

Properties of certain integer-valued analogues of Dedekind sums

by

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1. Introduction. B. C. Berndt [2] demonstrated that certain sums, first examined by G. H. Hardy [6], [7, pp. 362–392] and arising in the transformation formulas of the logarithms of the classical theta-functions, are analogous to the regular Dedekind sum. In this paper, we apply H. Rademacher's questions [12], [13, p. 112] about Dedekind sums to the study of these analogous sums. To introduce these sums, let $\Im(z) > 0$, $q = e^{\pi iz}$,

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \theta_4(z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

Let $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and define $Vz = (az + b)/(cz + d)$. We also define the standard modular groups

$$\Gamma(1) = \{V : a, b, c, d \in \mathbb{Z}, c > 0 \text{ with } ad - bc = 1\},$$

$$\Gamma_\theta = \{V \in \Gamma(1) : a \equiv d, b \equiv c \pmod{2}\},$$

$$\Gamma^0(2) = \{V \in \Gamma(1) : b \equiv 0 \pmod{2}\}.$$

In [2], Berndt proved the following results. If $V \in \Gamma_\theta$, then

$$\log \theta(Vz) = \log \theta(z) + \frac{1}{2} \log(cz + d) - \frac{1}{4} \pi i + \frac{1}{4} \pi i S(d, c),$$

where

$$(1.1) \quad S(d, c) = \sum_{j=1}^{c-1} (-1)^{j+1+[dj/c]}.$$

If $V \in \Gamma^0(2)$, then

$$\log \theta_4(Vz) = \log \theta_4(z) + \frac{1}{2} \log(cz + d) - \frac{1}{4} \pi i - \frac{1}{4} \pi i S_4(d, c),$$

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where

$$(1.2) \quad S_4(d, c) = \sum_{j=1}^{c-1} (-1)^{\lfloor dj/c \rfloor}.$$

We define the empty sums $S(d, 1) = S_4(d, 1) = 0$. In this paper, we examine properties of the sums $S(d, c)$ and $S_4(d, c)$ outside of the context of the transformation formulas. For results related to the transformation formulas, see L. A. Goldberg's thesis [5].

The most well-known property of the Dedekind sum $s(d, c)$ defined by

$$s(d, c) = \sum_{j=1}^c \left(\left(\frac{j}{c} \right) \right) \left(\left(\frac{dj}{c} \right) \right)$$

with

$$((x)) = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ x - [x] - 1/2 & \text{otherwise,} \end{cases}$$

is the reciprocity law, first proved by Dedekind [3]:

RECIPROCITY LAW FOR $s(d, c)$. *If $(c, d) = 1$ and $c, d > 0$, then*

$$(1.3) \quad s(c, d) + s(d, c) = -\frac{1}{4} + \frac{d}{12c} + \frac{c}{12d} + \frac{1}{12cd}.$$

In [2], Berndt proves the reciprocity theorem for $S(d, c)$:

RECIPROCITY LAW FOR $S(d, c)$. *If $(c, d) = 1$, $c + d$ is odd and $c, d > 0$, then*

$$(1.4) \quad S(c, d) + S(d, c) = 1.$$

There is no reciprocity formula for $S_4(d, c)$ in this same sense. (See [2] for so-called hybrid reciprocity formulas, and the author's thesis [9] for a reciprocity formula modulo 8).

Rademacher [12] defined

$$f(d/c) = s(d, c)$$

as a function on the rationals. The function $f(x)$ is well-defined, since it can be shown from the definition that $s(qd, qc) = s(d, c)$ for any positive integer q . Rademacher posed the following questions about $f(x)$. The first was raised in [12], the others in [13, p. 112]:

- 1) Is $f(x)$ unbounded in any neighborhood of any rational number?
- 2) Are the points $(x, f(x))$ dense in $\mathbb{R} \times \mathbb{R}$?
- 3) Given two adjacent Farey fractions, h_1/k_1 and h_2/k_2 with $f(h_1/k_1) > 0$ and $f(h_2/k_2) > 0$, does it follow that

$$f\left(\frac{h_1 + h_2}{k_1 + k_2}\right) \geq 0?$$

In [12], Rademacher answers the first question in the affirmative. D. Hickerson [8] answers the second, also in the affirmative, using a continued fraction representation for $s(d, c)$. Several authors [1], [11], [14] have answered the third in the negative. All of these authors make extensive use of (1.3) to prove their results. Because of the constant right-hand side of (1.4) and the fact that the parities of c and d are more restricted, one cannot employ the methods of Rademacher and Hickerson. Thus to study the corresponding problems for the sums defined in (1.1) and (1.2) we need to devise new methods. In Section 3, we answer the analogous questions about the sums associated with the theta-functions. The first two results of Section 2 are used extensively in the sequel.

