to specifying an element of $L(\mathcal{M})$. But, by Theorem 5, the sequence

$$0 \longrightarrow \mathcal{O}(U) \longrightarrow \mathcal{O}(U \setminus Z) \longrightarrow L(\mathcal{M}) \longrightarrow 0$$

is exact. In particular, every element of $L(\mathcal{M})$ is the image of some element of $\mathcal{O}(U \setminus Z)$, so that the Mittag-Leffler problem is always soluble.

3.1.2. *The Weierstrass theorem.* The theorem of Weierstrass may be regarded as a multiplicative analogue of the Mittag-Leffler theorem. However, the preliminaries from elementary complex analysis are a little more delicate. We shall summarize these preparatory results in a lemma, with a brief indication of proof. For a compact subset K of $\mathbb{C}$, we shall use the standard notation $R(K)$ for the uniform closure in $C(K)$ of the rational functions without poles in K. For any open subset U of $\mathbb{C}$, $\mathcal{O}^*(U)$ denotes the multiplicative group of nowhere-zero holomorphic functions on U (i.e. the group of units of $\mathcal{O}(U)$).

LEMMA 4. (i) *Let K be a compact subset of $\mathbb{C}$, and let Λ be a subset of $\mathbb{C} \setminus K$ that meets every bounded component of $\mathbb{C} \setminus K$. Then for every nowhere-zero $f \in R(K)$ there are $g \in R(K)$ and a rational function r, having all its zeros and poles in Λ, such that $f = re^g$ on K.*

(ii) *Let U be an open subset of $\mathbb{C}$, and Λ a subset of $\mathbb{C} \setminus U$ that meets every bounded component of $\mathbb{C} \setminus U$. Let f be a nowhere-zero holomorphic function on U. Then there are sequences (g_n), (r_n), where each $g_n \in \mathcal{O}(U)$ and each r_n is a rational function having all its zeros and poles in Λ, such that $r_n e^{g_n} \to f$ on U (in the usual Fréchet topology of $\mathcal{O}(U)$).*

(iii) *Let U, V be open subsets of $\mathbb{C}$ with $U \subset V$ and such that every bounded component of $\mathbb{C} \setminus U$ meets $\mathbb{C} \setminus V$. Then $\mathcal{O}^*(V)|U$ is dense in $\mathcal{O}^*(U)$.*

Proof. (i) We first use Runge's theorem (see reference in §3.1.1) to find a rational function s having all its poles in Λ and such that $\|f-s\|_K < \inf_K |f|$. It is then elementary that s is nowhere-zero on K and also $fs^{-1} \in e^{R(K)}$. So $s = p/q$, say, where p and q are polynomials, such that q has all its zeros in Λ, while p has no zeros in K. We then factorize $p(z) = (z-\alpha_1)\dots(z-\alpha_m)$, say (where $\alpha_1, \dots, \alpha_m$ are not necessarily distinct, and p has leading coefficient 1, without loss of generality). If, for any $\alpha \in \mathbb{C} \setminus K$, we define $u_\alpha \in R(K)$ by $u_\alpha(z) = z - \alpha$ ($z \in K$), then it is well known, and elementary, that: (a) if α and β belong to the same component of $\mathbb{C} \setminus K$, then $u_\alpha u_\beta^{-1} \in e^{R(K)}$, and (b) if α belongs to the unbounded component of $\mathbb{C} \setminus K$, then $u_\alpha \in e^{R(K)}$. We then consider the factorization $p = u_{\alpha_1} \dots u_{\alpha_m}$ of p. If, for example, α_1 is in the unbounded component of $\mathbb{C} \setminus K$, then u_{α_1} is an exponential on K; if α_1 lies in a bounded component of $\mathbb{C} \setminus K$, then there is a point of Λ, say λ_1, in the same component, so that $u_{\alpha_1} = u_{\lambda_1} e^{h_1}$, for some $h_1 \in R(K)$. Treating each factor in a similar way, we obtain, say, $p = p_1 e^h$, where p_1 is a polynomial with all its zeros in Λ and $h \in R(K)$. Take $r = p_1/q$ and the proof of (i) is complete.

(ii) The deduction of (ii) from (i) uses the form of Runge's theorem in [17], Chapter 5, §3, Theorem 1, together with the following simple exercise: if K is a compact subset of U such that no component of $U \setminus K$ is relatively compact in U (which is precisely the condition for $\mathcal{O}(U)|K$ to be dense in $R(K)$), then every bounded component of $\mathbb{C} \setminus K$ contains a component (necessarily bounded) of $\mathbb{C} \setminus U$. Under the hypothesis of (ii), it then follows that every bounded component of $\mathbb{C} \setminus K$ contains a point of Λ. The deduction of (ii) is now simple.

(iii) This is an easy consequence of (ii).

From this last lemma, it then follows that, if we write $U = \bigcup_{n \ge 1} U_n$, satisfying the same conditions (i) and (ii) given in the discussion of the Mittag-Leffler theorem in §3.1.1, then the IL-sequence $\mathcal{O}^* \equiv (\mathcal{O}^*(U_n); d_n)$ (where each d_n is again a restriction mapping) is a Mittag-Leffler sequence. (This is in contrast to the general situation for commutative Fréchet algebras; see the discussion in §3.6.) Then, with $Z = (a_k)_{k \ge 1}$ a discrete subset of U, as in §3.1.1, we define the *Weierstrass group* of the pair (U, Z) to be $W(U; Z) = \mathcal{O}^*(U \setminus Z)/\mathcal{O}^*(U)$. Analogously to the discussion of the Mittag-Leffler theorem, we use the stability of $\mathcal{O}^*$ to deduce that $W(U; Z) \cong \varprojlim W(U_n; Z_n)$. From this it follows easily that:

(i) given any sequence $n(k)$ of non-negative integers, there is an $f \in \mathcal{O}(U)$, having a zero of order $n(k)$ at a_k (for all k for which $n(k) > 0$), and having no other zeros in U;

(ii) given any sequence $n(k)$ of integers, there is a meromorphic function g on U such that, for each k, g has a zero of order $n(k)$ at a_k if $n(k) > 0$, a pole of order $-n(k)$ at a_k if $n(k) < 0$ and such that g is holomorphic and non-zero at every other point of U.

We just remark that (i) follows directly from the stability argument (since, just as in §3.1.1, each Z_n is finite, and the finite case is trivial), while (ii) follows by applying (i) twice and taking a quotient of two holomorphic functions.

3.2. *The quotient of a Fréchet algebra by a closed ideal.* Let A be a Fréchet algebra, $A \cong \varprojlim(A_n; d_n)$ an Arens–Michael representation of A. Let I be a closed, two-sided ideal of A. With the notation of §1, let $I_n = \overline{\pi_n(I)}$ (closure in A_n). Then, by Lemma 1, $I = \bigcap_{n \ge 1} \pi_n^{-1}(I_n)$; i.e. the Arens–Michael isomorphism, say $\alpha : A \to \varprojlim(A_n; d_n)$, induces an isomorphism $I \cong \varprojlim(I_n; \overline{d}_n)$, where $\overline{d}_n = d_n|I_{n+1} : I_{n+1} \to I_n$. Let $\widetilde{d}_n : A_{n+1}/I_{n+1} \to A_n/I_n$ be the homomorphism induced by d_n. Then we have the following simple result:

THEOREM 6. *With the above notation, the Arens–Michael isomorphism induces an isomorphism*

$$A/I \cong \varprojlim(A_n/I_n; \widetilde{d}_n).$$

Proof. By Lemma 1, the IL-sequence $\mathcal{I} = (I_n; \overline{d}_n)$ is a Mittag-Leffler sequence; it is therefore stable, by Theorem 4(i). Let $\mathcal{A} = (A_n; d_n)$; the result then follows from Theorem 5, by considering (with an obvious notation) the short exact sequence $0 \to \mathcal{I} \to \mathcal{A} \to \mathcal{A}/\mathcal{I} \to 0$.

Remark. The result of Theorem 6 is not just elementary algebra of inverse limits—in the sense that the result does not extend to an arbitrary complete LMC algebra (i.e. an inverse limit of Banach algebras over a general directed index set, see [15]). Indeed, using a remark in the final paragraph of [13], §31, 6, we may find an example of a complete, commutative LMC algebra A with a closed ideal I such that A/I is not complete. (The example in [13] is of a locally convex topological vector space; it may be converted into an LMC algebra by giving it the zero mutiplication, and then adjoining an identity if wished.) Since an inverse limit of Banach spaces is necessarily complete, this is an example in which $A/I \neq \varprojlim A_\lambda/I_\lambda$.

3.3. *Finitely generated ideals.* In [4], Theorem 4.2, Arens proved one of the most interesting results in the general theory of Fréchet algebras, concerning finitely generated ideals. In fact, the result in [4] was proved for a more general class of inverse limits of complete metric rings; the additional generality has some interest and we will also, in this section, work with such rings. We shall give an extension of the result of Arens, but only in the case of *commutative rings*. (In the non-commutative case we would have nothing to add to the Arens result, beyond a different way of organizing the proof.) By an *F-ring* we mean a complete metrizable topological ring. We remark that an arbitrary ring becomes an F-ring when given the discrete topology; so the topological-algebraic results which follow include purely algebraic results as special cases.

