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Errata to the paper "On a functional equation satisfied by certain Dirichlet series"

(Acta Arith. 71 (1995), 265–272)

by

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We have to point out that formula (5) in [1] is wrong, as well as the formula for $\Phi_L(s)$ given in the statement of the Theorem in [1]. The following lemma will take the place of formula (5) in [1].

Lemma 0.1. *The following formula for derivatives of higher order of $z^\nu I_\nu(z)$ holds*:

$$\frac{d^p}{dz^p}(z^\nu I_\nu(z)) = \sum_{l=0}^{[p/2]} (2l-1)!!\binom{p}{2l} z^{\nu-l} I_{\nu-(p-l)}(z) \tag{0.1}$$

if we put $(-1)!! = 1$.

P r o o f. From the well known formula (see [2])

$$\frac{d}{dz}(z^\nu I_\nu(z)) = z^\nu I_{\nu-1}(z) \tag{0.2}$$

we derive, by induction, that if $p \ge 1$ then

$$\frac{d^p}{dz^p}(z^\nu I_\nu(z)) = \sum_{l=0}^{[p/2]} \beta_{p,l} z^{\nu-l} I_{\nu-(p-l)}(z). \tag{0.3}$$

By a direct computation we get $\beta_{p,0} = 1$ for all $p \ge 1$. Comparing

$$\frac{d^{p+1}}{dz^{p+1}}(z^\nu I_\nu(z)) = \sum_{t=0}^{[(p+1)/2]} \beta_{p+1,t} z^{\nu-t} I_{\nu-(p+1-t)}(z)$$

with

$$\frac{d}{dz}\left(\frac{d^p}{dz^p}\,(z^\nu I_\nu(z))\right)$$

developed by (0.2) from (0.3), we obtain the following recurrence formula:

$$\beta_{p+1,t} = (p - 2t + 2)\beta_{p,t-1} + \beta_{p,t}, \tag{0.4}$$

where $p \geq 1$, $0 \leq t \leq [(p+1)/2]$ and $\beta_{p,i} = 0$ if $i > [p/2]$ or $i < 0$. From (0.4) for $t \geq 2$ due to the well known formula

$$\sum_{k=0}^{m} \binom{n+k}{n} = \binom{n+m+1}{n+1}$$

we obtain, for all $p \geq 1$,

$$\beta_{p+1,t} = (2t-1)!! \binom{p+1}{2t}. \tag{0.5}$$

We note that $\beta_{1,0} = 1$, so (0.5) holds if $p = 0$. If $t = 0$, taking $(-1)!! = 1$ the above formula holds by a direct computation. For $t = 1$, (0.5) follows directly from (0.4). ∎

By using formula (0.1) we obtain the corrected form for the function $\Phi_L(s)$ given in the statement of the Theorem in [1].

In the proof of the Theorem of [1] we have to replace page 270, from the fifth line starting with "By Cauchy's theorem..." up to the end of the page, with the following:

By Cauchy's theorem we have

$$I_N(s) = -\sum_{\substack{-N \leq 2n \leq N \\ n \neq 0}} \operatorname{Res}\left(H(z) I_{s-1/2}\left(\frac{\delta}{2} z\right) z^{s-1/2}; 2\pi n i \right).$$

If we put

$$A(z) = I_{s-1/2}\left(\frac{\delta}{2} z\right) z^{s-1/2},$$

its Taylor series at $s = 2\pi n i$, $n \neq 0$, is

$$A(z) = \sum_{m=0}^{\infty} \frac{1}{m!} A^{(m)}(2\pi n i)(z - 2\pi n i)^m.$$

Then we have

$$\operatorname{Res}(H(z)A(z); 2\pi n i) = \sum_{\substack{p+l=-1 \\ p \geq -(d+1) \\ l \geq 0}} \frac{1}{l!} \alpha_p^n A^{(l)}(2\pi n i) = \sum_{p=0}^{d} \frac{1}{p!} \alpha_{-p-1}^n A^{(p)}(2\pi n i).$$

By (0.1),

$$A^{(p)}(z) = \sum_{l=0}^{[p/2]} (2l-1)!! \binom{p}{2l} \left(\frac{\delta}{2}\right)^{p-l} z^{s-1/2-l} I_{s-1/2-(p-l)}\left(\frac{\delta}{2} z\right).$$

Therefore

$$I_N(s) = - \sum_{\substack{-N \le 2n \le N \\ n \ne 0}} \sum_{p=0}^{d} \sum_{l=0}^{[p/2]} \frac{(2l-1)!!}{p!} \binom{p}{2l} \left(\frac{\delta}{2}\right)^{p-l} \times \alpha_{-p-1}^{n} (2n\pi i)^{s-1/2-l} I_{s-1/2-(p-l)}(\delta n\pi i).$$

By (2) and (3) of [1] the series

$$\sum_{\substack{n \in \mathbb{Z} \\ n \ne 0}} \alpha_{-p-1}^{n} (2\pi n i)^{s-1/2-l} I_{s-1/2-(p-l)}(\delta n\pi i)$$

converges absolutely and uniformly on compact subsets of $\sigma < 0$. Thus, for $\sigma < 0$, we have

$$I(s) = - \sum_{\substack{n \in \mathbb{Z} \\ n \ne 0}} \sum_{p=0}^{d} \sum_{l=0}^{[p/2]} \frac{(2l-1)!!}{p!} \binom{p}{2l} \left(\frac{\delta}{2}\right)^{p-l} \times \alpha_{-p-1}^{n} (2\pi n i)^{s-1/2-l} I_{s-1/2-(p-l)}(\delta n\pi i).$$

Then we derive the final formula for $\Phi_L(s)$ in $\sigma > 1$:

$$\Phi_L(s) = I(1-s) = - \sum_{p=0}^{d} \sum_{l=0}^{[p/2]} \sum_{\substack{n \in \mathbb{Z} \\ n \ne 0}} \frac{(2l-1)!!}{p!} \binom{p}{2l} \left(\frac{\delta}{2}\right)^{p-l} \times \alpha_{-p-1}^{n} (2\pi n i)^{1/2-s-l} I_{1/2-s-(p-l)}(\delta n\pi i).$$

References

[1] E. Carletti and G. Monti Bragadin, *On a functional equation satisfied by certain Dirichlet series*, Acta Arith. 71 (1995), 265–272.

[2] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals*, *Series and Products*, fifth ed., Academic Press, 1993.

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