2. Main results

PROPOSITION 1. *Let $(c, d) = 1$ with $c > 0$.*

- (i) *If c and d are both odd, then $S(d, c) = 0$.*
- (ii) *If d is even, then $S_4(d, c) = 0$.*

PROOF. The proof of (i) is presented directly.

Since d and c are odd, $d = c - 2n$ for some $n \in \mathbb{Z}$. Then $(c, n) = 1$ since $(c, d) = 1$. Therefore, using the definition of $S(d, c)$ and the fact that $[\alpha] + [-\alpha] = -1$ for $\alpha \notin \mathbb{Z}$, we see that

$$\begin{aligned}
 (2.1) \quad S(d, c) &= S(c - 2n, c) = \sum_{j=1}^{c-1} (-1)^{j+1+[(c-2n)j/c]} \\
 &= \sum_{j=1}^{c-1} (-1)^{2j+1+[-2nj/c]} = \sum_{j=1}^{c-1} (-1)^{[2nj/c]}.
 \end{aligned}$$

Next we show that the sum of the j th and the $(c - j)$ th exponents in (2.1) is odd and thus they have opposite parity. Therefore the corresponding terms cancel each other. Now, since $[\alpha] + [-\alpha] = -1$ for $\alpha \notin \mathbb{Z}$,

$$\left[\frac{2nj}{c} \right] + \left[\frac{2n(c-j)}{c} \right] = \left[\frac{2nj}{c} \right] + 2n + \left[-\frac{2nj}{c} \right] = 2n - 1.$$

There are an even number of terms, so we conclude that $S(d, c) = 0$. The deduction of (ii) is made by observing that the exponents have the same form as those in the last sum in (2.1).

REMARK. Some properties about these analogous sums, including Proposition 1, are proved by R. Sitaramachandrarao in [15].

THEOREM 2. *Let $(a, b) = 1$ with $0 \leq a \leq b$, and let x, y be nonnegative integers such that $ay - bx = \pm 1$ and $y > 0$. Then for $k \in \mathbb{N}$,*

$$(i) \quad S(x+ak, y+bk) = \sum_{l=0}^{k-1} (-1)^{(a+b)l} \{S(a, b) \pm (-1)^{a+b}\} + (-1)^{(a+b)k} S(x, y)$$

and

$$(ii) \quad S_4(x+ak, y+bk) = \sum_{l=0}^{k-1} (-1)^{al} \{S_4(a, b) \mp (-1)^a\} + (-1)^{ak} S_4(x, y).$$

The following lemmas will be used to prove Theorem 2.

LEMMA 3. Let $(a, b) = 1$ with $0 \leq a \leq b$. Let x, y, m, l, k be nonnegative integers such that $ay - bx = \pm 1$, $y > 0$, $1 \leq m \leq b$, and $0 \leq l < k$. Then

$$(i) \quad \left[\frac{(x+ak)m \mp l}{y+bk} \right] = \left[\frac{(x+ak)m}{y+bk} \right].$$

If, in addition, $m < y$, then

$$(ii) \quad \left[\frac{(x+ak)m \mp k}{y+bk} \right] = \left[\frac{(x+ak)m}{y+bk} \right].$$

PROOF. For $l = 0$, there is nothing to prove. For $1 \leq l < k$, it is sufficient to show that

$$\left[\frac{(x+ak)m \mp (k-1)}{y+bk} \right] = \left[\frac{(x+ak)m}{y+bk} \right].$$

Suppose that

$$(2.2) \quad \left[\frac{(x+ak)m \mp (k-1)}{y+bk} \right] \neq \left[\frac{(x+ak)m}{y+bk} \right].$$

Then, since $b \geq 1$, $y \geq 1$ and consequently $k-1 < y+bk$, we have

$$(2.3) \quad \left[\frac{(x+ak)m \mp (k-1)}{y+bk} \right] = \left[\frac{(x+ak)m}{y+bk} \right] \mp 1.$$

If (2.3) holds, then there is an integer j with $1 \leq j \leq k-1$ such that

$$(2.4) \quad y+bk \mid (x+ak)m \mp j.$$

We show that (2.4) is impossible, so the assumption (2.2) is false.

