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Kummer's lemma for $\mathbb{Z}_p$-extensions over totally real number fields

by

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1. Introduction. Let p be an odd prime, and let K be the pth cyclotomic field. In 1847, E. E. Kummer proved the following famous theorem which plays a crucial role in the proof of the second case of Fermat's Last Theorem for regular primes:

Kummer's lemma. *Assume that p is a regular prime, namely, p does not divide the class number of K. Then every unit in K which is congruent to* 1 *modulo p is a pth power of another unit in K.*

L. C. Washington generalized this theorem to all primes p as follows ([7]):

Theorem A. *Let $M = \max\{v_p(L_p(1,\omega_p^i)) : 2 \le i \le p-3,\ \text{even}\}$, where v_p is the normalized p-adic valuation ($v_p(p) = 1$), ω_p is the Teichmüller character and $L_p(s,\omega_p^i)$ is the Kubota–Leopoldt p-adic L-function. Then every unit in K which is congruent to* 1 *modulo p^{M+1} is a pth power of another unit in K.*

He also proved the following similar theorem for prime power cyclotomic fields ([9]):

Theorem B. *Let $n \ge 1$, and let L be the p^nth cyclotomic field. Put*

$$M_n = p^{n-1}(p-1)\max\{v_p(\tau(\chi)L_p(1,\chi)) : 1 \ne \chi \in \mathrm{Gal}(L/\mathbb{Q})^\wedge,\ \text{even}\}.$$

Then every unit in L which is congruent to 1 *modulo $p^n\mathfrak{p}_n^{M_n-1}$ is a pth power of another unit in L, where $\mathfrak{p}_n$ is the unique prime of L above p and $\tau(\chi)$ is the Gauss sum for χ: $\tau(\chi) = \sum_{a=1}^{f_\chi}\chi(a)\zeta_{f_\chi}^a$ (f_χ is the conductor of χ).*

Let F be a number field and p a prime. If we assume that Leopoldt's conjecture is valid for F and p, then there exists an integral ideal $\mathfrak{M}$ of F whose prime factors are primes above p such that every unit in F which is congruent to 1 modulo $\mathfrak{M}$ is a pth power of another unit in F. In the present paper, we shall describe this ideal $\mathfrak{M}$ in terms of the p-adic zeta functions and the p-adic L-functions when F is the nth layer of a $\mathbb{Z}_p$-extension of a totally real number field. Especially, applying our result to the p^nth real cyclotomic field, we can improve Theorem B for sufficiently large n.

Our method is completely different from Washington's. In the proof of Theorems A and B, he used the cyclotomic units and Leopoldt's formula for $L_p(1,\chi)$ in which the cyclotomic units appear. But when we deal with totally real number fields, we do not know such special units as those connected to the value of the p-adic L-functions of totally real number fields. So we shall embed the unit group in the semi-local unit group and investigate its factor group applying Iwasawa's theory, especially Iwasawa's Main Conjecture proved by A. Wiles.

We shall prepare some preliminary results in Section 2, and we shall state and prove our main theorem in Section 3.

2. Exponents of some Λ-modules. Let p be a prime and $\mathcal{O}$ the integer ring of a finite extension field over $\mathbb{Q}_p$. Denote by Λ the ring of formal power series $\mathcal{O}[[T]]$. In this section, we shall estimate the exponent of some finite Λ-modules.

For $x \in \mathbb{Q}$ we denote by $\lceil x \rceil$ the smallest integer such that $\lceil x \rceil \geq x$. Let v_p denote the normalized p-adic valuation of $\overline{\mathbb{Q}}_p$; namely, $v_p(p) = 1$. For any finite $\mathbb{Z}_p$-module M, we write $\exp(M)$ for the exponent of M. Put $\omega_n = (1+T)^{p^n} - 1 \in \Lambda$ for $n \geq 0$.