Let A be a commutative F-ring, and suppose that, say, $\alpha : A \to L(\mathcal{A})$ is a topological-ring isomorphism of A with the inverse limit of a Mittag-Leffler sequence $\mathcal{A} = (A_n; d_n)_{n\geq 1}$ of commutative F-rings and continuous homomorphisms (see definition just before Theorem 5). (We could take $A = L(\mathcal{A})$, with α the identity mapping; but, in several places, writing an explicit isomorphism α is notationally more convenient.) For example, A might be a given commutative Fréchet algebra and α an Arens–Michael representation of A (in which case each A_n would be a Banach algebra). To be consistent with the standard notation for Fréchet algebras, define the continuous homomorphisms $\pi_n : A \to A_n$ by $\alpha(a) = (\pi_n(a))_{n\geq 1}$ $(a \in A)$. By Theorem 1, $\pi_n(A)$ is dense in A_n, for each n.

Following Arens, we say that an m-tuple $(x_1, \dots, x_m)$ of elements of a commutative ring R with identity is *regular* (in [4] it is *right regular*) if and only if there are elements $y_1, \dots, y_m$ of R such that $\sum_{k=1}^{m} x_k y_k = 1$. Then the result of Arens [4], Theorem 4.2 (but restricted to commutative rings), is as follows.

THEOREM 7 (Arens). *Suppose $a_1, \dots, a_m$ is a finite subset of A such that $\pi_n a_1, \dots, \pi_n a_m$ is a regular system in A_n, for each n. Then $a_1, \dots, a_m$ is a regular system in A.*

Theorem 8, below, gives a result about a larger class of finitely generated ideals; Theorem 7 will be an immediate consequence. We now introduce the notion that will generalize the Arens notion of regularity.

Let A be a commutative F-ring. If $a = (a_1, \dots, a_m), b = (b_1, \dots, b_m) \in A^m$, let $a.b \equiv \sum_{k=1}^{m} a_k b_k$. For $a \in A^m$ define $Z(a) = \{z \in A^m : z.a = 0\}$.

If we make A^m into a complete metric space in the obvious way (as the product of m copies of A), then it is clear that $Z(a)$ is a closed subspace of A^m. We shall define a notion of *A-independent* m-tuple $a \in A^m$, whose intended intuitive content is that "the only elements in $Z(a)$ are the obvious ones". Consider first the case of an ordered pair $a = (a_1, a_2)$. Then evidently $Z(a)$ contains all pairs $(a_2 b, -a_1 b)$, for arbitrary $b \in A$; also, since $Z(a)$ is closed, it contains the closure of this set. The notion of A-independence to be introduced will say that these are the *only* elements of $Z(a)$. The appropriate notion for general $m \geq 1$ is as follows.

For a commutative F-ring A and integer $m \geq 1$ let $S_m(A)$ be the set of all skew-symmetric $(m \times m)$-matrices with entries from A. If $a \in A^m$ (regarded as a $(1 \times m)$-vector), let

$$S(a) = \{aX : X \in S_m(A)\}.$$

Remark that $S(a) \subseteq Z(a)$ since, for each $X \in S_m(A)$, $(aX.a) = aXa^{t} = 0$, by the skew-symmetry of X. Hence also $\overline{S(a)} \subseteq Z(a)$. We say that the m-tuple $a \in A^m$ is *A-independent* if and only if $\overline{S(a)} = Z(a)$.

Remarks. 1. In the case $m = 2$, the definition of A-independence reduces to that discussed in the previous paragraph.

2. In the case $m = 1$, since $S_1(A) = 0$, a single element a_1 (or, more pedantically, the 1-tuple (a_1)) is A-independent if and only if $Z(a_1) = 0$. For example, if A is an integral domain then every non-zero element of A is A-independent. On the other hand, a_1 is regular in the sense of [4] if and only if it is invertible (and A must have an identity).

3. Unfortunately, the notion of A-independence does not have good permanence properties. For example, we may easily find examples of (i) an algebra A and an A-independent n-tuple $(a_1, \dots, a_n) \in A^n$ but with $(a_1, \dots, a_m)$ not A-independent for any $1 \leq m \leq n-1$; (ii) an example where, for instance, (a) is A-independent but (a, b) is not; (iii) an example in which A-independence is not preserved by a continuous homomorphism; (iv) A-independence is not preserved by passing either to a subalgebra or a superalgebra. These negative remarks make the result of Lemma 6(iii) below a little surprising.

4. There is a connection between the present notion of A-independence and the type of regularity that arises in the definition of the Taylor spectrum of a commuting n-tuple of operators introduced in [19]. We shall not use this connection, so merely remark that if we identify an element $a \in A$ with the corresponding multiplication operator on A (recall that A is here commutative), then regularity of an m-tuple $a \in A^m$ in the Taylor sense is defined as the exactness of a certain Koszul complex. To require (in the present notation) that $S(a) = Z(a)$ is equivalent to the exactness of this complex *just at the penultimate term*. Moreover, since we only require that

$\overline{S(a)} = Z(a)$, the notion of an A-independent m-tuple is even slightly less restrictive.

Next we show that A-independence really does generalize regularity, in the commutative case.

PROPOSITION 2. *Let A be a commutative F-ring with identity, let $a \in A^m$ and suppose that there is some $u \in A^m$ such that $u.a = 1$. Then $S(a) = \overline{S(a)} = Z(a)$, so that, in particular, a is A-independent.*

Proof. Let $u = (u_1, \ldots, u_m)$, so that $\sum_{k=1}^m u_k a_k = 1$. Let $z \in Z(a)$ and define $x_{ij} = u_i z_j - u_j z_i$ for $i, j = 1, \ldots, m$. Then the matrix $X = (x_{ij})$ is skew-symmetric and for $j = 1, \ldots, m$,

$$(aX)_j = \sum_i a_i x_{ij} = \sum_i a_i(u_i z_j - u_j z_i) = \Big(\sum_i a_i u_i\Big) z_j - \sum \Big(a_i z_i\Big) u_j = z_j,$$

i.e. $z = aX$, so that $z \in S(a)$.

EXAMPLE. Let $A = \mathcal{O}(\mathbb{C}^N)$, the Fréchet algebra of all entire functions in N variables. We shall show that, for each $m = 1, \ldots, N$, the m-tuple $(z_1, \ldots, z_m)$ of coordinate projections is A-independent (it is not, of course, regular in the sense of [4]); in fact, even $S(z_1, \ldots, z_m) = Z(z_1, \ldots, z_m)$ for each m.

The proof will be by induction on m. For $m = 1$, the result is clear, since A is an integral domain. Now let $2 \le m \le N$ and suppose that the result holds for $m - 1$ variables. Let $f = (f_1, \ldots, f_m) \in Z(z_1, \ldots, z_m)$, i.e. suppose that

$$(*) \qquad \sum_{k=1}^m f_k z_k = 0.$$

By rearranging its Taylor series, we may uniquely write

$$f_m = \sum h_{i_1,\ldots,i_{m-1}}(z_m, \ldots, z_N) z_1^{i_1} \ldots z_{m-1}^{i_{m-1}},$$

where the summation is over all $(i_1, \ldots, i_{m-1}) \in (\mathbb{Z}^+)^{m-1}$ and each coefficient function is an entire function in the variables $z_m, \ldots, z_N$.

Put $z_1 = \ldots = z_{m-1} = 0$ in $(*)$; then $f_m(0, \ldots, 0, z_m, \ldots, z_N) = 0$, i.e. $h_{0,\ldots,0}(z_m, \ldots, z_N) = 0$. So $f_m = a_1 z_1 + \ldots + a_{m-1} z_{m-1}$, where $a_1, \ldots, a_{m-1} \in A$. Relation $(*)$ may thus be written as $\sum_{k=1}^{m-1} (f_k + a_k z_m) z_k = 0$.

By the induction hypothesis, there is a matrix $X = (x_{kj}) \in S_{m-1}(A)$ such that $f_k + a_k z_m = \sum_{j=1}^{m-1} x_{kj} z_j$ $(k = 1, \ldots, m-1)$. Together with the equation $f_m = a_1 z_1 + \ldots + a_{m-1} z_{m-1}$, this may be written as

$$\begin{pmatrix} f_1 \\ \vdots \\ f_{m-1} \\ f_m \end{pmatrix} = \left(\begin{array}{ccc|c} & X & & -a_1 \\ & & & \vdots \\ & & & -a_{m-1} \\ \hline a_1 & \ldots & a_{m-1} & 0 \end{array}\right) \begin{pmatrix} z_1 \\ \vdots \\ z_{m-1} \\ z_m \end{pmatrix},$$

which completes the proof.

NOTATION. If, say, $T : A \to B$ is a homomorphism (of commutative rings), we shall use the *same* symbol T for the maps from e.g. A^m to B^m (or from $S_m(A)$ to $S_m(B)$) defined by applying the original homomorphism T to each component (or entry). Thus, for example, if $a = (a_1, \ldots, a_m) \in A^m$ then $T(a)$ means $(Ta_1, \ldots, Ta_m)$.

If A is an F-ring, then A^m and $S_m(A)$ are also F-rings when topologized as the cartesian product of the appropriate finite number of copies of A.