If (2.4) holds, then there exists an integer $t > 0$ such that

$$(2.5) \quad (y+bk)t = (x+ak)m \mp j.$$

Now

$$(2.6) \quad a(y+bk) - b(x+ak) = \pm 1,$$

so we also have

$$(2.7) \quad -aj(y+bk) - (-bj)(x+ak) = \mp j.$$

And from [10, p. 213], any solution of the Diophantine equation

$$A(y+bk) - B(x+ak) = \mp j$$

has the form

$$(2.8) \quad A = -aj + n(x + ak), \quad B = -bj + n(y + bk), \quad n \in \mathbb{Z},$$

Thus from (2.5), (2.7), and (2.8), for a particular $n_0 \in \mathbb{Z}$, we have

$$m = -bj + n_0(y + bk) \quad \text{and} \quad t = -aj + n_0(x + ak).$$

We rearrange the expression for m to get

$$(2.9) \quad m = b(n_0k - j) + n_0y.$$

If $n_0 > 0$, then from (2.9), since $j < k$, we are forced to conclude $m > b$. If $n_0 \leq 0$, then we must have $m < 0$. In any case, since $1 \leq m \leq b$ by hypothesis, we have a contradiction to (2.4).

If $m < y$, similar arguments rule out all values for n_0 when $j = k$, and we make the second conclusion in the statement of the lemma.

LEMMA 4. *Let a, b, m, x, y be nonnegative integers with $(a, b) = 1$, $1 \leq m < b$, $ay - bx = \pm 1$, and $y > 0$. Then for $k \in \mathbb{N}$,*

$$\left[\frac{(x + ak)m}{y + bk} \right] = \left[\frac{am}{b} \right].$$

PROOF. Note that since $(a, b) = 1$ and $m < b$, $b \nmid am$. Using this fact, $ay - bx = \pm 1$, and the inequality $m/(y + bk) < 1$, we conclude that

$$\begin{aligned} \left[\frac{(x + ak)m}{y + bk} \right] &= \left[\frac{(x + ak)m}{y + bk} - \frac{am}{b} + \frac{am}{b} \right] = \left[\frac{am}{b} \mp \frac{m}{b(y + bk)} \right] \\ &= \left[\frac{am}{b} \mp \frac{m/(y + bk)}{b} \right] = \left[\frac{am}{b} \right]. \end{aligned}$$

LEMMA 5. *Let a, b, x, y, m be nonnegative integers with $(a, b) = 1$, $ay - bx = \pm 1$, $y > 0$, and $1 \leq m < y$. Then for $k \in \mathbb{N} \cup \{0\}$,*

$$(2.10) \quad \left[\frac{(x + ak)m}{y + bk} \right] = \left[\frac{xm}{y} \right].$$

PROOF. We use induction on k . When $k = 0$,

$$\left[\frac{(x + a \cdot 0)m}{y + b \cdot 0} \right] = \left[\frac{xm}{y} \right].$$

Assuming (2.10) for $k \geq 1$, we see that, since $ay - bx = \pm 1$,

$$\begin{aligned} \left[\frac{xm}{y} \right] &= \left[\frac{(x + ak)m}{y + bk} \right] \\ &= \left[\frac{(x + ak)m}{y + bk} + \frac{m(x + a(k + 1))}{y + b(k + 1)} - \frac{m(x + a(k + 1))}{y + b(k + 1)} \right] \end{aligned}$$

$$\begin{aligned}
&= \left[\frac{m(x + a(k + 1))}{y + b(k + 1)} \mp \frac{m}{(y + bk)(y + b(k + 1))} \right] \\
&= \left[\frac{m(x + a(k + 1))}{y + b(k + 1)} \mp \frac{m/(y + bk)}{y + b(k + 1)} \right].
\end{aligned}$$

Now since $m/(y + bk) < 1$ and, from (2.6), $y + b(k + 1) \nmid m(x + a(k + 1))$,

$$\left[\frac{xm}{y} \right] = \left[\frac{m(x + a(k + 1))}{y + b(k + 1)} \right].$$

Proof of Theorem 2. We prove (i). The proof of (ii) follows the same steps.