Lemma 1. *Let $n \geq 0$, and let $f \in \Lambda$ be a power series which is prime to ω_n. Then*

$$\exp(\Lambda/(f,\omega_n)\Lambda) \leq p^{n+\lceil \max\{v_p(f(\zeta-1)) : \zeta^{p^n}=1\}\rceil}.$$

Proof. Let $\widetilde{\mathcal{O}} = \mathcal{O}[\{\zeta \in \overline{\mathbb{Q}}_p : \zeta^{p^n} = 1\}]$, and let $\widetilde{\Lambda} = \widetilde{\mathcal{O}}[[T]]$. From $\omega_n(T) = \prod_{\zeta^{p^n}=1}(T-(\zeta-1))$, we have the following exact sequence:

$$0 \to \widetilde{\Lambda}/\omega_n\widetilde{\Lambda} \xrightarrow{\varphi} \bigoplus_{\zeta^{p^n}=1} \widetilde{\Lambda}/(T-(\zeta-1))\widetilde{\Lambda} \to C \to 0, \tag{1}$$

where φ is induced by the natural projection and C is a finite $\widetilde{\Lambda}$-module. Since

$$\mathrm{Im}(\varphi) \supseteq \bigoplus_{\zeta^{p^n}=1} \left(T-(\zeta-1), \frac{\omega_n(T)}{T-(\zeta-1)}\right)\widetilde{\Lambda}/(T-(\zeta-1))\widetilde{\Lambda},$$

and

$$\widetilde{\Lambda}\Big/\left(T-(\zeta-1), \frac{\omega_n(T)}{T-(\zeta-1)}\right)\widetilde{\Lambda} \simeq \widetilde{\mathcal{O}}/\omega_n'(\zeta-1)\widetilde{\mathcal{O}} \simeq \widetilde{\mathcal{O}}/p^n\widetilde{\mathcal{O}},$$

we have

$$\exp(C) \leq p^n. \tag{2}$$

From (1) and the assumption of the lemma, we get the exact sequence

$$0 \to C^f \to \widetilde{\Lambda}/(f,\omega_n)\widetilde{\Lambda} \to \bigoplus_{\zeta^{p^n}=1} \widetilde{\Lambda}/(f, T-(\zeta-1))\widetilde{\Lambda} \to C/fC \to 0, \tag{3}$$

where $C^f = \{x \in C : fx = 0\}$. We note that

$$\widetilde{\Lambda}/(f, T-(\zeta-1))\widetilde{\Lambda} \simeq \widetilde{\mathcal{O}}/f(\zeta-1)\widetilde{\mathcal{O}}.$$

Hence we see that

$$\exp\Big(\bigoplus_{\zeta^{p^n}=1} \widetilde{\Lambda}/(f, T-(\zeta-1))\widetilde{\Lambda}\Big) \le p^{\lceil \max\{v_p(f(\zeta-1)):\zeta^{p^n}=1\}\rceil}.$$

It follows from (2), (3) and this inequality that

$$\exp(\widetilde{\Lambda}/(f,\omega_n)\widetilde{\Lambda}) \le p^{n+\lceil \max\{v_p(f(\zeta-1)):\zeta^{p^n}=1\}\rceil}.$$

Since $\widetilde{\Lambda}/(f,\omega_n)\widetilde{\Lambda} \simeq (\Lambda/(f,\omega_n)\Lambda) \otimes_{\mathcal{O}} \widetilde{\mathcal{O}} \simeq (\Lambda/(f,\omega_n)\Lambda)^{\oplus \operatorname{rank}_{\mathcal{O}} \widetilde{\mathcal{O}}}$ as $\mathbb{Z}_p$-modules, we obtain the lemma. ∎

LEMMA 2. *Let M be any finitely generated torsion Λ-module without nontrivial finite Λ-submodule, and let $f \in \Lambda$ be a generator of the characteristic ideal of the Λ-module M. Then*

$$\exp(M/gM) \le \exp(\Lambda/(f,g)\Lambda)$$

for any $g \in \Lambda$ which is prime to f.

Proof. From the assumption of the lemma, we have $fM = 0$. Hence we obtain $(f,g)(M/gM) = (fM+gM)/gM = 0$, which implies Lemma 2. ∎

Combining Lemmas 1 and 2, we obtain the following:

PROPOSITION 1. *Let M and $f \in \Lambda$ be as in Lemma* 2, *and let $n \ge 0$. Assume that f is prime to ω_n. Then*

$$\exp(M/\omega_n M) \le p^{n+\lceil \max\{v_p(f(\zeta-1)):\zeta^{p^n}=1\}\rceil}.$$