LEMMA 5. *Let $T : A \to B$ be a continuous homomorphism of commutative F-rings. Let $(a_1, \ldots, a_m) \in A^m$, let $b_k = T(a_k)$, for $k = 1, \ldots, m$, and let $b = (b_1, \ldots, b_m)$. Then:*

(i) $T(Z(a)) \subseteq Z(b)$, $T(S(a)) \subseteq S(b)$;
(ii) *if $T(A)$ is dense in B, then $T(S(a))$ is dense in $S(b)$;*
(iii) *if $T(A)$ is dense in B and if b is B-independent, then $T(S(a))$ is dense in $Z(b)$ and so, also, $T(Z(a))$ is dense in $Z(b)$.*

Proof. (i) This is a very elementary exercise.

(ii) Let $z \in S(b)$; so $z = bX$ for some $X \in S_m(B)$. But, since $T(A)$ is dense in B, it is trivial that $T(S_m(A))$ is dense in $S_m(B)$. We may thus pick, say, $Y \in S_m(A)$ such that $T(aY)$ is in any chosen neighbourhood of bX.

(iii) This is immediate from (ii), since $S(a) \subseteq Z(a)$ and we are now assuming that $S(b)$ is dense in $Z(b)$.

We now return to the situation in which $\alpha : A \to L(\mathcal{A})$ is an isomorphism of the commutative F-ring A with a Mittag-Leffler sequence $\mathcal{A} = (A_n; d_n)_{n \ge 1}$ of commutative F-rings and continuous homomorphisms.

LEMMA 6. *With the notation just described, let $a \in A^m$, for some $m \ge 1$.*

(i) *The isomorphism α induces an isomorphism*

$$\overline{S(a)} \cong \varprojlim(\overline{S(\pi_n(a))}; d_n).$$

(ii) $\overline{S(a)} \subseteq Z(a) \subseteq \alpha^{-1}(\varprojlim(Z(\pi_n(a)); d_n))$.

(iii) *If, for each $n \ge 1$, $\pi_n(a)$ is A_n-independent, then a is A-independent and*

$$Z(a) \cong \varprojlim(Z(\pi_n(a)); d_n).$$

Proof. (i) From Lemma 5(ii) it follows that $\pi_n(S(a))$ is dense in $S(\pi_n(a))$ for each n. By Lemma 1, $\overline{S(a)} \cong \varprojlim \overline{S(\pi_n(a))}$.

(ii) Trivial.

(iii) If, for each n, $\pi_n(a)$ is A_n-independent, then $\overline{S(\pi_n(a))} = Z(\pi_n(a))$ for each n. Hence the result is immediate from (i) and (ii).

THEOREM 8. *Let $a \in A^m$ be such that $\pi_n(a)$ is A_n-independent, for every $n \geq 1$. Let $J(a) = \sum_{k=1}^m Aa_k$ and, for each n, let $J_n(a) = \sum_{k=1}^m A_n\pi_n(a_k)$. Then the isomorphism α induces isomorphisms*

(i) $J(a) \cong \varprojlim(J_n(a); \overline{d}_n)$;

(ii) $A/J(a) \cong \varprojlim(A_n/J_n(a); \widetilde{d}_n)$.

(Here $\overline{d}_n = d_n|J_{n+1}(a)$ and $\widetilde{d}_n : A_{n+1}/J_{n+1}(a) \to A_n/J_n(a)$ is the homomorphism induced between the quotient rings.)

Proof. (i) For each n, define the mapping $\sigma_n : A_n^m \to A_n$ by

$$\sigma_n(x_1, \ldots, x_m) = \sum_{k=1}^m x_k\pi_n(a_k) \quad ((x_1, \ldots, x_m) \in A_n^m).$$

Then σ_n is an additive-group homomorphism, with $\operatorname{im}\sigma_n = J_n(a)$ and $\ker\sigma_n = Z(\pi_n(a))$. We write $j_n : Z(\pi_n(a)) \to A_n^m$ for inclusion.

Let $\mathcal{Z}$, $\mathcal{A}^m$, $\mathcal{J}(a)$ be the IL-sequences $(Z(\pi_n(a)); d_n|Z(\pi_{n+1}(a)))$, $(A_n^m; d_n)$, $(J_n(a); \overline{d}_n)$ respectively; then we evidently have a short exact sequence in **ILG**, with an obvious notation:

$$0 \longrightarrow \mathcal{Z} \xrightarrow{j} \mathcal{A}^m \xrightarrow{\sigma} \mathcal{J}(a) \longrightarrow 0.$$

By Lemmas 5 and 6, $\mathcal{Z}$ is a Mittag-Leffler sequence with $L(\mathcal{Z}) \cong Z(a)$. The sequence $\mathcal{A}^m$ is just the mth power of the Mittag-Leffler sequence $\mathcal{A}$, so it is trivially itself a Mittag-Leffler sequence with $L(\mathcal{A}^m) \cong A^m$. By Theorem 4, $\mathcal{Z}$ is a stable IL-sequence and so, by Theorem 5, the inverse-limit functor induces an isomorphism

$$L(\mathcal{J}(a)) \cong A^m/Z(a) \cong J(a).$$

(ii) We now consider, with an obvious notation, the short exact sequence in **ILG**

$$0 \longrightarrow \mathcal{J}(a) \xrightarrow{i} \mathcal{A} \xrightarrow{q} \mathcal{A}/\mathcal{J}(a) \longrightarrow 0,$$

where i is a sequence of inclusions, and q is a sequence of quotient maps. By (i), $\mathcal{J}(a)$ is a quotient of the Mittag-Leffler sequence $\mathcal{A}^m$, and so is stable by Theorem 4 and Proposition 1(i). Also from (i) we have $L(\mathcal{J}(a)) \cong J(a)$. The result therefore follows from Theorem 5.

Remark. From this last result we have an immediate proof of Theorem 7, in the case of a commutative F-ring. For, if $a \in A^m$ is such that, for each n, $\pi_n(a)$ is *regular*, then $J_n(a) = A_n$ for each n, and also $\pi_n(a)$ is A-independent, by Proposition 2. Theorem 8(ii) then gives $A/J(a) \cong 0$, i.e. $J(a) = A$, so that a is regular. This completes the proof of Theorem 7.

3.4. *Formal power series and Banach algebras*. We recall (see e.g. [2], §1) that an element x of a commutative (unital) Banach algebra A is said to have *finite closed descent* (*FCD*) if and only if Ax^{N+1} is dense in Ax^N for some integer $N \geq 0$; we write $\delta(x) = N$ if N is the least integer with this property. We refer to [2], Lemma 1, for a summary of the elementary properties of elements of finite closed descent. In particular, if $\delta(x) = N$ then also Ax^n is dense in Ax^N for all $n \geq N$; moreover, the ideal $I(x) \equiv \bigcap_{n\geq 1} Ax^n$ is dense in Ax^N. If we define $L_x : A \to A$ by $L_x(y) = xy$ $(y \in A)$, then L_x maps $I(x)$ bijectively onto itself. It follows that $I(x)$ is naturally isomorphic to the inverse limit of the sequence $A \xleftarrow{L_x} A \xleftarrow{L_x} A \xleftarrow{L_x} \ldots$ ([2], Corollary 1).

In [1], one main ingredient in proving the embedding of $\mathcal{F} \equiv \mathbb{C}[[X]]$ in certain Banach algebras was the following result, of interest in its own right (where we write $\pi_x : A \to A/I(x)$ for the quotient homomorphism):

THEOREM 9. *Let A be a commutative, unital Banach algebra and let $x \in A$. The following are equivalent*:

(i) *x has FCD*;

(ii) *there is a (unique) unital homomorphism $\theta_x : \mathcal{F} \to A/I(x)$ such that $\theta_x(X) = \pi_x(x)$.*

(See [1], Lemma 3, [2], Proposition 1.)

We shall show that the proof of (i)⇒(ii) in Theorem 9 may be rather attractively expressed using the ideas of the present paper. The next lemma also shows some new properties of elements of finite closed descent.

LEMMA 7. *Let A be a commutative, unital Banach algebra and let $x \in A$ have FCD. Then the sequences $\mathcal{A}$ and $\mathcal{B}$ in* **ILGA** *defined by*

$$\mathcal{A}: \quad A \xleftarrow{L_x} A \xleftarrow{L_x} A \xleftarrow{L_x} \ldots,$$

$$\mathcal{B}: \quad Ax \xleftarrow{j_1} Ax^2 \xleftarrow{j_2} Ax^3 \xleftarrow{j_3} \ldots$$

(*where each j_n is an inclusion*) *are both stable.*

Proof. (i) Let $\delta(x) = N$ and consider the sequence, say

$$\mathcal{A}_N: \quad A \xleftarrow{L_x^N} A \xleftarrow{L_x^N} A \xleftarrow{L_x^N} \ldots,$$

which is a subsequence of $\mathcal{A}$. Put $I_N = \overline{Ax^N}$; then $I_N \xleftarrow{L_x^N} I_N \xleftarrow{L_x^N} \ldots$ is a Mittag-Leffler sequence and so stable by Theorem 4. But also $L_x^N(A) \subseteq I_N$, so that $\mathcal{A}_N$ is stable by Corollary 1. But then $\mathcal{A}$ is stable by Lemma 3.

(ii) We have a short exact sequence in **ILGA**, namely

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{\varrho} \mathcal{B} \longrightarrow 0,$$

where $\varrho_n(a) = ax^n \in Ax^n$ $(n \geq 1)$ and $\mathcal{K}$ is simply the sequence of kernels. But $\mathcal{A}$ is stable by (i), so the stability of $\mathcal{B}$ follows from Proposition 1(i).