From (1.1),

$$(2.11) \quad S(x + ak, y + bk) = \sum_{j=1}^{y+bk-1} (-1)^{j+1+\left[\frac{(x+ak)j}{y+bk}\right]} = R_1 + R_2,$$

where

$$R_1 := \sum_{j=1}^{bk} (-1)^{j+1+\left[\frac{(x+ak)j}{y+bk}\right]}$$

and

$$R_2 := \sum_{j=bk+1}^{bk+y-1} (-1)^{j+1+\left[\frac{(x+ak)j}{y+bk}\right]}.$$

First,

$$\begin{aligned}
(2.12) \quad R_1 &= \sum_{l=0}^{k-1} \sum_{j=bl+1}^{b(l+1)} (-1)^{j+1+\left[\frac{(x+ak)j}{y+bk}\right]} \\
&= \sum_{l=0}^{k-1} \sum_{m=1}^b (-1)^{m+bl+1+\left[\frac{(x+ak)(m+bl)}{y+bk}\right]}.
\end{aligned}$$

Now since $ay - bx = \pm 1$,

$$\begin{aligned}
(2.13) \quad \left[\frac{(x + ak)(m + bl)}{y + bk} \right] &= \left[\frac{(x + ak)m + xbl + abkl}{y + bk} \right] \\
&= \left[\frac{(x + ak)m + (ay \mp 1)l + abkl}{y + bk} \right] \\
&= \left[\frac{(x + ak)m \mp l + al(y + bk)}{y + bk} \right] \\
&= \left[\frac{(x + ak)m \mp l}{y + bk} \right] + al.
\end{aligned}$$

So from (2.12), (2.13), and Lemma 3(i),

$$R_1 = \sum_{l=0}^{k-1} (-1)^{(a+b)l} \sum_{m=1}^b (-1)^{m+1+\left[\frac{(x+ak)m}{y+bk}\right]}.$$

Now, separating the inner sum and then applying Lemma 4, we find that

$$(2.14) \quad R_1 = \sum_{l=0}^{k-1} (-1)^{(a+b)l} \left\{ \sum_{m=1}^{b-1} (-1)^{m+1+\left[\frac{am}{b}\right]} + (-1)^{b+1+\left[\frac{(x+ak)b}{y+bk}\right]} \right\}.$$

Simplifying the exponent of the last term inside the braces in (2.14), we see that

$$(2.15) \quad \begin{aligned} b + 1 + \left[\frac{(x + ak)b}{y + bk}\right] &= b + 1 + \left[\frac{bx + akb}{y + bk}\right] \\ &= b + 1 + \left[\frac{ay \mp 1 + akb}{y + bk}\right] \\ &= b + 1 + a + \left[\frac{\mp 1}{y + bk}\right]. \end{aligned}$$

Then we have, from (2.14), (2.15) and the definition of $S(a, b)$,

$$(2.16) \quad R_1 = \sum_{l=0}^{k-1} (-1)^{(a+b)l} \{S(a, b) \pm (-1)^{a+b}\}.$$

To evaluate R_2 , we use $ay - bx = \pm 1$, Lemma 3(ii), Lemma 5, and the definition of $S(x, y)$ to deduce that

$$(2.17) \quad \begin{aligned} R_2 &= \sum_{j=bk+1}^{bk+y-1} (-1)^{j+1+\left[\frac{(x+ak)j}{y+bk}\right]} \\ &= \sum_{m=1}^{y-1} (-1)^{m+bk+1+\left[\frac{(x+ak)(m+bk)}{y+bk}\right]} \\ &= \sum_{m=1}^{y-1} (-1)^{m+bk+1+ak+\left[\frac{(x+ak)m \mp k}{y+bk}\right]} \\ &= (-1)^{(a+b)k} \sum_{m=1}^{y-1} (-1)^{m+1+\left[\frac{(x+ak)m}{y+bk}\right]} \\ &= (-1)^{(a+b)k} \sum_{m=1}^{y-1} (-1)^{m+1+\left[\frac{xm}{y}\right]} = (-1)^{(a+b)k} S(x, y). \end{aligned}$$