Proof. This follows immediately from Lemmas 1 and 2. ∎

3. Kummer's lemma for totally real number fields. Let p be an odd prime, and let K be a totally real abelian extension of a totally real number field k. We assume that $p \nmid [K:k]$. For any number field F, we denote by F_∞/F the cyclotomic $\mathbb{Z}_p$-extension, and let F_n be its nth layer. Put $\Gamma = \mathrm{Gal}(k_\infty/k)$, $\Gamma_n = \mathrm{Gal}(k_n/k)$ and $\Delta = \mathrm{Gal}(K/k)$. Since $p \nmid [K:k]$, the natural restriction induces the isomorphism $\mathrm{Gal}(K_\infty/K) \simeq \Gamma$. So we identify these Galois groups. For any finite group G, we let $\widehat{G} = \mathrm{Hom}(G, \overline{\mathbb{Q}}_p^\times)$. In this section, we shall generalize Kummer's lemma for each K_n as follows:

Main Theorem. *Let $n \geq 0$. Assume that Leopoldt's conjecture is valid for K_n and p. Put*

$$M = \lceil \max\{v_p(w\varrho), v_p(L_p(1,\psi)) + d_\psi^{-1}, v_p(L_p(1,\chi\psi')) : 1 \neq \chi \in \widehat{\Delta},\ \psi, \psi' \in \widehat{\Gamma}_n, \psi \neq 1\}\rceil + n + 1,$$

where $L_p(s,*)$ is the p-adic L-function of k, ϱ is the residue of the p-adic zeta function $\zeta_p(s,k)$ at $s = 1$, w is the number of p-power-th roots of unity contained in $K(\zeta_p)$, $d_\psi = \varphi(m_\psi)$ is the value of the Euler function at the order m_ψ of $\psi \in \widehat{\Gamma}_n$. Then every unit ε in K_n such that

$$\varepsilon \equiv 1 \bmod p^M \prod_{\mathfrak{p}|p} \mathfrak{p}^{[e_\mathfrak{p}/(p-1)]+1}$$

is a pth power of another unit in K_n, where $[\ ]$ is the greatest integer function, $\mathfrak{p}$ is a prime of K_n and $p = \prod_{\mathfrak{p}|p} \mathfrak{p}^{e_\mathfrak{p}}$.

Remark. P. Colmez proved in [2] that

$$\varrho = \frac{2^{d-1} h_k R_p(k)}{\sqrt{d(k)}} \prod_{\mathfrak{p}|p} (1 - \mathrm{N}(\mathfrak{p})^{-1}),$$

where $d = [k : \mathbb{Q}]$, h_k is the class number of k, $R_p(k)$ is the p-adic regulator of k, $d(k)$ is the discriminant of k and $\mathfrak{p}$ is a prime of k.

To prove the Main Theorem, we need some propositions. For any number field F, we denote by $L(F)$ and $M(F)$ the maximal unramified pro-p abelian extension field over F and the maximal pro-p abelian extension field over F which is unramified outside p, respectively. For a prime $\mathfrak{p}$ of F, let $U_{F_\mathfrak{p}}$ be the group of local units of $F_\mathfrak{p}$ which are congruent to 1 modulo $\mathfrak{p}$, and let $U_F = \prod_{\mathfrak{p}|p} U_{F_\mathfrak{p}}$. Denote by E_F the group of units of F which are congruent to 1 modulo all primes dividing p. We shall embed E_F in U_F diagonally, and we shall regard E_F as a subgroup of U_F. We denote by $\overline{E}_F$ the closure of E_F in U_F. Then class field theory shows that $\mathrm{Gal}(M(F)/L(F)) \simeq U_F/\overline{E}_F$.

Proposition 2. *Let p be a prime and let F be a totally real number field. Assume that Leopoldt's conjecture is valid for F and p. Put $p^e = \exp(\mathrm{Gal}(M(F)/F_\infty L(F)))$, where F_∞/F is the cyclotomic $\mathbb{Z}_p$-extension. Then every unit in F which is congruent to 1 modulo $p^{e+1}\prod_{\mathfrak{p}|p} \mathfrak{p}^{[e_\mathfrak{p}/(p-1)]+1}$ is a pth power of another unit in F, where $[\ ]$ is the greatest integer function, $\mathfrak{p}$ is a prime of F and $p = \prod_{\mathfrak{p}|p} \mathfrak{p}^{e_\mathfrak{p}}$.*