Now let $x \in A$ have FCD, let $q_n : A \to A/Ax^n$ be a quotient mapping ($n \geq 1$). Then, since $I(x) = \bigcap_{n\geq 1} Ax^n$, there is a naturally induced injective homomorphism, say $T : A/I(x) \to \varprojlim(A/Ax^n; i_n)$, where $i_n : A/Ax^{n+1} \to A/Ax^n$ is the homomorphism induced by the inclusion $Ax^{n+1} \subseteq Ax^n$.

COROLLARY 2. *With the above notation, and with x having FCD, the mapping $T : A/I(x) \to \varprojlim(A/Ax^n; i_n)$ is an isomorphism.*

Proof. For each n we have the canonical short exact sequence

$$0 \longrightarrow Ax^n \longrightarrow A \longrightarrow A/Ax^n \longrightarrow 0.$$

By Lemma 7(ii), the sequence $\mathcal{B} = (Ax^n; j_n)$ (where each j_n is an inclusion) is stable. Of course, $L(\mathcal{B}) \cong \bigcap_{n\geq 1} Ax^n = I(x)$ so, by Theorem 5, T effects an isomorphism, as stated.

We may now at once deduce

Proof of (i)⇒(ii) of Theorem 9. For each $n \geq 1$, there is clearly the unique unital homomorphism $\theta_n : \mathcal{F} \to A/Ax^n$ such that $\theta_n(X) = q_n(x)$, given by

$$\theta_n\Big(\sum_{k=0}^{\infty} \lambda_k X^k\Big) = q_k\Big(\sum_{k=0}^{n-1} \lambda_k x^k\Big).$$

Clearly, $\theta_n = i_n\theta_{n+1}$ ($n \geq 1$), so the sequence (θ_n) defines a homomorphism, say, $\theta : \mathcal{F} \to \varprojlim(A/Ax^n; i_n)$. But, by Corollary 2, this last algebra is isomorphic (by the natural isomorphism) to $A/I(x)$, which completes the proof.

3.5. *Elements of locally finite closed descent in a Fréchet algebra.* For this application, which is concerned with extending the results of §3.4 to Fréchet algebras, the reader is referred to [2] (especially Lemma 7; this was the example that motivated the ideas of the present paper.)

3.6. *The group of units of a Fréchet algebra: Arens–Royden theorems.* Let A be a unital Fréchet algebra (not necessarily commutative), and let $\alpha : A \to \varprojlim(A_n; d_n)$ be an Arens–Michael representation of A as an inverse limit of unital Banach algebras. One of the most basic results about Fréchet algebras is that an element $x \in A$ is invertible if and only if $\pi_n(x)$ is invertible in A_n for every n (see [15], Theorem 5.2(c)). (From the result of Arens, [4], Theorem 4.2, there is also the deeper fact that x is right-invertible if and only if $\pi_n(x)$ is right-invertible in A_n for every n.) Thus, writing $G(A)$, $G(A_n)$ for the groups of units (i.e. invertible elements) of A, A_n respectively, we find that α induces a topological-group isomorphism, $\alpha : G(A) \cong \varprojlim(G(A_n); d_n)$ (where $d_n|G(A_{n+1})$ is written simply as d_n).

Each $G(A_n)$ is an open subset of A_n, and so is complete-metrizable (in a metric that is topologically equivalent to the metric induced by the norm of A_n). Thus, say $\mathcal{G} = (G(A_n); d_n)_{n\geq 1}$ is an IL-sequence of complete metrizable topological groups and continuous homomorphisms and we know that $L(\mathcal{G}) \cong G(A)$. However, $\mathcal{G}$ is not, in general, a Mittag-Leffler sequence, i.e. it is not always true that $d_n(G_{n+1})$ is dense in G_n. The following example is instructive.

EXAMPLE: *A continuous surjection $T : A \to B$ of Banach algebras, with $T(G(A))$ not dense in $G(B)$.* Let $\Delta = \{z \in \mathbb{C} : |z| \leq 1\}$, $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$. Let $A = C(\Delta)$, the uniform algebra of all continuous, complex-valued functions on Δ, let $B = C(\Gamma)$ and let $T : A \to B$ be the restriction map. Then $T(A) = B$. However, $T(G(A))$ is precisely the component of the identity in $G(B)$ (i.e. it consists of all nowhere-zero continuous functions on Γ that have continuous logarithms); this set is a proper open-and-closed subgroup $G(B)$, so that $T(G(A))$ is not dense in $G(B)$.

In fact, this last simple example illustrates the general situation. For a Fréchet algebra A, we write $G_0(A)$ for the component of the identity in the topological group $G(A)$ and let $E(A)$ be the set of all finite products of exponentials. It is at least evident that $E(A)$ is a subgroup of $G_0(A)$. We recall that, if A is a unital *Banach* algebra, then $E(A) = G_0(A)$. In the case of a unital Fréchet algebra, $E(A)$ is dense in $G_0(A)$ (see Lemma 9(ii) below) but, as shown by Davie [6], there are examples (even with A commutative) in which $E(A) \neq G_0(A)$ (see the discussion of Arens–Royden theorems below).

LEMMA 8. *Let A, B be unital Fréchet algebras and let $T : A \to B$ be a continuous homomorphism with $T(A)$ dense in B. Then $T(E(A))$ is dense in $E(B)$.*

Proof. Let $h \in E(B)$, say $h = e^{b_1} \ldots e^{b_k}$, for some finite subset $\{b_1, \ldots \ldots, b_k\}$ of B. Since $T(A)$ is dense in B, we can, by choosing $a_1, \ldots, a_k$ in A, with $T(a_j)$ close to b_j ($j = 1, \ldots, k$), and setting $g = e^{a_1} \ldots e^{a_k}$, make $T(g) = e^{T(a_1)} \ldots e^{T(a_k)}$ close to h.

COROLLARY 3. *If, in Lemma 8, B is a Banach algebra (with A any unital Fréchet algebra), then $T(G_0(A))$ is dense in $G_0(B)$.*

Proof. This is immediate from the lemma, since $E(A) \subseteq G_0(A)$, while $E(B) = G_0(B)$.

LEMMA 9. *Let A be a unital Fréchet algebra, with $\alpha : A \to \varprojlim(A_n; d_n)$ an Arens–Michael representation. Then α induces isomorphisms*

(i) $G(A) \cong \varprojlim(G(A_n); d_n)$;

(ii) $G_0(A) \cong \varprojlim(G_0(A_n); d_n)$, *and, moreover, the sequence representing $G_0(A)$ is a Mittag-Leffler sequence.*

Further,

(iii) *$E(A)$ is dense in $G_0(A)$.*

Proof. (i) is just the classical result of Michael referred to above.

(ii) In the relative topologies of $G(A)$, $G(A_n)$ respectively, $G_0(A)$ is closed in $G(A)$ and $G_0(A_n)$ is closed in $G(A_n)$. By Corollary 3, for each n, $\pi_n(G_0(A))$ is dense in $G_0(A_n)$. By Lemma 1 (applied to the IL-sequence $(G(A_n);d_n)$), α induces an isomorphism $G_0(A) \cong \varprojlim(G_0(A_n);d_n)$. Moreover, it follows from Corollary 3 that $(G_0(A_n);d_n)$ is a Mittag-Leffler sequence.

(iii) Since, by Lemma 8, $\pi_n(\overline{E(A)})$ is dense in $G_0(A_n)$, for each n, we deduce from Lemma 1 also that $\overline{E(A)} = G_0(A)$.

THEOREM 10. *Let A be a unital Fréchet algebra, with $\alpha: A \to \varprojlim(A_n;d_n)$ an Arens–Michael representation. Then α induces an isomorphism*

$$G(A)/G_0(A) \cong \varprojlim(G(A_n)/G_0(A_n);\widetilde{d}_n).$$

(As usual, we have written $\widetilde{d}_n : G(A_{n+1})/G_0(A_{n+1}) \to G(A_n)/G_0(A_n)$ for the homomorphism induced on the quotient groups by d_n.)

Proof. With a hopefully obvious notation, we consider the short exact sequence in **ILG**

$$0 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}_0 \longrightarrow 0.$$

By Lemma 9(ii), $\mathcal{G}_0$ is a Mittag-Leffler sequence, hence stable by Theorem 4. Since by Lemma 9 we have $L(\mathcal{G}_0) \cong G_0(A)$ and $L(\mathcal{G}) \cong G(A)$, the result follows by Theorem 5.

Remark. Theorem 10 suggests connections with the Arens–Royden theorem (the exposition in [11], Chapter III, §7 is especially well-suited to the present section). Let A be a commutative Fréchet algebra. Then there are, in fact, *two* theorems that generalize the Arens–Royden theorem for commutative Banach algebras, one describing $G(A)/G_0(A)$ and the other describing $G(A)/E(A)$.