Upon combining (2.11), (2.16), and (2.17), we reach the conclusion

$$S(x + ak, y + bk) = \sum_{l=0}^{k-1} (-1)^{(a+b)l} \{S(a, b) \pm (-1)^{a+b}\} + (-1)^{(a+b)k} S(x, y).$$

COROLLARY 6. Let $(a, b) = 1$ with $0 \leq a \leq b$, and let x, y be nonnegative integers such that $ay - bx = \pm 1$ and $y > 0$.

(i) If a and b are both odd, then

$$S(x + ak, y + bk) = \pm k + S(x, y).$$

(ii) If a is even, then

$$S_4(x + ak, y + bk) = \mp k + S_4(x, y).$$

PROOF. To prove (i), we use Theorem 2(i), the fact that $a + b$ is even, and Proposition 1(i). The proof of (ii) is similar.

COROLLARY 7. Let $(a, b) = 1$ with $0 \leq a \leq b$, and let x, y be nonnegative integers such that $ay - bx = \pm 1$ and $y > 0$.

(i) If $a + b$ is odd and x and y are both odd, then

$$S(x + ak, y + bk) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ S(a, b) \mp 1 & \text{if } k \text{ is odd.} \end{cases}$$

(ii) If a is odd and x is even, then

$$S_4(x + ak, y + bk) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ S_4(a, b) \pm 1 & \text{if } k \text{ is odd.} \end{cases}$$

PROOF. To prove (i), we note that, from Theorem 2(i) and Proposition 1(i),

$$\begin{aligned} S(x + ak, y + bk) &= \sum_{l=0}^{k-1} (-1)^l \{S(a, b) \mp 1\} \\ &= \begin{cases} 0 & \text{if } k \text{ is even,} \\ S(a, b) \mp 1 & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

The proof of (ii) is similar.

REMARK. In the cases when k is even in Corollary 7 the sums on the left are equal to zero by Proposition 1.

COROLLARY 8. Let $(a, b) = 1$ with $0 \leq a \leq b$, and let x, y be nonnegative integers such that $ay - bx = \pm 1$ and $y > 0$.

(i) If $a + b$ is odd and $x + y$ is odd, then

$$S(x + ak, y + bk) = \begin{cases} S(a, b) \mp 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

(ii) If a is odd and x is odd, then

$$S_4(x + ak, y + bk) = \begin{cases} S_4(a, b) \pm 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

PROOF. We prove (i). The proof of (ii) follows the same steps. If $k = 1$, recall that $ay - bx = \pm 1$, so that, because of the parities of a, b, x , and y , the result follows from Proposition 1(i). If $k \geq 2$, (i) follows from Theorem 2(i) and Proposition 1(i) since

$$\begin{aligned} S(x + ak, y + bk) &= S(x + a + a(k - 1), y + b + b(k - 1)) \\ &= \sum_{l=0}^{k-2} (-1)^l \{S(a, b) \mp 1\} + (-1)^{k-1} S(x + a, y + b) \\ &= \sum_{l=0}^{k-2} (-1)^l \{S(a, b) \mp 1\} \\ &= \begin{cases} S(a, b) \mp 1 & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

COROLLARY 9. Let $(a, b) = 1$ with $0 \leq a \leq b$, and let x, y be nonnegative integers such that $ay - bx = \pm 1$ and $y > 0$.