Proof. It follows from the validity of Leopoldt's conjecture for F and p that $\mathrm{rank}_{\mathbb{Z}_p} \mathrm{Gal}(M(F)/L(F)) = 1$. Then, from the split exact sequence

$$0 \to \mathrm{Gal}(M(F)/F_\infty L(F)) \to \mathrm{Gal}(M(F)/L(F)) \\ \to \mathrm{Gal}(F_\infty L(F)/L(F)) \to 0,$$

we have

$$\mathrm{Gal}(M(F)/F_\infty L(F)) = \mathrm{Tor}_{\mathbb{Z}_p}(\mathrm{Gal}(M(F)/L(F))) \simeq \mathrm{Tor}_{\mathbb{Z}_p}(U_F/\overline{E}_F).$$

So we find that $\exp(\mathrm{Tor}_{\mathbb{Z}_p}(U_F/\overline{E}_F)) = p^e$. Let $\varepsilon = u^{p^{e+1}} \in U_F^{p^{e+1}} \cap E_F$ ($u \in U_F$) be any element. Then $u \bmod \overline{E}_F \in \mathrm{Tor}_{\mathbb{Z}_p}(U_F/\overline{E}_F)$, hence we have $\varepsilon = u^{p^{e+1}} \in \overline{E}_F^p \cap E_F$. The validity of Leopoldt's conjecture for F and p implies that $\varepsilon \in \overline{E}_F^p \cap E_F = E_F^p$. On the other hand, if $u_\mathfrak{p} \in U_{F_\mathfrak{p}}$ satisfies $u_\mathfrak{p} \equiv 1 \bmod p^{e+1}\mathfrak{p}^{[e_\mathfrak{p}/(p-1)]+1}$, then $u_\mathfrak{p} \in U_{F_\mathfrak{p}}^{p^{e+1}}$, for $\mathfrak{p} \mid p$. Thus we have completed the proof of Proposition 2. ∎

By virtue of Proposition 2, we know that what we have to do is estimating $\exp(\mathrm{Gal}(M(K_n)/K_\infty L(K_n)))$ for every $n \ge 0$. We shall do this below applying Iwasawa's theory.

Let $\Lambda = \mathbb{Z}_p[\Delta][[T]]$. We fix a topological generator $\gamma \in \Gamma$ and we identify $\mathbb{Z}_p[\Delta][[\Gamma]]$ with Λ by the isomorphism $\mathbb{Z}_p[\Delta][[\Gamma]] \simeq \Lambda$, $\gamma \leftrightarrow (1+T)$. Let $\widetilde{\gamma} \in \mathrm{Gal}(K_\infty(\zeta_p)/K(\zeta_p))$ be the image of $\gamma \in \Gamma$ by the natural isomorphism $\Gamma \simeq \mathrm{Gal}(K_\infty(\zeta_p)/K(\zeta_p))$. We let $\kappa \in \mathbb{Z}_p^\times$ be the number such that $\zeta^{\widetilde{\gamma}} = \zeta^\kappa$ for any p-power-th root of unity ζ. For any $\mathbb{Z}_p[\Delta]$-module M and $\chi \in \widehat{\Delta}$, we denote by M^χ the χ-part of M. We identify $\mathbb{Z}_p[\Delta]^\chi$ with $\mathbb{Z}_p[\chi(\Delta)]$ via χ. We need the following theorem which is a variation of Iwasawa's Main Conjecture proved by A. Wiles (cf. [10], [1]):

Theorem. *The notation being as above, for each $\chi \in \widehat{\Delta}$, there exists $G_\chi \in \Lambda^\chi = \mathbb{Z}_p[\chi(\Delta)][[T]]$ such that*

$$G_\chi(\kappa^s - 1) = \begin{cases} L_p(s,\chi) & \text{if } \chi \ne 1, \\ (\kappa^s - \kappa)\zeta_p(s,k) & \text{if } \chi = 1, \end{cases} \tag{4}$$

for $s \in \mathbb{Z}_p$, and

$$\mathrm{char}_{\Lambda^\chi}(\mathrm{Gal}(M(K_\infty)/K_\infty)^\chi) = G_\chi(\kappa(1+T)^{-1} - 1)\Lambda^\chi,$$

where $L_p(s,\chi)$ and $\zeta_p(s,k)$ are the p-adic L-function and p-adic zeta function of k, respectively, and $\mathrm{char}_{\Lambda^\chi}(\mathrm{Gal}(M(K_\infty)/K_\infty)^\chi)$ *denotes the characteristic ideal of the finitely generated torsion Λ^χ-module* $\mathrm{Gal}(M(K_\infty)/K_\infty)^\chi$.