It may be useful to recall the example of Davie [6], referred to above, of a commutative Fréchet algebra with $G_0(A) \neq E(A)$. (I am grateful to Jean Esterle for drawing my attention to this example.) We shall denote Davie's example by D; write $\mathbb{N} = \{1,2,\ldots\}$ and then define

$$D = \{f \in C(\mathbb{C}) : f \text{ is bounded on } \mathbb{N}\},$$

where D is given the topology of uniform convergence on the compact subsets of $\mathbb{C}$ *and* on $\mathbb{N}$. Clearly, D is a commutative Fréchet algebra with identity. The functions $e^{2\pi iz}$ and $e^{-2\pi iz}$ are both in D, so that $e^{2\pi iz}$ is in $G(D)$; it is not, however, in $E(D)$, since none of its continuous logarithms on $\mathbb{C}$ belongs to D. However, it is even true that $G(D) = G_0(D)$. For, let $f \in G(D)$; then it is simple to see that, for each compact subset K of $\mathbb{C}$, there is a function $g_K \in D$ such that $f(z) = \exp g_K(z)$ for all $z \in K \cup \mathbb{N}$. Thus $f \in \overline{E(D)} = G_0(D)$. We remark that, if we define the topological space X to be the union of $\mathbb{C}$ and the Stone–Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$, identified along $\mathbb{N}$, then $A \cong C(X)$.

Let Φ_A be the set of all continuous characters on the commutative, unital Fréchet algebra A. We write w for the weak-* topology $\sigma(\Phi_A, A)$ on Φ_A. If $A = \varprojlim(A_n;d_n)$ is an Arens–Michael representation of A, let Φ_n be the set of all characters on A that are continuous with respect to the seminorm p_n on A (in standard notation, see §1). Then the mapping $\phi \mapsto \phi \circ \pi_n$ ($\phi \in \Phi_{A_n}$) is a homeomorphism from the character space of A_n (in its usual Gelfand topology) onto the subset Φ_n of Φ_A. Then $\Phi_A = \bigcup_{n=1}^{\infty} \Phi_n$ expresses Φ_A as the union of an increasing sequence of compact subsets; moreover, every w-compact subset of Φ_A is a subset of some Φ_n. However, as remarked by Davie [6], for many purposes it is more useful to give Φ_A the so-called *k-topology* κ: this topology is defined by specifying that a subset F of Φ_A is κ-closed if and only if $F \cap K$ is w-compact for every w-compact subset K of Φ_A (or, equivalently, if and only if $F \cap \Phi_n$ is w-compact for every n). It is then elementary that $w \leq \kappa$ and that $w|K = \kappa|K$ for every w-compact subset K of Φ_A. Also, a function $f : \Phi_A \to \mathbb{C}$ is κ-continuous if and only if $f|K$ is continuous on K, for every compact subset K of Φ_A. (Note that if, for example, $(\Phi_A; w)$ is locally compact, then $w = \kappa$.) One good reason for preferring κ to w is that, when Φ_A is given the topology κ, then $C(\Phi_A)$ is a Fréchet algebra, in the topology of uniform convergence on compact subsets of Φ_A.

We may now easily deduce the first form of the Arens–Royden theorem for Fréchet algebras. (As far as I know, this is a new result.)

THEOREM 11. *Let A be a commutative, unital Fréchet algebra, let Φ_A have the k-topology κ, and let $C = C(\Phi_A)$. Then the Gelfand mapping of A effects an isomorphism*

$$G(A)/G_0(A) \cong G(C)/G_0(C).$$

Proof. By the Arens–Royden theorem for Banach algebras (e.g. [11], Chapter III, Corollary 7.3), for each n, the Gelfand mapping effects an isomorphism $G(A_n)/G_0(A_n) \cong G(C_n)/G_0(C_n)$, where $C_n = C(\Phi_n)$. The result now follows at once by applying Theorem 10 to the Fréchet algebras A and C.

Remark. Theorem 11 shows that $G(A)/G_0(A)$ depends only on the topology of Φ_A (either w or κ, since w determines κ uniquely). The question of identifying $G(C)/G_0(C)$ with some other topological invariant of Φ_A is then pure topology; we remark that it is *not* in general equal to $H^1(\Phi_A;\mathbb{Z})$, which in fact occurs with the second extension of the Arens–Royden theorem (Theorem 12). We merely make the obvious remark that $G(C)/G_0(C)$ is isomorphic to $\varprojlim_K H^1(K;\mathbb{Z})$ as K runs through the compact subsets of Φ_A.

In §2 of [6], Davie proved a different version of the Arens–Royden theorem for Fréchet algebras, using $E(A)$ rather than $G_0(A)$. In fact, Davie's proof (contained in Theorems 2.2 and 2.3 of [6]) may be rather nicely presented using the idea of a stable sequence. We shall give this as another illustration of the ideas of the present paper.

THEOREM 12 (Davie [6]). *With the notation of Theorem* 11 (*in particular*, Φ_A *has the topology* κ), *the Gelfand mapping effects an isomorphism*

$$G(A)/E(A) \cong G(C)/E(C).$$

Proof. With standard notation $A = \varprojlim(A_n; d_n)$, we have, for each n, a short exact sequence of groups and homomorphisms

$$0 \longrightarrow A_n \xrightarrow{\theta_n} G(A_n) \times C_n \xrightarrow{q_n} G(C_n) \longrightarrow 0,$$

where

$$\theta_n(a) = (e^a, \widehat{a}) \quad (a \in A_n), \qquad q_n(b,g) = \widehat{b}e^{-g} \quad (b \in G(A_n),\ g \in C_n),$$

and $a \mapsto \widehat{a}$ is the Gelfand mapping of A_n.

We claim that the assertion of exactness is essentially a translation of the Arens–Royden theorem for the Banach algebra A_n. Firstly, θ_n is injective, for if $a \in \ker\theta_n$, then both $e^a = 1$ and $\widehat{a} = 0$, i.e. a is quasinilpotent; it follows easily that $a = 0$. Next, it is clear that $\operatorname{im}\theta_n \subseteq \ker q_n$; but also, if $(b,g) \in \ker q_n$, then $\widehat{b} = e^g$; so it follows (Arens–Royden for Banach algebras or, if preferred, the implicit function theorem for Banach algebras) that there is a unique element $a \in A_n$ such that both $b = e^a$ and $\widehat{a} = g$, i.e. $(b,g) \in \operatorname{im}\theta_n$. That q_n is surjective again follows from the Arens–Royden theorem applied to the Banach algebra A_n.

It is also clear that the sequences (θ_n), (q_n) define morphisms of the obvious sequences in **ILGA**. But the kernel sequence (A_n) is just the Arens–Michael representation of A, so is certainly a Mittag-Leffler sequence and is thus stable. By Theorem 5 (and using Lemma 9 to identify the inverse limits), it follows that the inverse-limit sequence

$$0 \longrightarrow A \xrightarrow{\theta} G(A) \times C \xrightarrow{q} G(C) \longrightarrow 0$$

is also exact. It is easily checked that the homomorphisms θ and q are given by

$$\theta(a) = (e^a, \widehat{a}) \quad (a \in A), \qquad q(b,g) = \widehat{b}e^{-g} \quad (b \in G(A),\ g \in C),$$

and then the exactness translates precisely into the stated theorem.

Remarks. 1. Following Davie [6], we may then deduce that $E(A)$ is the *path-component* of the identity in $G(A)$. Thus, the distinction between the two extensions of the Arens–Royden theorem is that Theorem 11 decribes the space of components of $G(A)$, while Theorem 12 describes the space of path-components. As noted in [6], $G(C)/E(C)$ (and hence also $G(A)/E(A)$) is isomorphic to $H^1(\Phi_A;\mathbb{Z})$ (see [11], Chapter III, Corollary 7.4).

2. The fact that the group homomorphism, say $\Gamma : G(A) \to G(C)$ (the restriction to $G(A)$ of the Gelfand representation of A), induces *isomorphisms* $G(A)/E(A) \cong G(C)/E(C)$ and $G(A)/G_0(A) \cong G(C)/G_0(C)$, implies that Γ also induces an isomorphism

$$G_0(A)/E(A) \cong G_0(C)/E(C).$$

At the end of the paper (Example 2 following Theorem 16), $G_0(A)/E(A)$ is identified with a certain cohomology group, which leads to a somewhat different proof of the isomorphism.

There are a few more comments on the relation between $E(A)$ and $G_0(A)$. Suppose, then, that A is a commutative, unital Fréchet algebra, and define the group homomorphism $E : A \to E(A)$ by $E(a) = e^{2\pi i a}$ $(a \in A)$; of course, E is a homomorphism from the additive group of A onto the multiplicative group $E(A)$. It is well known that, if A is a commutative *Banach* algebra, then $\ker E$ is the additive subgroup, say $\varepsilon(A)$, of A generated by the idempotents of A (in fact, the proof is included in the Fréchet-algebra case in Theorem 13). Every element of $\varepsilon(A)$ is expressible as a finite sum, $\sum_{j=1}^k n(j)e_j$, where each $n(j) \in \mathbb{Z}$ and $\{e_j : 1 \le j \le k\}$ is a finite set of pairwise orthogonal idempotents.

To describe $\ker E$ in the more general case of a commutative Fréchet algebra, we need another idea. If A is a Fréchet algebra, we call a sequence in A, say $(a_k)_{k\ge1}$, *locally finite* if and only if, for every continuous, submultiplicative seminorm p on A, there is an integer $N = N(p)$ such that $p(a_k) = 0$ for all $k \ge N$. (Equivalently, if $A \cong \varprojlim(A_n; d_n)$ is an Arens–Michael representation of A, then for each $n \ge 1$ there is $N = N(n)$ such that $\pi_n(a_k) = 0$ for all $k \ge N$.) It is clear that if (a_k) is a locally finite sequence in A then $\sum_{k\ge1} a_k$ is convergent in A. Note that, if A is a Banach algebra, then any locally finite sequence in A is eventually zero.