(i) If $a + b$ is odd and $x + y$ is odd, then

$$S(a, b) - S(x, y) = \pm 1.$$

(ii) If a is odd and x is odd, then

$$S_4(a, b) - S_4(x, y) = \mp 1.$$

PROOF. We apply Theorem 2(i) with $k = 1$ to get

$$(2.18) \quad S(x + a, y + b) = S(a, b) \pm (-1)^{a+b} + (-1)^{a+b} S(x, y).$$

By Corollary 8(i), with $k = 1$,

$$(2.19) \quad S(x + a, y + b) = 0.$$

The result follows from (2.18) and (2.19), since $a + b$ is odd. As usual, (ii) is proved similarly.

COROLLARY 10. Let $(a, b) = 1$ with $0 \leq a \leq b$, and let x, y be nonnegative integers such that $ay - bx = \pm 1$ and $y > 0$.

(i) If $a + b$ is odd, $x + y$ is odd and k is even, then

$$S(x + ak, y + bk) = S(x, y).$$

(ii) If a is odd, x is odd and k is even, then

$$S_4(x + ak, y + bk) = S_4(x, y).$$

PROOF. To prove (i), we have, from Corollary 8(i),

$$(2.20) \quad S(x + ak, y + bk) = S(a, b) \mp 1.$$

Then the result follows immediately from (2.20) and Corollary 9(i). The proof of (ii) is similar.

3. Rademacher’s questions. We are now in a position to address the questions of Rademacher as they relate to the sum $S(d, c)$. Let $(d, c) = 1$ and define $g(d/c) = S(d, c)$. Here we require that c and d be relatively prime, since $S(qd, qc) \neq S(d, c)$ in general. The analogues of Rademacher’s questions are:

- 1) Is $g(x)$ unbounded in any neighborhood of any real number?
- 2) Are the points $(x, g(x))$ dense in $\mathbb{R} \times \mathbb{Z}$?
- 3) Given two adjacent Farey fractions, h_1/k_1 and h_2/k_2 with $g(h_1/k_1) > 0$ and $g(h_2/k_2) > 0$, does it follow that

$$g\left(\frac{h_1 + h_2}{k_1 + k_2}\right) \geq 0?$$

Note that $g(x)$ is periodic with period 2, since

$$\begin{aligned} (3.1) \quad g\left(\frac{d}{c} + 2\right) &= S(d + 2c, c) = \sum_{j=1}^{c-1} (-1)^{j+1+[(d+2c)j/c]} \\ &= \sum_{j=1}^{c-1} (-1)^{3j+1+[dj/c]} = \sum_{j=1}^{c-1} (-1)^{j+1+[dj/c]} \\ &= S(d, c) = g\left(\frac{d}{c}\right). \end{aligned}$$

Also observe that

$$\begin{aligned} (3.2) \quad S(d + c, c) &= \sum_{j=1}^{c-1} (-1)^{j+1+[(d+c)j/c]} = \sum_{j=1}^{c-1} (-1)^{2j+1+[dj/c]} \\ &= - \sum_{j=1}^{c-1} (-1)^{[dj/c]} = -S_4(d, c). \end{aligned}$$

We now show that, as in the case of Dedekind sums, Question 1 can be answered affirmatively.

THEOREM 11. *The function $g(x)$ is unbounded above and below in any neighborhood of any real number.*

Proof. Let $r \in \mathbb{R}$ and $\varepsilon > 0$ be given. Because of (3.1), we may assume that $r \in [0, 2)$. If $r \in [0, 1]$, then there exists a/b such that $(a, b) = 1$, a and b are both odd, and

$$\left| r - \frac{a}{b} \right| < \varepsilon.$$

Let $x_1, y_1 \in \mathbb{N}$ be such that $ay_1 - bx_1 = 1$. Then for large enough $k \in \mathbb{N}$,

$$\left| r - \frac{x_1 + ak}{y_1 + bk} \right| < \varepsilon.$$

Now, applying Corollary 6(i), we see that

$$(3.3) \quad g\left(\frac{x_1 + ak}{y_1 + bk}\right) = k + O(1) \quad \text{as } k \rightarrow \infty.$$

Similarly, there exist $x_2, y_2 \in \mathbb{N}$ such that $ay_2 - bx_2 = -1$ and

$$\left| r - \frac{x_2 + ak}{y_2 + bk} \right| < \varepsilon.$$

Then, by Corollary 6(i) again, we have

$$(3.4) \quad g\left(\frac{x_2 + ak}{y_2 + bk}\right) = -k + O(1) \quad \text{as } k \rightarrow \infty.$$

Thus, from (3.3) and (3.4), $g(x)$ is arbitrarily large in either direction in this neighborhood.