Using the above theorem, we estimate the exponent of $\mathrm{Gal}(M(K_n)/K_\infty)$ in terms of p-adic L-functions:

Proposition 3. *Let notations be as above, and let $n \ge 0$ and $\chi \in \widehat{\Delta}$. Moreover, we assume that Leopoldt's conjecture is valid for K_n and p. Then*

$$\exp(\mathrm{Gal}(M(K_n)/K_\infty)^\chi) \le \begin{cases} p^{n+\lceil \max\{v_p(L_p(1,\chi\psi)):\psi\in\hat{\Gamma}_n\}\rceil} & \textit{if } \chi \neq 1, \\ p^{n+\lceil \max\{v_p(w\varrho), v_p(L_p(1,\psi))+d_\psi^{-1}:1\neq\psi\in\hat{\Gamma}_n\}\rceil} & \textit{if } \chi = 1, \end{cases}$$

where ϱ, w *and* d_ψ *are the same as in the statement of the Main Theorem.*

P r o o f. It is known that

$$\mathrm{Gal}(M(K_n)/K_\infty)^\chi = \mathrm{Gal}(M(K_\infty)/K_\infty)^\chi/\omega_n \,\mathrm{Gal}(M(K_\infty)/K_\infty)^\chi,$$

and that $\mathrm{Gal}(M(K_\infty)/K_\infty)^\chi$ has no non-trivial finite Λ^χ-submodule (cf. [4], [3]). Since $\mathrm{Gal}(M(K_n)/K_\infty)^\chi$ is finite by the validity of Leopoldt's conjecture for K_n and p, a generator of the $\mathrm{char}_{\Lambda^\chi}(\mathrm{Gal}(M(K_\infty)/K_\infty)^\chi)$ is prime to ω_n. Hence we have

$$\exp(\mathrm{Gal}(M(K_n)/K_\infty)^\chi) \le p^{n+\lceil \max\{v_p(G_\chi(\zeta\kappa-1)):\zeta^{p^n}=1\}\rceil} \tag{5}$$

by Proposition 1 and the above theorem. It is also known that

$$L_p(s,\chi\psi) = G_\chi(\psi(\gamma)^{-1}\kappa^s - 1) \tag{6}$$

for $1\neq\chi\in\widehat{\Delta}$ and $\psi\in\widehat{\Gamma}_n$, and

$$L_p(s,\psi) = G_1(\psi(\gamma)^{-1}\kappa^s - 1)/(\psi(\gamma)^{-1}\kappa^s - \kappa)$$

for $\psi\in\widehat{\Gamma}_n$ (see for example [6, (2.4), p. 7]). Since

$$v_p(G_1(\kappa-1)) = v_p\left(\varrho\lim_{s\to 1}\frac{\kappa^s-\kappa}{s-1}\right)$$
$$= v_p(\varrho\kappa\log_p(\kappa)) = v_p(\varrho(\kappa-1)) = v_p(w\varrho),$$
$$v_p(\psi(\gamma)^{-1}\kappa-\kappa) = d_\psi^{-1} \qquad (\psi\neq 1),$$

and

$$\{\psi(\gamma):\psi\in\widehat{\Gamma}_n\} = \{\zeta\in\overline{\mathbb{Q}}_p : \zeta^{p^n}=1\},$$

(5) concludes the proof of the proposition. ∎

P r o o f o f t h e M a i n T h e o r e m. Since

$$\exp(\mathrm{Gal}(M(K_n)/K_\infty L(K_n))) \le \exp(\mathrm{Gal}(M(K_n)/K_\infty))$$
$$= \max\{\exp(\mathrm{Gal}(M(K_n)/K_\infty)^\chi):\chi\in\widehat{\Delta}\},$$

the Main Theorem follows from Propositions 2 and 3. ∎

Let p be an odd prime, $k=\mathbb{Q}$ and $K=\mathbb{Q}(\zeta_p+\zeta_p^{-1})$. Now we shall apply the Main Theorem to p and K/k. Denote by $\mathfrak{p}_n$ the unique prime of $\mathbb{Q}(\zeta_{p^n})$ above p. We note that a unit in $\mathbb{Q}(\zeta_{p^n})$ which is congruent to 1 modulo p is real, and that a unit in $\mathbb{Q}(\zeta_{p^n}+\zeta_{p^n}^{-1})$ which is congruent to 1 modulo $p^n\mathfrak{p}_n^{2j+1}$ is congruent to 1 modulo $p^n\mathfrak{p}_n^{2j+2}$ for any integer $j\ge 0$. Since $G_1(T)$ is a unit power series in this case, we obtain the following corollary of the Main Theorem:

