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Fractional moments of the Riemann zeta-function

by

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To Professor Kannan Soundararajan
on his twenty-third birthday

1. Introduction. The object of this paper is to prove the following theorem.

Theorem 1. *Let $k = pq^{-1}$ where p and q are integers subject to $1 \le p \le q(\log(q+1))^{-1/2}$. Let $T \ge H \ge C_0 \log\log(T^k + 100)$ where $C_0 > 0$ is a certain large absolute constant. Then for $T \ge 10$, we have*

$$\frac{1}{H}\int_T^{T+H} |\zeta(1/2+it)|^{2k}\,dt > C_1(\log H)^{k^2} \tag{1}$$

where $C_1 > 0$ is a certain absolute constant (C_0 and C_1 are effective).

Remark 1. In place of $(\log(q+1))^{-1/2}$ we can have $C_2(\log(q+1))^{-1/2}$ where $C_2 > 0$ is any absolute constant. Then C_0 and C_1 depend on C_2.

Remark 2. The previous history of the theorem is as follows. First, E. C. Titchmarsh considered the case $H = T$, and k any positive integer, of (1) and proved that

$$\limsup_{T\to\infty}((\mathrm{LHS})(\mathrm{RHS})^{-1}) > 0.$$

Next I considered the case where k is half of any positive integer and proved (1) (however with C_1 depending possibly on k). Next D. R. Heath-Brown [1] considered the case $H = T$ and k any positive rational number and proved (1) (however with C_1 depending possibly on k). Next M. Jutila [4] considered the case $H = T$ and $k = q^{-1}$ and proved (1) with C_1 independent of k. For all these references see also my book [6]. Two other excellent reference books are [7] and [2].

Remark 3. We use only "Euler product" in the proof of Theorem 1 and so its analogue goes through for L-functions of algebraic number fields, Ramanujan's zeta-function and so on.

2. Some preliminaries to the proof

THEOREM 2 (H. L. Montgomery and R. C. Vaughan [5]). *Let $H > 0$, $N \geq 1$ be an integer, and $a_1, \ldots, a_N$ any N complex numbers. Then*

$$\int_0^H \Big| \sum_{n \leq N} a_n n^{it} \Big|^2 dt = \sum_{n \leq N} (H + O(n))|a_n|^2.$$

Moreover, the O-constant is absolute.

Remark 1. Montgomery and Vaughan obtained an economical O-constant (see [6], p. 21, for a proof with some absolute constant).

Remark 2. We use Theorem 2 with N something like $N = H^{7/8}$ ($H \geq 10$) and for this choice there are much simpler methods of proving what we want.

THEOREM 3 (K. Ramachandra [6]). *Let $z = x + iy$ be a complex variable with $|x| \leq 1/4$. Then*:

(a) $|\exp((\sin z)^2)| \leq 2$ *for all y.*
(b) *If $|y| \geq 2$ we have*

$$|\exp((\sin z)^2)| \leq 2(\exp \exp |y|)^{-1}.$$

Proof. See [6], p. 38.

THEOREM 4. *Let $q > 0$ and $a > 0$ be real numbers and n any positive integer. Consider the rectangle defined by*

$$0 \leq x \leq (2^n + 1)a, \qquad -R \leq y \leq R.$$

Let $f(z)$ and $\varphi(z)$ be two functions analytic inside this rectangle and let $|f(z)|$ and $|\varphi(z)|$ be continuous on its boundary. Let

$$I_x = \int_{-R}^{R} |\varphi(z)| \cdot |f(z)|^{1/q} \, dy$$

and let

$$Q(\alpha) = \max(|\varphi(z)| \cdot |f(z)|^{1/q})$$

taken over $0 \leq x \leq \alpha$, $y = \pm R$. Then with $b_n = 2^n + 1$ we have

$$I_a \leq (I_0 + U)^{1/2} (I_a + U)^{1/2 - 2^{-n-1}} (I_{ab_n} + U)^{2^{-n-1}}$$

where $U = 2^{2(n+1)} Q(ab_n) a$.