We now define the set $\varepsilon(A)$ to consist of all sums $\sum_{k\ge1} n(k)e_k$, where $(e_k)_{k\ge1}$ is a locally finite sequence of idempotents in A and each $n(k) \in \mathbb{Z}$; we may always, without loss of generality, suppose the idempotents in the sequence to be pairwise orthogonal. In case A is a Banach algebra, the new definition of $\varepsilon(A)$ reduces to the original one. Evidently, $\varepsilon(A)$ is an additive subgroup of A and, for each n, $\pi_n(\varepsilon(A)) \subseteq \varepsilon(A_n)$. It follows at once that $\varepsilon(A) \subseteq \ker E$. In fact, we have the following result.

THEOREM 13. *Let $A = \varprojlim(A_n; d_n)$ be a commutative, unital Fréchet algebra. Then* $\ker E = \varepsilon(A)$ *and the Arens–Michael isomorphism induces an isomorphism* $\varepsilon(A) \cong \varprojlim(\varepsilon(A_n); \overline{d}_n)$ *(where $\overline{d}_n = d_n|\varepsilon(A_{n+1})$).*

Proof. Let $a \in \ker E$; then the Gelfand transform $\widehat{a}$ is a continuous integer-valued function on Φ_A, so, for each $n \in \operatorname{im}\widehat{a}$, the set $\widehat{a}^{-1}(n)$ is open-and-closed in Φ_A (for the weak*-topology w). For any compact subset K of Φ_A, in particular for each Φ_n, $\widehat{a}$ can take only finitely many different values on K. Hence $\widehat{a}$ takes at most countably many different values on Φ_A; let them be $\{n(1), n(2), \ldots\}$. Let $U_k = \{\phi \in \Phi_A : \widehat{a}(\phi) = n(k)\}$; then (U_k) is a pairwise disjoint sequence of open-and-closed subsets of Φ_A. By the Shilov idempotent theorem (applied in each Banach algebra A_n, plus a routine argument), for each k there is a unique idempotent $e_k \in A$ with $\widehat{e}_k = 1_{U_k}$ (the characteristic function of U_k).

For each n, Φ_n meets at most finitely many of the sets U_k. So, for all $k > N(n)$, say, $\pi_n(e_k)$ is an idempotent in A_n with Gelfand transform zero; thus $\pi_n(e_k) = 0$ $(k > N(n))$, so $(e_k)_{k\ge1}$ is locally finite. Also, $a = \sum_{k=1}^{\infty} n(k)e_k$, for both terms of the equation have the same Gelfand transform, while their difference is in $\ker E$. This proves that $\varepsilon(A) = \ker E$.

Clearly, $\varepsilon(A) \subseteq \varprojlim(\varepsilon(A_n); \overline{d}_n)$. Conversely, if $a \in A$ and if $\pi_n(a) \in \varepsilon(A_n)$ for every n, then $\pi_n(E(a)) = 1_n$ for all n; thus $E(a) = 1$ and so $a \in \varepsilon(A)$. This completes the proof.

EXAMPLE. *The IL-sequence $\varepsilon(A) = (\varepsilon(A_n); \overline{d}_n)$ is not always stable.* With what is hoped to be an obvious notation, the short exact sequences of abelian groups and homomorphisms

$$0 \longrightarrow \varepsilon(A_n) \longrightarrow A_n \xrightarrow{E_n} E(A_n) \longrightarrow 0$$

define a short exact sequence in **ILGA**. By Theorem 13, $\varprojlim \varepsilon(A_n) \cong \varepsilon(A)$; also, $\varprojlim A_n \cong A$, and, by Lemma 9 ($E(A_n) = G_0(A_n)$ for each n), $\varprojlim E(A_n) \cong G_0(A)$. The corresponding inverse-limit sequence is thus

$$0 \longrightarrow \varepsilon(A) \longrightarrow A \xrightarrow{E} G_0(A) \longrightarrow 0.$$

In this last sequence $\operatorname{im} E = E(A)$, so if, for example, A is Davie's algebra D, then $E(A) \ne G_0(A)$. It follows that, for such an algebra, the kernel sequence $\varepsilon(A)$ is not stable. (See Example 2 following Theorem 16 for the determination of $H^1(\varepsilon(A))$.)

4. IL-sequences of abelian groups.

If we specialize the theory of §2 to IL-sequences of *abelian* groups, i.e., formally we discuss the category **ILGA**, then there is a reasonable measure of the non-stability of a sequence, in terms of a simple cohomology theory. As explained in §1, such a theory is discussed in [14], Appendix, in the language of derived functors. We shall describe an equivalent approach in the language of sheaves on a very special topological space.

Let $\mathbb{N}$ be the ordered set of natural numbers, $\mathbb{N} = \{1, 2, \ldots\}$, and, for each $n \in \mathbb{N}$, let $U_n = \{1, \ldots, n\}$. We define a topology σ on $\mathbb{N}$ by requiring that a subset $U \subseteq \mathbb{N}$ is σ-open if and only if *either* (i) $U = \emptyset$ or $\mathbb{N}$, *or* (ii) $U = U_n$ for some $n \in \mathbb{N}$. It is readily checked that σ is a non-Hausdorff (but T_0-) topology on $\mathbb{N}$; we shall write $\mathbb{N}^\sigma$ for the set $\mathbb{N}$ equipped with the topology σ. Notice that each $n \in \mathbb{N}$ has a unique smallest (open) neighbourhood, namely U_n.

Given a sequence $\mathcal{X} = (X_n; f_n)_{n\ge1}$ in **ILGA**, we define $\mathcal{F}^{\mathcal{X}}$, a presheaf of abelian groups on $\mathbb{N}^\sigma$, by setting $\mathcal{F}^{\mathcal{X}}(\emptyset) = 0$, $\mathcal{F}^{\mathcal{X}}(\mathbb{N}) = L(\mathcal{X})$ and, for each $n \in \mathbb{N}$,

$$\mathcal{F}^{\mathcal{X}}(U_n) = \Big\{(x_1, \ldots, x_n) \in \prod_{i=1}^{n} X_i : x_i = f_i(x_{i+1})\ (1 \le i \le n-1)\Big\} \cong X_n.$$

Observe that, since U_n is the unique smallest neighbourhood of n, the direct-limit process used in defining the *stalk* $\mathcal{F}_n^{\mathcal{X}}$ of $\mathcal{F}^{\mathcal{X}}$ at n is here trivial, so that, for each $n \in \mathbb{N}$, $\mathcal{F}_n^{\mathcal{X}} \cong \mathcal{F}^{\mathcal{X}}(U_n) \cong X_n$. It is then also clear that $\mathcal{F}^{\mathcal{X}}$ is even a *sheaf* (in the sense of Godement; otherwise expressed, it is a *complete presheaf*, which is then isomorphic to the corresponding sheaf). It should now also be clear, without a more formal discussion, that the notion of "exact sequence" in the category **ILGA** corresponds precisely to the sheaf-theoretic notion; also the inverse-limit functor L translates into the global section functor for the category of sheaves of abelian groups on the space $\mathbb{N}^\sigma$.

There is a sheaf cohomology theory for sheaves of abelian groups on a completely arbitrary topological space (originally due to Grothendieck), which may be constructed, as in [12], §4.3, using the notion of a *flabby resolution*. It may be recalled that a *flabby* sheaf is one in which all the restriction homomorphisms are surjective; each sheaf has a *flabby hull*. We shall explain the translation of this construction into the language of the category **ILGA**. Not surprisingly, since the space $\mathbb{N}^\sigma$ is so special, the flabby resolution process is very simple to describe.

Thus, let $\mathcal{X} = (X_n; f_n)_{n\ge1}$ be in **ILGA** and set $\widetilde{X}_n = \prod_{k=1}^{n} X_k$ $(n \ge 1)$. Define $\widetilde{f}_n : \widetilde{X}_{n+1} \to \widetilde{X}_n$ to be the projection onto the first n coordinates. Evidently, $\widetilde{\mathcal{X}} \equiv (\widetilde{X}_n; \widetilde{f}_n)$ is a sequence in **ILGA** in which each homomorphism $\widetilde{f}_n$ is surjective. It is also clear that $L(\widetilde{\mathcal{X}}) \cong \prod_{k=1}^{\infty} X_k$. For each n define the group homomorphism $\varepsilon_n : X_n \to \widetilde{X}_n$ by

$$\varepsilon_n(x) = ((f_1 \ldots f_{n-1})(x), \ldots, f_{n-1}(x), x) \quad (x \in X_n),$$

so that $\varepsilon_n(x)$ is the unique element of $\mathcal{F}^{\mathcal{X}}(U_n)$ that has the element x in its nth component. Then ε_n is an injective homomorphism with $\operatorname{im}\varepsilon_n = \mathcal{F}^{\mathcal{X}}(U_n)$. It is trivial to check that $\varepsilon_n f_n = \widetilde{f}_n\varepsilon_{n+1}$ for each n, so that the sequence $(\varepsilon_n)_{n\geq 1}$ defines an injective morphism $\varepsilon : \mathcal{X} \to \widetilde{\mathcal{X}}$ in the category **ILGA**. The pair $(\widetilde{\mathcal{X}};\varepsilon)$ will be called the *flabby hull* of $\mathcal{X}$. (It corresponds precisely to the sheaf-theoretic flabby hull, when translated into the language of $\mathcal{F}^{\mathcal{X}}$, $\mathcal{F}^{\widetilde{\mathcal{X}}}$.)