COROLLARY. *Let $n \geq 1$. Put*

$$N_n = p^{n-1}(p-1)\lceil \max\{v_p(L_p(1,\chi)) : 1 \neq \chi \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n}+\zeta_{p^n}^{-1})/\mathbb{Q})^\wedge\}\rceil + p^{n-1}.$$

Then every unit in $\mathbb{Q}(\zeta_{p^n})$ which is congruent to 1 *modulo $p^n\mathfrak{p}_n^{N_n}$ is a pth power of another unit in $\mathbb{Q}(\zeta_{p^n})$.*

In Theorem B of the introduction, we note that

$$v_p(\tau(\chi)) = \begin{cases} \dfrac{i}{p-1} & \text{if } \chi = \omega_p^{-i},\ 0 \leq i \leq p-2, \\ \frac{1}{2}v_p(f_\chi) & \text{if } f_\chi = p^c,\ c \geq 2 \end{cases} \tag{7}$$

(cf. [8, Prop. 6.13], [5]). By (4), (6) and the similar argument in [8, p. 127], we can see that $v_p(L_p(1,\chi\psi)) = \lambda_\chi/\varphi(m_\psi)$ with the constant λ_χ depending on χ if m_ψ is sufficiently large, where $1 \neq \chi \in \mathrm{Gal}(\mathbb{Q}(\zeta_p+\zeta_p^{-1})/\mathbb{Q})^\wedge$, $\psi \in \bigcup_{n\geq 0}\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})^\wedge$ and m_ψ is the order of ψ. Hence we find that $\max\{v_p(L_p(1,\chi)) : 1 \neq \chi \in \mathrm{Gal}(\mathbb{Q}(\zeta_{p^n}+\zeta_{p^n}^{-1})/\mathbb{Q})^\wedge\}$ is stabilized for sufficiently large n. This yields that $N_n = p^{n-1}(c(p-1)+1)$ with a certain constant c provided n is large enough. On the other hand, we have $M_n - 1 \geq (n/2)p^{n-1}(p-1) - 1$ for all $n \geq 2$ by (7), where M_n is as in Theorem B. Hence the Corollary is certainly stronger than Theorem B for sufficiently large n. Furthermore, for $n = 1$ the Corollary is equivalent to Theorem A in the introduction since every unit in $\mathbb{Q}(\zeta_p)$ which is congruent to 1 modulo p^j is congruent to 1 modulo $p^j\mathfrak{p}_1^2$ for any integer $j \geq 1$.

References

[1] J. Coates, *p-adic L-functions and Iwasawa's theory*, in: Algebraic Number Fields, Durham Symposium, 1975, A. Fröhlich (ed.), Academic Press, London, 1977, 269–353.

[2] P. Colmez, *Résidu en $s = 1$ des fonctions zêta p-adiques*, Invent. Math. 91 (1988), 371–389.

[3] R. Greenberg, *On the structure of certain Galois groups*, ibid. 47 (1978), 85–99.

[4] K. Iwasawa, *On $\mathbb{Z}_l$-extensions of algebraic number fields*, Ann. of Math. 98 (1973), 246–326.

[5] R. W. K. Odoni, *On Gauss sums* (mod p^n), Bull. London Math. Soc. 5 (1973), 325–327.

[6] W. Sinnott, *On p-adic L-functions and the Riemann–Hurwitz genus formula*, Compositio Math. 53 (1984), 3–17.

[7] L. C. Washington, *Units of irregular cyclotomic fields*, Illinois J. Math. 23 (1979), 635–647.

[8] —, *Introduction to Cyclotomic Fields*, Grad. Texts in Math. 83, Springer, New York, 1982.

[9] —, *Kummer's lemma for prime power cyclotomic fields*, J. Number Theory 40 (1992), 165–173.

[10] A. Wiles, *The Iwasawa conjecture for totally real fields*, Ann. of Math. 131 (1990), 493–540.

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