Proof. See [6], p. 97. (Here we have replaced the interval $(0, R)$ by $(-R, R)$ and the number q by $1/q$.)

THEOREM 5. *Let $w = u + iv$ and $s = \sigma + it$ be two complex variables*,

$$K(w) = \exp\left(\left(\sin \frac{w}{8A}\right)^2\right)$$

where $A>0$ is a large constant, and let

$$f(s,w)=(K(w))^q f_0(s+w)$$

where q (>0) is any real number. Let $K(w)$ and $f_0(s+w)$ satisfy the conditions of Theorem 4 with

$$\varphi(z)=K(z+a) \quad \text{and} \quad f(z)=f_0(s+z+a).$$

Then if we take $R=\tau$ we have, with $b_n=2^n+1$,

$$\begin{aligned}(2)\qquad \int\limits_{|v|\le\tau}|f(s,w)|_{u=0}^{1/q}\,dv &\le \Big(\int\limits_{|v|\le\tau}|f(s,w)|_{u=-a}^{1/q}\,dv+H^{-10}\Big)^{1/2}\\ &\quad\times\Big(\int\limits_{|v|\le\tau}|f(s,w)|_{u=0}^{1/q}\,dv+H^{-10}\Big)^{1/2-2^{-n-1}}\\ &\quad\times\Big(\int\limits_{|v|\le\tau}|f(s,w)|_{u=ab_n-a}^{1/q}\,dv+H^{-10}\Big)^{2^{-n-1}}\end{aligned}$$

provided $U\le H^{-10}$.

Theorem 6. *If the conditions of Theorem 5 are satisfied uniformly for t belonging to an interval $B\le t\le B+H_1$ with $0\le H_1\le H$, we have (2) with $\int_{|v|\le\tau}\ldots\,dv$ replaced by $\int_{(t)}\int_{|v|\le\tau}\ldots\,dv\,dt$ and H^{-10} replaced by H^{-9}. Moreover, if*

$$(3)\qquad \int\limits_{(t)}\int\limits_{|v|\le\tau}|f(s,w)|_{u=0}^{1/q}\,dv\,dt\ge H^{-9}$$

then

$$\begin{aligned}(4)\qquad &\int\limits_{(t)}\int\limits_{|v|\le\tau}|f(s,w)|_{u=0}^{1/q}\,dv\,dt\\ &\le 2\Big(\int\limits_{(t)}\int\limits_{|v|\le\tau}|f(s,w)|_{u=-a}^{1/q}\,dv\,dt+H^{-9}\Big)^{2^n/(2^n+1)}\\ &\quad\times\Big(\int\limits_{(t)}\int\limits_{|v|\le\tau}|f(s,w)|_{u=a2^n}^{1/q}\,dv\,dt+H^{-9}\Big)^{1/(2^n+1)}.\end{aligned}$$

P r o o f. Under the assumption (3) we can replace the second factor on the RHS of (2) by

$$\Big(2\int\limits_{(t)}\int\limits_{|v|\le\tau}|f(s,w)|_{u=0}^{1/q}\,dv\,dt\Big)^{1/2-2^{-n-1}}.$$

This gives Theorem 6.