If $r \in (1, 2)$, we apply a similar argument to $r - 1$. There exists a/b with $(a, b) = 1$, a even, and

$$\left| r - 1 - \frac{a}{b} \right| < \varepsilon.$$

Now let $x_1, y_1 \in \mathbb{N}$ with $ay_1 - bx_1 = 1$. Then for large enough $k \in \mathbb{N}$,

$$(3.5) \quad \left| r - 1 - \frac{x_1 + ak}{y_1 + bk} \right| < \varepsilon.$$

Then by Corollary 6(ii),

$$(3.6) \quad S_4(x_1 + ak, y_1 + bk) = -k + S_4(x_1, y_1).$$

Thus from (3.2) and (3.6), we get

$$S(x_1 + ak + y_1 + bk, y_1 + bk) = -S_4(x_1 + ak, y_1 + bk) = k - S_4(x_1, y_1).$$

In other words,

$$(3.7) \quad g\left(\frac{x_1 + ak + y_1 + bk}{y_1 + bk}\right) = k + O(1) \quad \text{as } k \rightarrow \infty.$$

And from (3.5),

$$\left| r - \frac{x_1 + ak + y_1 + bk}{y_1 + bk} \right| < \varepsilon.$$

So we conclude from (3.7) that $g(x)$ is unbounded from above for $x \in (1, 2)$. An identical argument shows that $g(x)$ is also unbounded from below on $(1, 2)$.

We now answer Question 2. Again, as with Dedekind sums, we answer in the affirmative.

THEOREM 12. *The points $(x, g(x))$ are dense in $\mathbb{R} \times \mathbb{Z}$.*

The proof of Theorem 12 depends upon two additional results. The first, due to Goldberg [5, pp. 109–110], is

THEOREM 13. *Let $(c, d) = 1$ with $d^2 \equiv -1 \pmod{c}$.*

- (i) *If c is odd and d is even, then $S(d, c) = 0$.*
- (ii) *If c and d are both odd, then $S_4(d, c) = 0$.*

The second is due to W. Duke, J. B. Friedlander, and H. Iwaniec [4]. Let $f(x)$ be an irreducible quadratic polynomial with integral coefficients and negative discriminant and let p be prime. Let p vary and take all numbers of the form ν/p , where $f(\nu) \equiv 0 \pmod{p}$ and $0 < \nu \leq p$. Arrange the numbers ν/p as a sequence n_1, \dots, n_m, \dots , so that the corresponding denominators p are in ascending order.

THEOREM 14. *The sequence n_1, \dots, n_m, \dots , as defined above, is uniformly distributed in the interval $(0, 1)$.*

Proof of Theorem 12. It is sufficient to show that the function $g(x)$ takes on every integer value in any neighborhood of a number r . Also, by (3.1), we need only consider $r \in [0, 2)$.

Let $\varepsilon > 0$, $r \in [0, 1]$, and our desired value $M \in \mathbb{Z}$ be given.

We want to find a_0/b_0 such that $(a_0, b_0) = 1$, b_0 is odd, a_0 is even, $a_0^2 \equiv -1 \pmod{b_0}$, and

$$(3.8) \quad \left| r - \frac{a_0}{b_0} \right| < \varepsilon.$$

This can be accomplished by applying Theorem 14 in a way that will guarantee that a_0 is even. First let $f(x) = 4x^2 + 1$. Theorem 14 asserts that there is a ν and an odd prime p such that $4\nu^2 \equiv -1 \pmod{p}$ and

$$\left| \frac{r}{2} - \frac{\nu}{p} \right| < \frac{\varepsilon}{2}.$$

Now set $a_0 = 2\nu$ and $b_0 = p$ and the conditions are satisfied.