There is thus a short exact sequence

$$(*)\qquad 0 \longrightarrow \mathcal{X} \xrightarrow{\varepsilon} \widetilde{\mathcal{X}} \xrightarrow{\pi} \widetilde{\mathcal{X}}/\operatorname{im}\varepsilon \longrightarrow 0$$

in **ILGA** (π is the quotient morphism). Write $\widetilde{\mathcal{X}}/\operatorname{im}\varepsilon = \mathcal{Y} = (Y_n; g_n)$, say; then $Y_n = (\prod_{k=1}^n X_k)/\mathcal{F}^{\mathcal{X}}(U_n)$ and $g_n : Y_{n+1} \to Y_n$ is the map induced on the quotient groups by the projection map $\widetilde{f}_n$. But it is then clear that each g_n is also surjective, i.e., the short exact sequence $(*)$ is already a complete flabby resolution of $\mathcal{X}$ (when translated into the parallel sheaf-theoretic language).

We now define

$$H^p(\mathcal{X}) = H^p(\mathbb{N}^\sigma; \mathcal{F}^{\mathcal{X}}) \qquad (p \geq 0).$$

From $(*)$, we then have (using e.g. [12], Théorème (4.7.1)(a)):

THEOREM 14. *Let $\mathcal{X}$ be a sequence in* **ILGA**. *Then $H^p(\mathcal{X}) = 0$ ($p \geq 2$) and $H^0(\mathcal{X}) \cong L(\mathcal{X})$.*

The basic theorem of sheaf cohomology (e.g. [12], Théorème (4.4.2)) then gives:

THEOREM 15. *Given any short exact sequence $0 \to \mathcal{X} \xrightarrow{\alpha} \mathcal{Y} \xrightarrow{\beta} \mathcal{Z} \to 0$ in* **ILGA**, *there is a (functorially determined) exact sequence of abelian groups and homomorphisms*,

$$0 \longrightarrow L(\mathcal{X}) \xrightarrow{L(\alpha)} L(\mathcal{Y}) \xrightarrow{L(\beta)} L(\mathcal{Z})$$
$$\xrightarrow{\sigma} H^1(\mathcal{X}) \xrightarrow{H^1(\alpha)} H^1(\mathcal{Y}) \xrightarrow{H^1(\beta)} H^1(\mathcal{Z}) \longrightarrow 0.$$

Remark on the notation. In this last theorem, L is the inverse-limit functor, as described in §1, H^1 is the first cohomology functor and σ is the connecting homomorphism arising from the given short exact sequence. Explicit descriptions of H^1 and of σ will be given after the next result.

We now wish to give an explicit description of the cohomology groups $H^1(\mathcal{X})$. Thus let $\mathcal{X} = (X_n; f_n)$ be a sequence in **ILGA**. Let $X = \prod_{n=1}^\infty X_n$ ($\cong L(\widetilde{\mathcal{X}})$) and define the mapping $\Delta : X \to X$ by

$$\Delta(x_1, x_2, \ldots) = (x_1 - f_1x_2, x_2 - f_2x_3, \ldots).$$

Then Δ is a group endomorphism and, clearly, $\ker\Delta = L(\mathcal{X})$.

PROPOSITION 3. *With the above notation,*

$$H^1(\mathcal{X}) \cong X/\operatorname{im}\Delta.$$

Proof. Let $\mathcal{X} = (X_n; f_n)_{n\geq 1}$ and let $(\widetilde{\mathcal{X}};\varepsilon)$ be its flabby hull. We write $\widetilde{\mathcal{X}}_0$ for the sequence $\widetilde{\mathcal{X}}$ augmented on the left by 0. Thus, say, $\widetilde{\mathcal{X}}_0$ is the sequence

$$\widetilde{\mathcal{X}}_0 : \quad 0 \xleftarrow{h_1} \widetilde{X}_1 \xleftarrow{h_2} \widetilde{X}_2 \xleftarrow{h_3} \ldots,$$

where $h_1 = 0$ and $h_n = \widetilde{f}_{n-1}$ ($n \geq 2$). We define a morphism $\delta : \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}_0$ by setting $\delta_1 = 0$ ($\delta_1 : X_1 \to 0$) and, for each $n \geq 2$,

$$\delta_n(x_1, \ldots, x_n) = (x_1 - f_1x_2, x_2 - f_2x_3, \ldots, x_{n-1} - f_{n-1}x_n).$$

It is readily verified that the sequence (δ_n) does define a morphism.

For every $n \geq 1$, $\ker\delta_n = \operatorname{im}\varepsilon_n$, so there is a short exact sequence in **ILGA**

$$0 \longrightarrow \mathcal{X} \xrightarrow{\varepsilon} \widetilde{\mathcal{X}} \xrightarrow{\delta} \operatorname{im}\delta \longrightarrow 0.$$

Since $\widetilde{\mathcal{X}}$ is flabby, $H^1(\widetilde{\mathcal{X}}) = 0$; hence part of the exact cohomology sequence given by Theorem 15 is

$$0 \longrightarrow L(\mathcal{X}) \xrightarrow{L(\varepsilon)} L(\widetilde{\mathcal{X}}) \xrightarrow{L(\delta)} L(\operatorname{im}\delta) \longrightarrow H^1(\mathcal{X}) \longrightarrow 0,$$

so that $H^1(\mathcal{X}) \cong L(\operatorname{im}\delta)/\operatorname{im}L(\delta)$.

For $n \geq 2$,

$$\operatorname{im}\delta_n = \Big\{(x_1 - f_1x_2, \ldots, x_{n-1} - f_{n-1}x_n) : (x_1, \ldots, x_n) \in \prod_{k=1}^n X_k\Big\} \cong \prod_{k=1}^{n-1} X_k.$$

It follows that $L(\delta) = \Delta$, while $L(\operatorname{im}\delta) \cong \prod_{k=1}^\infty X_k = X$, and the result is proved.

COROLLARY 4. *Let $\mathcal{X}$ be a sequence in* **ILGA**. *Then $\mathcal{X}$ is stable if and only if $H^1(\mathcal{X}) = 0$.*

Proof. This is immediate from the definition of stability and the proposition just proved.

Remarks. (i) Suppose that $\alpha : \mathcal{X} \to \mathcal{Y}$ is a morphism in **ILGA**. Then, with what is hoped to be an obvious notation, there is a natural mapping, say $\widetilde{\alpha} : X \to Y$, defined by $\widetilde{\alpha}(x_1, x_2, \ldots) = (\alpha_1(x_1), \alpha_2(x_2), \ldots)$. If we write Δ_X, Δ_Y for the endomorphisms of X, Y respectively (defined as was Δ just before the last proposition), then it is immediately checked that $\widetilde{\alpha}\Delta_X = \Delta_Y\widetilde{\alpha}$. Thus $\widetilde{\alpha}$ naturally induces a homomorphism $X/\operatorname{im}\Delta_X \to Y/\operatorname{im}\Delta_Y$; it is this homomorphism which is the translation of the functorially induced $H^1(\alpha)$ in terms of the concrete representation of the H^1-groups given by the proposition.

(ii) Similarly, if we have a short exact sequence $0 \to \mathcal{X} \xrightarrow{\alpha} \mathcal{Y} \xrightarrow{\beta} \mathcal{Z} \to 0$ in **ILGA**, then we may give a concrete description of the connecting homomorphism σ that appears in the statement of Theorem 15. We recall that $\sigma : L(\mathcal{Z}) \to H^1(\mathcal{X})$. So, given a sequence $z = (z_1, z_2, \ldots) \in L(\mathcal{Z})$, we have $z \in Z \equiv \prod_{n=1}^{\infty} Z_n$ and $\Delta_Z(z) = 0$. We then choose $y \in Y$ such that $\widetilde{\beta}(y) = z$ and then note that $\widetilde{\beta}(\Delta_Y(y)) = 0$, so that there is some $x \in X$ with $\widetilde{\alpha}(x) = \Delta_Y(y)$. It is then not hard to verify that the coset $x + \operatorname{im}\Delta_X$, an element of $H^1(\mathcal{X})$, is uniquely determined by the original $z \in L(\mathcal{Z})$. It is the coset $x + \operatorname{im}\Delta_X$ that is the element $\sigma(z)$.

COROLLARY 5. *Let $0 \to \mathcal{X} \xrightarrow{\alpha} \mathcal{Y} \xrightarrow{\beta} \mathcal{Z} \to 0$ be any short exact sequence in* **ILGA**, *in which $\mathcal{Y}$ is stable. Then the associated connecting homomorphism effects an isomorphism*

$$H^1(\mathcal{X}) \cong L(\mathcal{Z})/\operatorname{im} L(\beta).$$

Proof. Since $\mathcal{Y}$ is stable, we have $H^1(\mathcal{Y}) = 0$, by Corollary 4. But then part of the exact cohomology sequence given by Theorem 15 is

$$L(\mathcal{Y}) \xrightarrow{L(\beta)} L(\mathcal{Z}) \xrightarrow{\sigma} H^1(\mathcal{X}) \longrightarrow 0,$$

from which the result is immediate.