THEOREM 7. *LHS of* (4) *is*

$$\gg \int_{B+\tau}^{B+H_1-\tau} |f_0(\sigma+it)|^{1/q}\,dt,$$

where the interval for t is $(B, B+H_1)$, provided $2\tau \le H_1$. Also for any u on RHS of (4) *we have*

$$\int_{(t)}\int_{|v|\le\tau} \ldots\, dv\,dt \ll \int_{B-\tau}^{B+H_1+\tau} |f_0(\sigma+it+u)|^{1/q}\,dt.$$

Proof. LHS of (4) equals

$$\int_{B}^{B+H_1}\int_{|v|\le\tau} K(iv)|f_0(\sigma+it+iv)|^{1/q}\,dv\,dt$$

$$= \int_{(v)} K(iv)\Big(\int_{(t)} \ldots\, dt\Big)\,dv = \int_{(v)} K(iv)\Big(\int_{B+v}^{B+H_1-v} \ldots\, dt\Big)\,dv$$

$$> \int_{(v)} K(iv)\Big(\int_{B+\tau}^{B+H_1-\tau} \ldots\, dt\Big)\,dv = \Big(\int_{(v)} K(iv)\,dv\Big)\Big(\int_{B+\tau}^{B+H_1-\tau} \ldots\, dt\Big)$$

and this proves the first part of Theorem 7. The proof of the second part is similar.

Remark. Theorems 6 and 7 are stated here for the first time although they are already implicitly contained in [6]. These are new versions of the convexity.

THEOREM 8 (D. R. Heath-Brown and M. Jutila [1], [4]). *Let k (>0) be any real number. Then for $1/2 < \sigma \le 2$, we have*

$$\sum_{n=1}^{\infty}(d_k(n))^2 n^{-2\sigma} \le (\zeta(2\sigma))^{k^2} \le A_1^{k^2}(\sigma-1/2)^{-k^2},$$

where $A_1 > 0$ is an absolute constant. (*Here $d_k(n)$ are defined as usual by $(\zeta(s))^k = \sum_{n=1}^{\infty} d_k(n)n^{-s}$, $\operatorname{Re} s \ge 2$.*) *Also let $N \ge 2$ and $0 < k \le 1$. Then there exists an absolute constant $A_2 > 0$ for which*

$$\sum_{n\le N}(d_k(n))^2 n^{-2\sigma} \ge A_2(\sigma-1/2)^{-k^2}$$

provided

$$1/2 + A_3(\log N)^{-1} \le \sigma \le 2,$$

with an absolute constant A_3 (>0) which depends only on A_2.

Remark. We can allow any (absolute) constant upper bound for k and still prove the second part of the theorem.

Proof of Theorem 8. The first part follows from the inequality $(d_k(n))^2 \le d_{k^2}(n)$. The second part (due essentially to D. R. Heath-Brown [1]) can be proved as follows. For all $\delta > 0$, $(1+\delta)/2 \le \sigma < 2$, we have

$$\sum_{n\le N}(d_k(n))^2 n^{-2\sigma} \ge \sum_{n=1}^{\infty}(d_k(n))^2|\mu(n)|n^{-2\sigma}\left(1-\left(\frac{n}{N}\right)^{\delta}\right)$$
$$\ge \prod_p\left(1+\frac{k^2}{p^{2\sigma}}\right) - N^{-\delta}A_1\left(\sigma-\frac{\delta}{2}-\frac{1}{2}\right)^{-k^2}.$$

(Here and in the next line p is a symbol running over all primes and it should not be confused with p in Theorem 1.) Here the product over p is

$$\left[\exp\sum_p\left\{\log\left(1+\frac{k^2}{p^{2\sigma}}\right)-k^2\log\left(\frac{1}{1-p^{-2\sigma}}\right)\right\}\right](\zeta(2\sigma))^{k^2},$$

which exceeds $A_4(\sigma-1/2)^{-k^2}$. Thus

$$\sum_{n\le N} d_k(n)n^{-2\sigma} \ge A_4\left(\sigma-\frac{1}{2}\right)^{-k^2}\left\{1-\frac{A_1}{A_4}N^{-\delta}\left(\frac{\sigma-1/2}{\sigma-(1+\delta)/2}\right)^{k^2}\right\}.$$