If $M = 0$ we stop since, by Theorem 13(i), $g(a_0/b_0) = 0$. And we have found a number in the neighborhood giving the desired value. Suppose $M > 0$. Let $x_0, y_0 > 0$ be such that $a_0y_0 - b_0x_0 = -1$ and $x_0 + y_0$ is odd. Then from (3.8), for sufficiently large k ,

$$\left| r - \frac{x_0 + a_0k}{y_0 + b_0k} \right| < \varepsilon.$$

Now by Corollary 8(i) and Theorem 13(i), for some large even k_0 ,

$$(3.9) \quad g\left(\frac{x_0 + a_0k_0}{y_0 + b_0k_0}\right) = S(x_0 + a_0k_0, y_0 + b_0k_0) = S(a_0, b_0) + 1 = 1.$$

If $M = 1$ we stop, since we have found a number in the neighborhood giving the value M . If $M > 1$, set

$$a_1 = x_0 + a_0k_0 \quad \text{and} \quad b_1 = y_0 + b_0k_0.$$

Since $(a_1, b_1) = 1$, there exist $x_1, y_1 > 0$ such that $a_1y_1 - b_1x_1 = -1$ and $x_1 + y_1$ is odd. Then from (3.8), for sufficiently large k ,

$$\left| r - \frac{x_1 + a_1k}{y_1 + b_1k} \right| < \varepsilon.$$

From Corollary 8(i) and (3.9), with a sufficiently large even k_1 , we deduce that

$$g\left(\frac{x_1 + a_1k_1}{y_1 + b_1k_1}\right) = g\left(\frac{a_1}{b_1}\right) + 1 = 2.$$

We continue this process M times to find x_{M-1}, y_{M-1} and a_{M-1}, b_{M-1} such that $x_{M-1} + y_{M-1}$ and $a_{M-1} + b_{M-1}$ are odd, $a_{M-1}y_{M-1} - b_{M-1}x_{M-1} = -1$, and

$$S(a_{M-1}, b_{M-1}) = M - 1,$$

and for large enough k ,

$$\left| r - \frac{x_{M-1} + a_{M-1}k}{y_{M-1} + b_{M-1}k} \right| < \varepsilon.$$

Choosing a large enough even k_{M-1} and using Corollary 8(i), we find that

$$g\left(\frac{x_{M-1} + a_{M-1}k_{M-1}}{y_{M-1} + b_{M-1}k_{M-1}}\right) = g\left(\frac{a_{M-1}}{b_{M-1}}\right) + 1 = M - 1 + 1 = M.$$

If $M < 0$, then we apply the same argument, except that we find appropriate a, b, x, y with $ay - bx = 1$.

If $r \in (1, 2)$, we apply the argument used in the proof of Theorem 11, using Corollary 8(ii), Theorem 13(ii) and finally (3.2).

We now address Question 3. Here, unlike the situation with Dedekind sums, the question, as posed, has a simple answer. Suppose that h_1/k_1 and h_2/k_2 are adjacent Farey fractions with $g(h_1/k_1), g(h_2/k_2) \neq 0$. Recall that, from elementary properties of Farey fractions, $h_1k_2 - h_2k_1 = \pm 1$. Then by Proposition 1(i), $h_1 + k_1$ and $h_2 + k_2$ are odd. Thus $h_1 + h_2$ and $k_1 + k_2$ are both odd and by Proposition 1(i),

$$g\left(\frac{h_1 + h_2}{k_1 + k_2}\right) = S(h_1 + h_2, k_1 + k_2) = 0.$$

From Corollary 9(i), we deduce a more interesting result.

THEOREM 15. *Let $h_1/k_1 < h_2/k_2$ be adjacent Farey fractions. If $g(h_1/k_1) >$ (resp. $<$) 0, then*

$$g\left(\frac{h_2}{k_2}\right) \geq \text{(resp. } \leq) 0.$$

Proof. If h_2 and k_2 are both odd, then, by Proposition 1(i), $g(h_2/k_2) = 0$.

If $h_2 + k_2$ is odd, then, by Corollary 9(i),

$$S(h_2, k_2) = 1 + S(h_1, k_1) > (\text{resp. } \leq) 0.$$

Identical arguments can be used to show that Theorems 11, 12, and 15 hold for the function $g_4(x)$, defined by $g_4(d/c) = S_4(d, c)$ with $(d, c) = 1$.

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