The following result summarizes what has, mostly, already been proved.

THEOREM 16. *Let $\mathcal{X}$ be a sequence in* **ILGA**. *Then the following properties are equivalent:*

(i) *$\mathcal{X}$ is stable;*
(ii) *$H^1(\mathcal{X}) = 0$;*
(iii) *for every short exact sequence $0 \to \mathcal{X} \to \mathcal{Y} \to \mathcal{Z} \to 0$ in* **ILGA**, *the inverse-limit sequence*

$$0 \longrightarrow L(\mathcal{X}) \longrightarrow L(\mathcal{Y}) \longrightarrow L(\mathcal{Z}) \longrightarrow 0$$

is also exact.

Proof. The equivalence of (i) and (ii) is Corollary 5. That (i) implies (iii) follows from Theorem 5 (or, alternatively, we may deduce that (ii) implies (iii) from Theorem 15).

We shall complete the proof by showing that (iii) implies (ii). We consider the canonical flabby resolution

$$0 \longrightarrow \mathcal{X} \xrightarrow{\varepsilon} \widetilde{\mathcal{X}} \xrightarrow{\pi} \widetilde{\mathcal{X}}/\operatorname{im}\varepsilon \longrightarrow 0,$$

where $(\widetilde{\mathcal{X}}; \varepsilon)$ is the flabby hull of $\mathcal{X}$. By the assumption that (iii) holds, the sequence

$$(*) \qquad 0 \longrightarrow L(\mathcal{X}) \xrightarrow{L(\varepsilon)} L(\widetilde{\mathcal{X}}) \xrightarrow{L(\pi)} L(\widetilde{\mathcal{X}}/\operatorname{im}\varepsilon) \longrightarrow 0$$

is exact.

But, from Theorem 15, there is another exact sequence, part of which (since $H^1(\widetilde{\mathcal{X}}) = 0$) is

$$(**) \qquad L(\widetilde{\mathcal{X}}) \xrightarrow{L(\pi)} L(\widetilde{\mathcal{X}}/\operatorname{im}\varepsilon) \xrightarrow{\sigma} H^1(\mathcal{X}) \longrightarrow 0.$$

From $(*)$, $L(\pi)$ is surjective and then, from $(**)$, $H^1(\mathcal{X}) = 0$.

EXAMPLES. 1. At the end of §1, we gave an example of a *non-stable* IL-sequence $\mathcal{I}$, of closed ideals in a Mittag-Leffler sequence $\mathcal{A}$ of Banach algebras, where $\mathcal{A}$ was a standard Arens–Michael representation of the algebra $A = \mathcal{O}(\mathbb{C})$ of all entire functions in one variable. Using the short exact sequence occurring in that example, together with Corollary 5, we may deduce the more precise statement that

$$H^1(\mathcal{I}) \cong \mathbb{C}[[z]]/\mathcal{O}(\mathbb{C}).$$

(Here, as in the original discussion of the example, an entire function is identified with the formal power series provided by its Taylor series about 0.)

2. In the example at the end of §3.6, concerning the sequence $\varepsilon(A)$ of kernels of exponential maps, it was shown that the sequence need not be stable. Applying Corollary 5 to the short exact sequence in that example at once gives that, for every commutative Fréchet algebra A,

$$H^1(\varepsilon(A)) \cong G_0(A)/E(A).$$

Using this isomorphism, we may give (in the notation of Theorem 11) the alternative proof of the result referred to in Remark 2, following Theorem 12.

THEOREM 17. *Let A be a commutative unital Fréchet algebra, let Φ_A have the k-topology κ, and let $C = C(\Phi_A)$. Then the Gelfand mapping induces an isomorphism $G_0(A)/E(A) \cong G_0(C)/E(C)$.*

Proof. Remark first that, if B is any commutative unital Banach algebra, then the Gelfand mapping for B induces an isomorphism between $\varepsilon(B)$, the additive subgroup generated by the idempotents of B, and the corresponding group $\varepsilon(C(\Phi_B))$ (essentially using well-known results recalled in the proof of Theorem 13).

If we now apply this result to each Banach algebra A_n in an Arens–Michael representation of A, then it is clear that we obtain an isomorphism in the category **ILGA** between the sequences $\varepsilon(A)$ and $\varepsilon(C)$. Thus $H^1(\varepsilon(A)) \cong H^1(\varepsilon(C))$, from which the theorem follows by the remark of the last paragraph.

Remark. The group $G_0(A)/E(A)$ thus depends only on the topology of Φ_A; it is an invariant that is "invisible" to an Arens–Michael representation, since $G_0(A_n)/E(A_n)$ is trivial for each n.

References

[1] G. R. Allan, *Embedding the algebra of formal power series in a Banach algebra*, Proc. London Math. Soc. (3) 25 (1972), 329–340.
[2] —, *Fréchet algebras and formal power series*, Studia Math. 119 (1996), 271–288.
[3] R. F. Arens, *A generalization of normed rings*, Pacific J. Math. 2 (1952), 455–471.
[4] —, *Dense inverse limit rings*, Michigan Math. J. 5 (1958), 169–182.
[5] N. Bourbaki, *Théorie des ensembles*, Hermann, Paris, 1970.
[6] A. M. Davie, *Homotopy in Fréchet algebras*, Proc. London Math. Soc. (3) 23 (1971), 31–52.
[7] S. Eilenberg and N. E. Steenrod, *Foundations of Algebraic Topology*, Princeton Univ. Press, Princeton, N.J., 1952.
[8] R. Engelking, *General Topology*, revised and completed edition, Heldermann, Berlin, 1989.
[9] J. Eschmeier and M. Putinar, *Spectral Decompositions and Analytic Sheaves*, London Math. Soc. Monographs (N.S.) 10, Clarendon Press, Oxford, 1996.
[10] J. Esterle, *Mittag-Leffler methods in the theory of Banach algebras and a new approach to Michael's problem*, in: Contemp. Math. 32, Amer. Math. Soc., 1984, 107–129.
[11] T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, 1969.
[12] R. Godement, *Topologie algébrique et théorie des faisceaux*, Publ. Inst. Math. Univ. Strasbourg XIII, Hermann, Paris, 1964.
[13] G. Köthe, *Topological Vector Spaces I*, Springer, Berlin, 1969.
[14] W. S. Massey, *Homology and Cohomology Theory*, Marcel Dekker, Basel, 1978.
[15] E. A. Michael, *Locally multiplicatively convex topological algebras*, Mem. Amer. Math. Soc. 11 (1953; third printing 1971).
[16] J. Milnor, *On axiomatic homology theory*, Pacific J. Math. 12 (1962), 337–341.
[17] R. Narasimhan, *Complex Analysis in One Variable*, Birkhäuser, Boston, 1985.
[18] V. P. Palomodov, *Homological methods in the theory of locally convex spaces*, Uspekhi Mat. Nauk 26 (1) (1971), 3–65 (in Russian); English transl.: Russian Math. Surveys 26 (1971), 1–64.
[19] J. L. Taylor, *A joint spectrum for several commuting operators*, J. Funct. Anal. 6 (1970), 172–191.

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
16 Mill Lane
Cambridge CB2 1SB, U.K.
E-mail: G.R.Allan@pmms.cam.ac.uk

Received May 13, 1996
Revised version August 6, 1996 (3671)

INFORMATION FOR AUTHORS

Manuscripts should be typed on one side only, with double or triple spacing and wide margins, and submitted in duplicate, including the original typewritten copy.

An **abstract** of not more than 200 words and the AMS Mathematics Subject Classification are required.

Formulas must be typewritten. A complete **list of all handwritten symbols** with indications for the printer should be enclosed.

Figures must be prepared in a form suitable for direct reproduction. Sending EPS, PCX, TIF or CorelDraw files will be most helpful. The author should indicate on the margin of the manuscript where figures are to be inserted.

References should be arranged in alphabetical order, typed with double spacing, and styled and punctuated according to the examples given below. Abbreviations of journal names should follow Mathematical Reviews. Titles of papers in Russian should be translated into English.

Examples:

[6] D. Beck, *Introduction to Dynamical Systems*, Vol. 2, Progr. Math. 54, Birkhäuser, Basel, 1978.
[7] R. Hill and A. James, *An index formula*, J. Differential Equations 15 (1982), 197–211.
[8] J. Kowalski, *Some remarks on $J(X)$*, in: Algebra and Analysis, Proc. Conf. Edmonton 1973, E. Brook (ed.), Lecture Notes in Math. 867, Springer, Berlin, 1974, 115–124.
[Nov] A. S. Novikov, *An existence theorem for planar graphs*, preprint, Moscow University, 1980 (in Russian).

Authors' **affiliation** should be given at the end of the manuscript.

Authors receive only page **proofs** (one copy). If the proofs are not returned promptly, the article will be printed in a later issue.

Authors receive 50 **reprints** of their articles. Additional reprints can be ordered.

The publisher strongly encourages submission of manuscripts written in TeX. **On acceptance of the paper,** authors will be asked to send discs (preferably PC) plus relevant details to the Editorial Committee, or transmit the file by electronic mail to:

STUDIA@IMPAN.GOV.PL