Here we set $\delta = \sigma - 1/2$ and obtain for the RHS the lower bound

$$A_4\left(\sigma-\frac{1}{2}\right)^{-k^2}\left\{1-\frac{A_1}{A_4}N^{-\delta}2^{k^2}\right\} \ge A_4\left(\sigma-\frac{1}{2}\right)^{-k^2}\left(1-\frac{2A_1}{A_4}N^{1/2-\sigma}\right)$$
$$\ge A_4\left(\sigma-\frac{1}{2}\right)^{-k^2}\left(1-\frac{2A_1}{A_4}e^{-A_3}\right)$$
$$= (A_4-2A_1e^{-A_3})\left(\sigma-\frac{1}{2}\right)^{-k^2}$$

and this proves the second part of Theorem 8.

THEOREM 9. *Let $f(z)$ be analytic in $|z|\le r$. Then for any real $k>0$, we have*

$$|f(0)|^k \le \frac{1}{\pi r^2}\iint_{|z|\le r}|f(z)|^k\,dx\,dy.$$

Proof. See [6], p. 34.

3. Proof of Theorem 1 (first step). The main object of this section is to prove the following theorem. (From now on we assume that $k = p/q$ where p and q are integers subject to $1 \le p \le q(\log(q+1))^{-1/2}$.)

Theorem 10. *Let $T \geq H$ and H exceed a certain large positive absolute constant. Then*

$$\max_{\sigma \geq 1/2+q(\log H)^{-1}} \left(\frac{1}{H} \int_{T}^{T+H} |\zeta(\sigma+it)|^{2k}\, dt \right) \geq C_2 (\log H)^{k^2} \tag{5}$$

where $C_2 > 0$ is an absolute constant (not to be confused with C_2 of Remark 1 below Theorem 1).

Remark. If $q \geq (\log H)^{1/100}$, then $(\log H)^{k^2}$ lies between two positive constants and also for $\sigma \geq 2$,

$$|\zeta(\sigma+it)|^{-1} \leq \zeta(2) < 1 + \sum_{n=2}^{\infty} (n(n-1))^{-1} = 2$$

and so $|\zeta(\sigma+it)| \geq 1/2$. Hence $|\zeta(\sigma+it)|^{2k} \geq 2^{-4} = 1/16$. Thus Theorem 10 is obvious in this case.

From now on till the end of this section we assume that $1 \leq q \leq (\log H)^{1/100}$ and that for all $\sigma \geq 1/2 + q(\log H)^{-1}$, we have

$$\frac{1}{H} \int_{T}^{T+H} |\zeta(\sigma+it)|^{2k} dt < C_2 (\log H)^{k^2} \tag{6}$$

where C_2 (> 0) is a small constant. (Finally, we arrive at a contradiction.)

Note that assuming (6) it suffices to either get a contradiction or to prove Theorem 10 with

$$\frac{1}{H} \int_{T}^{T+H} |\zeta(\sigma+it)|^{2k} dt$$

replaced by

$$\frac{1}{H-H_0} \int_{T+H_0}^{T+H-H_0} |\zeta(\sigma+it)|^{2k} dt$$

(and C_2 replaced by C_2^* (a small positive constant)) where H_0 lies between two (small absolute) positive constant multiples of H. Note also that the maximum over any region is greater than or equal to the maximum taken over a sub-region.

Lemma 1. *For $\sigma \geq 1/2 + (q+2)(\log H)^{-1}$, $T+1 \leq t \leq T+H-1$, we have*

$$|\zeta(s)|^{2k} \leq H^2. \tag{7}$$

Proof. Take the circle $|z| \leq (\log H)^{-1}$, apply Theorem 9 to $f(z) = \zeta(s+z)$ and (7) follows.

We next apply Theorems 5, 6 and 7 with

$$f_0(z) = (\zeta(z))^{2p} - (P_N(z))^{2q} \tag{8}$$

where

$$P_N(z) = \sum_{n\le N} d_k(n) n^{-z}, \quad N = H^{7/8}. \tag{9}$$

From now on we assume $\sigma \ge 1/2 + (q+2)(\log H)^{-1}$.

LEMMA 2. *For H_2 with $0 \le 2H_2 \le H$, the quantity*

$$\int_{T+H_2}^{T+H-H_2} |(\zeta(\sigma+it))^{2p} - (P_N(\sigma+it))^{2q}|^{q^{-1}}\, dt \tag{10}$$

lies between

$$\int_{T+H_2}^{T+H-H_2} |P_N(\sigma+it)|^2\, dt - C_2 H(\log H)^{k^2} \tag{11}$$

and

$$\int_{T+H_2}^{T+H-H_2} |P_N(\sigma+it)|^2\, dt + C_2 H(\log H)^{k^2}. \tag{12}$$

Proof. For any two complex numbers z_1 and z_2 we show that

$$|z_1|^{q^{-1}} - |z_2|^{q^{-1}} \le |z_1 - z_2|^{q^{-1}} \le |z_1|^{q^{-1}} + |z_2|^{q^{-1}}.$$

The latter inequality follows on raising both sides to the power q and using $|z_1| + |z_2| \ge |z_1 - z_2|$. The former is similar: we have to use $|z_1| \le |z_2| + |z_1 - z_2|$.

LEMMA 3. *If $H_2 \le (1000)^{-1}H$, the quantity $\int_{T+H_2}^{T+H-H_2} |P_N(\sigma+it)|^2\, dt$ lies between $C_3 H(\sigma - 1/2)^{-k^2}$ and $C_4 H(\sigma-1/2)^{-k^2}$, where $C_3 > 0$ and $C_4 > 0$ are absolute constants (independent of C_2) provided $\sigma \le 2$.*

Proof. Apply Theorems 2 and 8.

LEMMA 4. *Let $\sigma_0 = 1/2 + 10q(\log H)^{-1}$, $a = Dq(\log H)^{-1}$, $s = \sigma_0 + a + it$, where $D > 0$ is any large absolute constant and $T + H_3 \le t \le T + H - H_3$, where H_3 is a small positive constant multiple of H. Then with τ equal to a small positive constant multiple of H, we have*

$$\int_{(t)} \int_{|v|\le\tau} |f(s,w)|_{u=0}^{1/q}\, dv\, dt \ge H^{-9}, \tag{13}$$

$$\int_{(t)} \int_{|v|\le\tau} |f(s,w)|_{u=0}^{1/q}\, dv\, dt \ge C_5 H(\log H)^{k^2} D^{-k^2}, \tag{14}$$

$$\int\limits_{(t)}\int\limits_{|v|\le\tau} |f(s,w)|^{1/q}_{u=-a}\,dv\,dt + H^{-9} \le C_6 H(\log H)^{k^2} \tag{15}$$

and

$$\int\limits_{(t)}\int\limits_{|v|\le\tau} |f(s,w)|^{1/q}_{u=a2^n}\,dv\,dt + H^{-9} \le C_7 H^{1-a2^n/(100q)}, \tag{16}$$

where $a2^n$ lies between 10 *and* 20. *Here C_5, $C_6 \ge 1$ and $C_7 \ge 1$ are positive constants* (*since C_2 can be fixed to be small*) *and D^{-k^2} exceeds a certain positive absolute constant times C_2 for the validity of* (14).

Proof. This follows from Theorem 3 and assumption (6) and its consequence (7). Note that q^{k^2} lies between two absolute positive constants. We give some details in proving (16). We have

$$|f(s,w)|^{1/q}_{u=a2^n} \le |K(w+a)|\cdot|(\zeta(s+w+a))^{2p} - (P_N(s+w+a))^{2q}|^{1/q}_{u=a2^n}$$

with $N = H^{7/8}$ and

$$|K(w+a)| \ll \left(\exp\exp\frac{|v|}{8A}\right)^{-1}.$$

Also

$$\begin{aligned}
&|(\zeta(s+w+a))^{2p} - (P_N(s+w+a))^{2q}|_{u=a2^n}\\
&\quad = |((\zeta(s+w+a))^{p/q})^{2q} - (P_N(s+w+a))^{2q}|_{u=a2^n}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{where } N = H^{7/8})\\
&\quad \le |(\zeta(s+w+a))^{p/q} - P_N(s+w+a)|_{u=a2^n}(100)^{2p+2q}\\
&\quad \ll \Big(\sum_{n\ge N} n^{-10}\Big)(100)^{2p+2q} \ll N^{-9}(100)^{2p+2q} = H^{-63/8}(100)^{2p+2q}.
\end{aligned}$$

Thus

$$|f(s,w)|^{1/q}_{u=a2^n} \ll \left(\exp\exp\frac{|v|}{8A}\right)^{-1} H^{-63/(8q)}.$$

Finally

$$\frac{63}{8q} \ge \frac{a2^n}{100q} \quad \text{since} \quad a2^n \le \frac{6300}{8}.$$

These calculations prove (16).

Lemma 5. *We have*

$$\begin{aligned}
&C_5 D^{-4} H(\log H)^{k^2}\\
&\qquad \le 2(C_6 H(\log H)^{k^2})^{2^n/(2^n+1)}(C_7 H^{1-2^n a/(100q)})^{1/(2^n+1)}.
\end{aligned}$$

Proof. This follows from Theorem 6 and Lemma 4.

Lemma 6. *We have*

$$H^{-2^n a(2^n+1)^{-1}(100q)^{-1}} \le H^{-D(200\log H)^{-1}} \le e^{-D/200}$$

and $\frac{1}{2}(2^n+1) \le 2^n < 2^n+1$.

Proof. Trivial.

Lemmas 5 and 6 end up with the contradiction

$$C_5 D^{-4} \le 2C_6C_7e^{-D/200}$$

provided we fix $C_2 = D^{-100}$ and choose D to be large enough. Thus Theorem 10 is completely proved.

4. Deduction of Theorem 1 from Theorem 10 (second and final step). Actually our proof of Theorem 10 with a trivial modification gives

$$\max_{\sigma\ge 1/2+q(\log H)^{-1}} \left(\frac{1}{H}\int\limits_{T+H_4}^{T+H-H_4} |\zeta(\sigma+it)|^{2k}\,dt\right) > C_8(\log H)^{k^2} \tag{17}$$

where $C_8 > 0$ is absolute and H_4 is a small (absolute) positive constant times H. We first prove

Theorem 11. *If* $q \ge (\log H)^{1/100}$ *then* (1) *is true.*

Proof. We argue as we did after proving Lemma 1 but with $f_0(z) = \zeta(z)$, $\sigma_0 = 1/2$, $a = 10$, $n = 2$. Note that $(\log H)^{k^2}$ lies between two absolute positive constants. We use $|\zeta(\sigma+it)| \ll t^{1/2}$ uniformly for $\sigma \ge 1/2$, $t \ge 10$ and we see that we need the condition

$$(\exp\exp(C_9H))^{-1}T^k \le H^{-11} \qquad (C_9 > 0 \text{ is an absolute constant}),$$

which is precisely the condition $H \ge C_0\log\log(T^k+100)$ of Theorem 1. We need the condition $H \le T$ for the bound on $|\zeta(\sigma+it)|$ mentioned above.

We only have to prove the following theorem.

Theorem 12. *Let* $q \le (\log H)^{1/100}$. *Then* (1) *is true.*

Proof. We use (17). We fix a to be the largest $\sigma \le 2$ with the property

$$\frac{1}{H}\int\limits_{T+H_4}^{T+H-H_4} |\zeta(\sigma+it)|^{2k}\,dt > C_8(\log H)^{k^2}$$

and σ_0 to be 1/2. We argue as before with $f_0(z) = \zeta(z)$, where n is such that $a2^n$ lies between 10 and 20. Note that in this case

$$\int\limits_{(t)}\int\limits_{|v|\le\tau} |f(s,w)|^{1/q}_{u=a2^n}\,dv\,dt + H^{-9}$$

does not exceed an absolute constant times H. We use $|\zeta(\sigma+it)| \ll t^{1/2}$ for $\sigma \geq 1/2$ and $t \geq 10$ and we see that we need the condition

$$(\exp\exp(C_{10}H))^{-1}T^k \leq H^{-11} \qquad (C_{10} > 0 \text{ is an absolute constant}),$$

which is precisely the condition $H \geq C_0 \log\log(T^k+100)$ of Theorem 1. We need the condition $H \leq T$ for the bound on $|\zeta(\sigma+it)|$ mentioned above.

5. Concluding remarks. The new kernel $K(w)$ is very useful. We note that for $|u| \leq 200$ it satisfies the relation

$$\int_{-\infty}^{\infty} |K(u+iv)|\,dv = \Big(\int_{-\infty}^{\infty} K(iv)\,dv\Big)\Big(1+O\Big(\frac{1}{A}\Big)\Big) \tag{18}$$

(for large A), which is not hard to verify. Using this we can prove the following theorem.

Theorem 13. *Let $a_1, a_2, \ldots$ be any infinite sequence of complex numbers and $\lambda_1, \lambda_2, \ldots$ any sequence of real numbers satisfying $a_1 = \lambda_1 = 1$, $\lambda_{n+1}-\lambda_n$ bounded both above and below by positive constants, and $|a_n|$ bounded above by a positive constant power of n. Suppose that*

$$F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$$

(*which is certainly analytic in a half plane*) *can be continued in* ($\sigma \geq 1/2$, $T-H \leq t \leq T+2H$) *and there satisfies the condition that M defined by $M = \max|F(s)|$ satisfies* $\log\log(M+100) = o(T)$. *Let k be any positive real number which is less than an absolute* (*arbitrary*) *constant. Let ε (>0) be any constant. Then there exists a constant $C_{11} = C_{11}(\varepsilon)$ (>0) independent of k such that for all $T \geq 2H \geq C_{11}(\varepsilon)\log\log(M^{2k}+100)$, we have*

$$\min_{\sigma\geq 1/2}\Big(\frac{1}{H}\int_{T}^{T+H} |F(\sigma+it)|^{2k}\,dt\Big) \geq 1-\varepsilon. \tag{19}$$

Proof. We argue as in the proof of Theorem 11 taking $f_0(z) = F(z)$, $\sigma_0 = 1/2$, a equal to a large constant depending on ε and $n = 2$. This leads to the proof of theorem on using (18).

The application to the Riemann zeta-function is obvious. It runs as follows. (We use $|\zeta(\sigma+it)| \ll t^{1/2}$ for $\sigma \geq 1/2$, $t \geq 10$.)

Theorem 14. *Let k be any positive number which is bounded above and ε (>0) any constant. Then there exists a constant $C_{12}(\varepsilon)$ (>0) independent of k such that for all H satisfying $T \geq H \geq C_{12}(\varepsilon)\log\log(T^k+100)$, we* *have*

$$\min_{\sigma\ge 1/2}\left(\frac{1}{H}\int_{T}^{T+H}|\zeta(\sigma+it)|^{2k}\,dt\right)\ge 1-\varepsilon. \tag{20}$$

By taking $H=T$ we recover the following special case.

THEOREM 15 (A. Ivić and A. Perelli [3]). *We have*, *for all* $k>0$,

$$\frac{1}{T}\int_{T}^{2T}|\zeta(1/2+it)|^{2k}\,dt\ge 1+o(1) \tag{21}$$

uniformly in k *as* $T\to\infty$.

Remark. The proof of Theorem 15 by Ivić and Perelli is completely different.

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